第25卷第4期 数 学 研 究 与 评 论 Vol.25, No.4
2005年11月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Nov., 2005

Article ID: 1000-341X(2005)04-0647-04 **Document code:** A

A Set-Theoretical Lemma That Implies an Abstract Form of Gödel's Theorem

Edward Arroyo[1], *L. C. Hsu*[2]

(1. Department of Computer Science, Francis Marion University, Florence, SC, USA;
2. Department of Mathematics, Dalian University of Technology, Liaoning 116024, China)

Abstract: We propose a simple set-theoretical lemma that implies Gödel's Incompleteness Theorem. Also mentioned are some related consequences.

Key words: Enumerably infinite set; Gödel's Incompleteness Theorem; turing machines.
MSC(2000): 03E05, 03E75
CLC number: O152

1. A Lemma and an Abstract Form of Gödel's Theorem

In what follows **N** denotes the set of positive integers.

Lemma 1 *Let* $\{S_i\}_{i\geq 1}$ *be an infinite sequence of sets, each containing enumerably infinite distinct elements, i.e.* $S_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}, i \in \mathbf{N}$. *If we let* $W = \cup_{i=1}^{\infty} S_i$, *then we can draw two conclusions:*

(i) There can be constructed an enumerable set S *from* $\{S_i\}$ *that differs from every* S_i.

(ii) There exists a set of positive integers I *and an injection* $\sigma : I \to \mathbf{N}$ *such that*

$$\overline{S} = \bigcup_{i\in I} \{x_{i\sigma(i)}\},$$

where $\overline{S}$ *is the compliment of* S *with respect to* W *and* $x_{i\sigma(i)} \in S_i$.

Proof Let us choose an enumeration $\tau : \mathbf{N} \to W$ of W. Define the subset $T \subseteq \mathbf{N}$ by

$$T = \{k|\tau(k) \in S_k\} = \{k|\tau(k) = x_{kl}, \text{ for some } l \in \mathbf{N}\}.$$

Then

$$\overline{T} = \mathbf{N}\backslash T = \{k|\tau(k) \notin S_k\}.$$

Let

$$X = \tau(T), \quad \overline{X} = \tau(\overline{T}).$$

We claim that $\overline{X} \neq S_i$ for any $i \in \mathbf{N}$. If $\overline{X} = S_{i_0}$ for some $i_0 \in \mathbf{N}$, then

$$i_0 \in \overline{T} \Rightarrow \tau(i_0) \in \overline{X} = S_{i_0},$$

Received date: 2005-03-14

which is false by the definition of $\overline{T}$. On the other hand, if $i_0 \notin \overline{T}$, i.e., $i_0 \in T$, then

$$T(i_i) \notin \overline{X} = S_{i_0}$$

and this implies that $i_0 \in \overline{T}$, a contradiction. Hence, it must be the case $\overline{X} \neq S_i$ for any $i \in \mathbf{N}$. Let $S = \overline{X}$. Then S satisfies condition (i) of the lemma. It remains to show that $\overline{S} = X$ satisfies condition (ii).

Observe that

$$x_{ij} \in \overline{S} \Longleftrightarrow x_{ij} = \tau(i)$$

and so for each $i \in T$, $\overline{S}$ contains precisely one element x_{ij} of S_i in accordance with the definition $X = \tau(T)$, i.e., $j = \sigma(i)$. Hence the lemma is proved. □

In order to connect Lemma 1 with Gödel's theorem, we recall a few basic concepts related to Turing machines (cf. Arbib[1], 1st edition: p.19-20; 2nd edition: p.136-141).

Definition 1 *By an effective procedure on a sequence of symbols we mean a computation process that can be executed by a Turing machine.*

Definition 2 *An integer-set is "recursive" if there exists an effective procedure for deciding whether or not every $n \in \mathbf{N}$ belongs to it. [This concept has various equivalent statements.]*

Definition 3 *An integer-set is "recursively enumerable" if there exists an effective procedure for generating its elements, one after another.*

It is easily shown that the following proposition holds (cf. Arbib[1], 1st edition: Th. 1.6.2; Martin[2], Thms. 10.5-10.7).

Poposition 1 *An integer-set S is recursive if and only if both S and its compliment $\overline{S}(= \mathbf{N} \backslash S)$ are recursively enumerable.*

We also know that the collection of all Turing machines is effectively enumerable, and consequently the collection of all recursively enumerable sets of positive integers is also effectively enumerable. Starting with these basic facts we may now deduce an abstract form of Gödel's Incompleteness Theorem as a consequence of Lemma 1.

Theorem 1 (Abstract form of Gödel's Theorem) *There exists a recursively enumerable set of positive integers which is not recursive.*

Proof It suffices to show the existence of a recursively enumerable set X whose complement $\overline{X}$ is not recursively enumerable.

All recursively enumerable sets are enumerable so that they may be displayed as $S_1, S_2, S_3, \cdots$. Thus by Lemma 1 there can be construced a set $S = \overline{X}$ from $\{S_i\}$, which differs from every S_i, so that S is not recursively enumerable. However, $\overline{S} = X$ consists of a single element from each S_i, where i ranges over some subset of $\mathbf{N}$. Therefore, $\overline{S} = X$ is recursively enumerable and the theorem is proved. □

It may be worth mentioning that the deduction of Gödel's Incompleteness Theorem, that "every consistent adequate arithmetic logic is incomplete", from the above "abstract form" only requires a third of a printed page to sketch (cf. Arbib[1], 2nd edition: p.166). Of course, all the technical terms involved in the theorem are assumed to be known.

2. A few remarks and consequences

Remark 1 Apparently the conclusion of Lemma 1 simply asserts that, starting with $W = \cup_{i=1}^{\infty} S_i$ and by suitably deleting some single element from each S_i's, where i ranges over some subset of $\mathbf{N}$, the remaining set, say S, will be different from every $S_i (i \in \mathbf{N})$. More precisely, S may be expressed in a set-theoretical subtraction form

$$S = \bigcup_{i=1}^{\infty} S_i \backslash \bigcup_{j \in J} \{x_{j\sigma(j)}\}. \tag{*}$$

where $\sigma : \mathbf{J} \to \mathbf{N}$ is a certain injection. The key point of Lemma 1 is to show the existence of the integer set $\mathbf{J}$ as required. The technique used in the proof is a form of "diagonal argument", a common technique in discrete mathematics.

Remark 2 If in the proof of Lemma 1 we use a diagonal process to define the enumeration $\tau : \mathbf{N} \to W$ of W, then it is easily seen that τ induces a partial ordering on the elements x_{ij} of W. Let the ordering relation be denoted by $\prec$. Then for $j < k$,

$$x_{ij} = \tau(p) \prec x_{ik} = \tau(q) \Rightarrow p < q;$$

$$x_{ji} = \tau(r) \prec x_{ki} = \tau(s) \Rightarrow r < s.$$

In this way we see that the natural ordering of τ is consistent with every ordered set S_i o f W. This leads us to state following.

Proposition 2 *Using a diagonal process as an enumeration $\tau : \mathbf{N} \to W$ of W, the enumerable set S as asserted by (i) of Lemma 1 could be an ordered set whose ordering is induced by τ, and which is consistent with the ordering of the elements of each set S_i.*

Sometimes the construction of the required set S as asserted in Lemma 1 may be conveniently achieved by omitting from each of given sets a fixed subset of the union of the given sets. More precisely, we have:

Proposition 3 *Suppose that we are given an infinite sequence of sets $\{S_i\}_{i \geq 0}$, each containing enumerably infinite distinct elements, i.e.*

$$S_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}, \quad i \geq 0.$$

Let $W = \cup_{i=1}^{\infty} S_i$ and assume that $S_0 \subseteq W$. Let $\{S_i'\}_{i \geq 1}$ be defined by $S_i' = S_i \backslash S_0, i \geq 1$. Then

(i) There can be constructed an enumberable set S from $S_0 \cup \{S_i'\}$ that differs from every S_i.

(ii) There exists a set of positive integers I and an injection $\sigma : I \to \mathbf{N}$ such that

$$\overline{S} = \bigcup_{i \in I} \{x_{i\sigma(i)}\},$$

where $\overline{S}$ is the compliment of S with respect to W and $x_{i\sigma(i)} \in S_i$.

Proof Let $W' = \cup_{i=1}^{\infty} S_i'$. Hence, $W = S_0 \cup W'$. By using Lemma 1 applied to $\{S_i'\}$ in such a way that S' differs from every S_i and $\overline{S'} = W' \backslash S'$ satisfies the condition

$$\overline{S'} = \bigcup_{i \in I} \{x_{i\sigma(i)}\}$$

for some set of positive integers I and some injection $\sigma : I \to \mathbf{N}$. Observe that each $x_{i\sigma(i)} \in S_i' \subseteq S_i$. Let $S = S_0 \cup S'$ and observe that

$$\overline{S} = W \backslash S = (S_0 \cup W') \backslash (S_0 \cup S') = W' \backslash S' = \overline{S}'.$$

Hence, the S satisfies condition (i) since S' does and $\overline{S}$ satisfies condition (ii) since $\overline{S}'$ does. □

Finally, it may be worth noticing that Lemma 1 can even be reformulated as a metamathematical proposition that may have applications from the methodological viewpoint.

Let P denote a certain property that applies to a sequence or an ordered set of enumerably infinite distince elements, say, $S = \{x_1, x_2, x_3, \cdots\}$. This is to say, all the x-elements of S have the P-property. In this case, S is called a P-set. Then, as a consequence of Lemma 1, we have:

Proposition 4 *Suppose that we are given a definite property P. If all the P-sets $\{S_i\}, i \geq 1$, of distince x-elements are enumerably infinite, i.e.*

$$S_i = \{x_{i1}, x_{i2}, x_{i3}, \cdots\}, \quad i \in \mathbf{N},$$

then, according to the set-theoretical subtraction form $(*)$, there can be constructed a set S of x-elements from $\{S_i\}$ that differs from every S_i, so that S is not a P-set.

Apparently, the abstract form of Gödel's theorem may be regarded as a consequence of the last Proposition. It is hoped that this proposition may have other applications in those parts of discrete mathematicas that deal with infinitely many objects.

References:

[1] ARBIB M A. *Brains, Machines, and Mathematics* [M]. McGraw-Hill Book Co., New York, 1964.

[2] MARTIN J. *Introduction to Languages and the Theory of Computation* [M]. McGraw-Hill Book Company, Boston, New York, London, 2003, 365–371.

[3] SMULLYAN R M. *Gödel's Incompleteness Theorem* [M]. Oxford Univ. Press, Oxford (U.K.), New York, 1992, 56–64.

[4] MACLANE S. *Mathematics Form and Function* [M]. Springer-Verlag, New York, Berlin, Tokyo, 1986, 379–383.

蕴含 Gödel 定理抽象形式的集合论引理

爱德华 · 阿罗约 [1]， 徐利治 [2]

(1. Francis Marion 大学计算机科学系， Florence, SC, USA; 2. 大连理工大学应用数学系，辽宁 大连 116024)

摘要: 本文证明了一个集合论性质的引理，由它可直接得出 Gödel 不完全性定理的抽象形式. 文中还述及该引理的有关诸推论.

关键词: 可数无穷集； Gödel 不完全性定理；图灵机.