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Properties of Fuzzy M-Semigroups with t-Norms

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Abstract: In this papr, we introduce the notion of T-fuzzy M-subsemigroups of M-semigroups by using a t-norm T and obtain some interesting properties. Further we show that the direct product of a T-fuzzy M-subsemigroup of R and a fuzzy M-subsemigroup of S is a T-fuzzy M-subsemigroup of M. Moreover, we prove that T-fuzzy M-subsemigroup of M is exhibited as the direct product of T-fuzzy M-subsemigroups of R and S respectively.

Key words: M-semigroups; t-norm; (imaginable) T-fuzzy M-subsemigroups; T-product.
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1. Introduction

In the theory of semigroups, the properties of the direct product of two semigroups are an important topic to study. In particular, Petrich has studied the structure of prime ideals in the direct product of two semigroups[7]. Plemmas and Robbert have further studied the structure of maximal ideals in the direct product of two semigroups[9]. On the other hand, Tamura, Markel and Latimer[11,12] have discussed M-groupoid as the direct product of a right zero semigroup and a groupoid with identity. It was shown that Warner in [13] that an M-semigroup is isomorphic to the direct product of a right zero semigroup and semigroup with two sided identity. In 1965, Zadeh has initiated fuzzy set theory, which turned out to be of far reaching implications. A detailed study on fuzzy subgoupoid and fuzzy subgroup has been made by Rosenfeld[10]. Kuroki[3−5] has contributed fuzzy theory results in semigroups. Recently, AL.Narayanan and AR.Meenakshi have defined the concept of fuzzy M-subsemigroup of an M-semigroup and they have a canonical example for fuzzy M-subsemigroup[8]. In this paper, by using a t-norm T, we introduce the notion of T-fuzzy M-subsemigroups of M-semigroups and obtain some interesting properties of fuzzy M-subgroups. In this papr, we introduce the notion of T-fuzzy M-subsemigroups of M-semigroups by using a t-norm T and obtain some interesting properties. Further we show that the direct product of a T-fuzzy M-subsemigroup of R and a fuzzy M-subsemigroup of S is a T-fuzzy M-subsemigroup of M. Moreover, we prove that T-fuzzy M-subsemigroup of M is exhibited as the direct product of T-fuzzy M-subsemigroups

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of R and S respectively.

2. Preliminaries

Definition 2.1[8] *Let S and R be semigroups. Then we define the direct product of S and R by the set $S \times R$ of all pairs (s, r) of the elements s of S and r of R with the coordinate multiplication $(s, r)(s_1, r_1) = (ss_1, rr_1)$, for $s, s_1 \in S$ and $r, r_1 \in R$.*

Definition 2.2[2] *A semigroup S is said to be a right (left) zero semigroup if for all $x, y \in S, xy = y (xy = x)$.*

Definition 2.3[8] *Let S be a semigroup. If there exists an element $e \in S$ such that $xe = ex = x$ for every $x \in S$, then S is called a semigroup with two sided identity.*

Definition 2.4[12] *A semigroup M is called an M-semigroup if the following two conditions are satisfied:*

(i) There exists at least one left identity $e \in M$ such that $ex = x$, for all $x \in M$;

(ii) For every $x \in M$, there is a unique left identity, say e_x such $xe_x = x$, that is, e_x is two sided identity.

To be more precise, we call an M-semigroup a left M-semigroup if we consider only the left identities. In the same manner, we can define a right M-semigroup. Throughout this paper, M-semigroup always means left M-semigroup unless otherwise specified.

Lemma 2.5[13] *An M-semigroup M is isomorphic to the direct product of a right zero semigroup R and a semigroup S with two sided identity. That is, $M \cong R \times S$, where the right singular semigroup R is the set of all left identities of M and $S = Me$ for a left identity $e \in M$.*

Lemma 2.6[13] *The direct product of a right zero semigroup R and a semigroup S with two sided identity is an M-semigroup.*

Lemma 2.7[6] *An M-semigroup M can be always decomposable into the union of some isomorphic M-subsemigroups of M.*

Hereafter, we use M, R and S to denote the (left) M-semigroup, right singular semigroup and semigroup with two sided identity respectively, unless otherwise mentioned.

Definition 2.8[8] *Let M be an M-semigroup. A fuzzy subset $\mu : M \to [0, 1]$ is a fuzzy M-subsemigroup of M if the following conditions are satisfied:*

(i) $\mu(xy) \geq \min(\mu(x), \mu(y))$ for all $x, y \in M$;

(ii) $\mu(e) = 1$ for every left identity $e \in M$.

Definition 2.9[1] *By a t-norm T, we mean a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:*

(T1) $T(x, 1) = x$;

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$;

(T3) $T(x, y) = T(y, x)$;

(T4) $T(x, T(y, z)) = T(T(x, y), z)$

for all $x, y, z \in [0, 1]$.

For a t-norm T on $[0, 1]$, we denote it by $\Delta_T = \{\alpha \in [0, 1] | T(\alpha, \alpha) = \alpha\}$.

Note 2.10 Every t-norm T has the following property: $T(\alpha, \beta) \leq \min\{\alpha, \beta\}$.

Definition 2.11 *Let T be a t-norm. Then we say that the fuzzy set μ in X satisfies the imaginable property if* $\mathrm{Im}\mu \subseteq \Delta_T$.

Definition 2.12 *The fuzzy union of two fuzzy subsets μ and ν of M, denoted by $\mu \cup \nu$, is a fuzzy subset of M defined by $(\mu \cup \nu)(x) = T(\mu(x), \nu(x))$ for all $x \in M$.*

Definition 2.13 *Let μ and ν be two fuzzy subsets. Then we define the T-product $\mu \circ \nu$ by*

$$(\mu \circ \nu)(x) = \begin{cases} \sup_{x=yz}(T(\mu(y), \mu(z))) & \text{if } x \text{ is of the form } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.14[11] *Let M be an M-semigroup and μ a fuzzy subset of M. Then the set $\mu_t = \{x \in M | \mu(x) \geq t\}$ is called a level subset of M with respect to μ.*

3. T-fuzzy M-subsemigroups

Definition 3.1 *Let M be an M-semigroup. A fuzzy subset $\mu : M \to [0, 1]$ is a fuzzy M-subsemigroup of M with respect to a t-norm T (in brief, T-fuzzy M-subsemigroup of M) if it satisfies the following conditions:*

(i) $\mu(xy) \geq T(\mu(x), \mu(y))$ *for all* $x, y \in M$;

(ii) $\mu(e) = 1$ *for every left identity* $e \in M$.

Definition 3.2 *A T-fuzzy M-subsemigroup of an M-semigroup M is said to be imaginable if it satisfies the imaginable property.*

Example 3.3 Let $M = \{e, f, a, b\}$ be an M-semigroup with the following Cayley table:

$\cdot$	e	f	a	b
e	e	f	a	b
f	e	f	a	b
a	a	b	e	f
b	a	b	e	f

Define a fuzzy subset $\mu : M \to [0, 1]$ by $\mu(e) = 1 = \mu(f), \mu(a) = \mu(b) = 0$ and let $T_m : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$. Then T_m is a t-norm. By routine calculation , we see that μ is a T_m-fuzzy M-subsemigroup of M and $\mathrm{Im}\mu \subseteq \Delta_{T_m}$.

Theorem 3.4 *(i) Let T be a t-norm. Then every imaginable T-fuzzy M-subsemigroup of M is a fuzzy M-subsemigroup of M.*

(ii) If μ is an imaginable T-fuzzy M-subsemigroup of M, then every non-empty level subset μ_t of μ is a M-subsemigroup of M.

Proof (i) Let μ be an imaginable T-fuzzy M-subsemigroup of M. Then $\mu(xy) \geq T(\mu(x), \mu(y))$ for all $x, y \in M$. Since μ is imaginable , we have

$$\min(\mu(x), \mu(y)) = T(\min(\mu(x), \mu(y)), \min(\mu(x), \mu(y))) \leq T(\mu(x), \mu(y)) \leq \min(\mu(x), \mu(y)).$$

Hence , it follows that $\mu(xy) \geq T(\mu(x), \mu(y)) = \min(\mu(x), \mu(y))$. This shows that μ is a fuzzy M-subsemigroup of M.

(ii) Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows that $\mu(xy) \geq T(\mu(x), \mu(y)) \geq T(t, t) = t$, and thus $xy \in \mu_t$. Hence μ_t is a sub-semigroup of M. For every $e \in M$, we have $\mu(e) = 1 \geq t$, and so $e \in \mu_t$. Let $x \in \mu_t$. Then $ex \in \mu_t \subseteq M$. This leads to $ex = x$, and so μ_t is an M-subsemigroup of M.

The following example illustrates that there exists a t-norm T such that a fuzzy M-subsemigroup of M may not be an imaginable T-fuzzy M-subsemigroup of M.

Example 3.5 Let $M = \{e, f, a, b\}$ be an M-semigroup with the following Cayley table:

$\cdot$	e	f	a	b
e	e	f	a	b
f	f	e	e	f
a	a	e	e	a
b	b	b	b	e

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(e) = 1$, $\mu(f) = \mu(a) = 0.6$, $\mu(b) = 0.3$. Then, we see that μ is a fuzzy M-subsemigroup of M. Let $\gamma \in [0, 1]$ and define the binary operation T_γ on $[0, 1]$ as follows:

$$T_\gamma(\alpha, \beta) = \begin{cases} \alpha \wedge \beta & \text{if} \quad \max(\alpha, \beta) = 1, \\ 0 & \text{if} \quad \max(\alpha, \beta) < 1, \alpha + \beta \leq 1 + \gamma, \\ \gamma & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in [0, 1]$. Then T_γ is a t-norm on $[0, 1]$. Hence, $T_\gamma(\mu(f), \mu(f)) = T_\gamma(0.6, 0.6) = \gamma \neq \mu(f)$ whenever $\gamma < 0.2$, and so $\mu(f) \notin \Delta_{T_\gamma}$, i.e., $\mathrm{Im}\mu \nsubseteq \Delta_{T_\gamma}$ whenever $\gamma < 0.2$. This shows that μ is not an imaginable T-fuzzy M-subsemigroup of M whenever $\gamma < 0.2$.

Definition 3.6 *A mapping $f : M \to M'$ of M-semigroups is called a homomorphism if*

(i) $f(xy) = f(x)f(y)$ for all $x, y \in M$;

(ii) $f(e) = e'$ for every left identity $e \in M$, where e' is a left identity of M'.

Let $f : M \to M'$ be a mapping of M-semigroups. For a fuzzy set μ in M', the inverse image of μ under f, denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 3.7 *Let $f : M \to M'$ be a homomorphism of M-semigroups. If μ is a T-fuzzy M-subsemigroup of M', then $f^{-1}(\mu)$ is a T-fuzzy M-subsemigroup of M.*

Proof For any $x, y \in M$, we have

$$f^{-1}(\mu)(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \geq T(\mu(f(x)), \mu(f(y))) = T(f^{-1}(\mu)(x), f^{-1}(\mu)(y));$$

$$f^{-1}(\mu)(e) = \mu(f(e)) = \mu(e') = 1.$$

This completes the proof.

Let μ be a fuzzy set in X and f a mapping defined on X. Then we call the fuzzy set $f(\mu)$ in $f(X)$ defined by $f(\mu)(y) = \sup\{\mu(x) | x \in f^{-1}(\mu)\}$ for all $y \in f(X)$ the image of μ under f. A fuzzy set μ in X is said to have the sup property if for every subset $T \subseteq X$, there exists $t_0 \in T$ such that $\mu(x_0) = \sup\{\mu(t) | t \in T\}$.

Theorem 3.8 *An onto homomorphic image of a fuzzy M-subsemigroup with the sup property is a fuzzy M-subsemigroup.*

Proof Let $f : M \to M'$ be an onto homomorphism of M-semigroups and let μ be a fuzzy M-subsemigroup of M with the sup property . Now, for any $u, v \in M'$, we let $x_0 \in f^{-1}(u)$ and $y_0 \in f^{-1}(v)$ be such that

$$\mu(x_0) = \sup\{\mu(t) | t \in f^{-1}(u)\}, \mu(y_0) = \sup\{\mu(t) | t \in f^{-1}(v)\}, \quad \text{respectively.}$$

Then we can deduce that

$$\begin{aligned} f(\mu)(uv) &= \sup\{\mu(z) | z \in f^{-1}(uv)\} \geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min\{\sup\{\mu(t) | t \in f^{-1}(u)\}, \sup\{\mu(t) | t \in f^{-1}(v)\}\} \\ &= \min\{f(\mu)(u), f(\mu)(v)\} \\ f(\mu)(e') &= \sup\{\mu(e) | e \in f^{-1}(e')\} = 1. \end{aligned}$$

Hence, $f(\mu)$ is a fuzzy M-subsemigroup of M.

The above theorem can be further strengthened. We first give the following definition:

Definition 3.9 *A t-norm T on $[0,1]$ is called a continuous t-norm if T is a continuous function from $[0,1] \times [0,1] \to [0,1]$ with respect to the usual topology.*

We observe that the function "min" is always a continuous t-norm .

Theorem 3.10 *Let T be a continuous t-norm and $f : M \to M'$ an onto homomorphism of M-semigroups. If μ is a T-fuzzy M-subsemigroup of M, then $f(\mu)$ is a T-fuzzy M-subsemigroup of M'.*

Proof Let $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1y_2)$, where $y_1, y_2 \in M'$. Consider the set $A_1A_2 = \{x \in M | x = a_1a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2\}$.

If $x \in A_1A_2$, then $x = x_1x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that we have $f(x) = f(x_1x_2) = f(x_1)f(x_2) = y_1y_2$, that is, $x \in f^{-1}(y_1y_2) = A_{12}$. Thus $A_1A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f(\mu)(y_1y_2) &= \sup\{\mu(x) | x \in f^{-1}(y_1y_2)\} = \sup\{\mu(x) | x \in A_{12}\} \geq \sup\{\mu(x) | x \in A_1A_2\} \\ &\geq \sup\{\mu(x_1x_2) | x_1 \in A_1, x_2 \in A_2\} \geq \sup\{T(\mu(x_1), \mu(x_2)) | x_1 \in A_1, x_2 \in A_2\}. \end{aligned}$$

Since T is continuous, for every $\varepsilon > 0$, we see that if $\sup\{\mu(x_1)|x_1 \in A_1\} - x_1^* \le \delta$ and $\sup\{\mu(x_2)|x_2 \in A_2\} - x_2^* \le \delta$, then $T(\sup\{\mu(x_1)|x_1 \in A_1\}, \sup\{\mu(x_2)|x_2 \in A_2\}) - T(x_1^*, x_2^*) \le \varepsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$, such that $\sup\{\mu(x_1)|x_1 \in A_1\} - \mu(a_1) \le \delta$ and $\sup\{\mu(x_2)|x_2 \in A_2\} - \mu(a_2) \le \delta$. Then we have

$$T(\sup\{\mu(x_1)|x_1 \in A_1\}, \sup\{\mu(x_2)|x_2 \in A_2\}) - T(\mu(a_1), \mu(a_2)) \le \varepsilon.$$

Consequently , we have

$$\begin{aligned} f(\mu(y_1y_2)) &\ge \sup\{T(\mu(x_1), \mu(x_2))|x_1 \in A_1, x_2 \in A_2\} \\ &\ge T(\sup\{\mu(x_1)|x_1 \in A_1\}, \sup\{\mu(x_2)|x_2 \in A_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)) \end{aligned}$$

and $f(\mu)(e') = \sup\{\mu(e)|e \in f^{-1}(e')\} = 1$. This shows that $f(\mu)$ is a T-fuzzy M-subsemigroup of M' .

4. T-products of M-semigroups

Any element $x \in M \cong R \times S$ can be written as the product (binary operation in M) of two elements: One element in R is called the two sided identity(right identity) of x,denoted it by e_x, and the other element is xe in $S(= Me)$. That is , $x = xe_x = x(ee_x) = (xe)e_x$. Hence, we can express an element x in M by the element (e_x, xe) in $R \times S$.

Definition 4.1 *Let M be an M-semigroup. Then, by a fuzzy subset $\mu : M \to [0, 1]$, we mean a fuzzy subsemigroup of M with respect to a t-norm T (briefly, T-fuzzy subsemigroup of M) if it satisfies $\mu(xy) \ge T(\mu(x), \mu(y))$ for all $x, y \in M$.*

Clearly, every T-fuzzy M-subsemigroup is T-fuzzy subsemigroup.

Theorem 4.2 *Let μ be an imaginable T-fuzzy M-subsemigroup of $M \cong R \times S$ such that $\mathrm{Im}\mu = \{1, \alpha\}, 0 \le \alpha < 1$. Then μ can be written as a union of T-fuzzy subsemigroups.*

Proof Let $M_t = \{x \in M|\mu(x) = 1\}$, we have

$$\mu(x) = \begin{cases} 1 & \text{if } x \in M_t \\ \alpha, & \text{if } x \in M - M_t. \end{cases}$$

We claim that M_t is an M-subsemigroup of M. In fact , if we choose $x, y \in M_t$, then we have $\mu(x) = \mu(y) = 1$. Since μ is a T-fuzzy M-subsemigroup of M, we have

$$\mu(xy) \ge T(\mu(x), \mu(y)) = T(1, 1) = 1.$$

Hence $\mu(xy) = 1$, so that $xy \in M_t$ and hence M_t is a subsemigroup of M. By the definition of a T-fuzzy M-subsemigroup, we have $\mu(e) = 1$ for every left identity $e \in R$. Hence $e \in M_t$. Since M_t is a subsemigroup, we can choose $e, x \in M_t$, such that $ex \in M_t$. Clearly $ex \in M$. This leads to $ex = x$, and so M_t is an M-subsemigroup of M. By applying Lemma 2.7, there exist

M-subsemigroups M_te, M_tf, $\cdots$ of M_t, such that $M_t = M_te \bigcup M_tf \bigcup \cdots$, where $e, f, \cdots \in R$. For each $e \in R$, define the fuzzy subsets μ_e on M by

$$\mu_e(x) = \begin{cases} 1 & \text{if } x \in M_te, \\ \alpha & \text{otherwise.} \end{cases}$$

Then it is easy to prove that every μ_e is a T-fuzzy subsemigroup of M-semigroup M. In fact, if $x, y \in M_te$, then $xy \in M_te$. Consequently, $\mu_e(xy) = 1 \geq T(\mu_e(x), \mu(y))$. If x and $y \notin M_te$, then $\mu_e(x) = \alpha, \mu_e(y) = \alpha$ and $xy \notin M_te$. It follows that $\mu_e(xy) = \alpha$ and $\mu_e(xy) = T(\mu_e(x), \mu_e(y)) = T(\alpha, \alpha) = \alpha$, that is, $\mu_e(xy) = T(\mu_e(x), \mu_e(y))$. If $x \in M_te$ and $y \notin M_te$, then $\mu_e(x) = 1, \mu_e(y) = \alpha$ and $T(\mu_e(x), \mu_e(y)) = T(1, \alpha) = \alpha$. Hence $\mu_e(xy) = T(\mu_e(y), \mu_e(y))$. Now it is straight forward to verify that $\mu = \mu_e \bigcup \mu_f \bigcup \cdots$.

Theorem 4.3 *Let μ be an imaginable T-fuzzy M-subsemigroup of $M \cong R \times S$ such that* $\text{Im}\mu = \{1, \alpha_1, \cdots, \alpha_n\}$, *where* $1 > \alpha_1 > \cdots > \alpha_n \geq 0$. *Then μ can be written as a union of T-fuzzy subsemigroups of M.*

Proof Let $M_1 = \{x \in M | \mu(x) = 1\}$. Then by using the argument in Theorem 4.2, we see that M_1 is an M-subsemigroup of M. By applying Lemma 2.7, we have $M_1 = M_1e \bigcup M_1f \bigcup \cdots$. Since the image of μ is $\{1, \alpha_1, \cdots, \alpha_n\}$, the level subset of μ are $M_1, M_{\alpha_1}, \cdots, M_{\alpha_n}$, where $M_{\alpha_i} = \{x \in M | \mu(x) \geq \alpha_i\}$. For each $e \in R$, define the fuzzy subset μ_e on M as

$$\mu_e(x) = \begin{cases} 1 & \text{if } x \in M_1e, \\ \alpha_1 & \text{if } x \in M_{\alpha_1} - M_1e, \\ \alpha_2 & \text{if } x \in M_{\alpha_2} - M_{\alpha_1}, \\ \cdots & \cdots \\ \alpha_n & \text{if } x \in M_{\alpha_n} - M_{\alpha_{n-1}}. \end{cases}$$

Then μ_e is a T-fuzzy subsemigroup of M. It is straight forward to verify that $\mu = \mu_e \bigcup \mu_f \bigcup \cdots$. □

Lemma 4.4 *Let μ_1 be a T-fuzzy M-subsemigroup of R and μ_2 be a T-fuzzy M-subsemigroup of S. Then for any $e, f \in R$ and $x_1, x_2 \in S$, we have*

$$T(T(\mu_1(e), \mu_1(f)), T(\mu_2(x_1), \mu_2(x_2))) = T(T(\mu_1(e), \mu_2(x_1)), T(\mu_1(f), \mu_2(x_2))).$$

Proof We have

$$\begin{aligned} &T(T(\mu_1(e), \mu_1(f)), T(\mu_2((x_1), \mu_2(x_2))) = T(\mu_1(e), T(\mu_1(f), T(\mu_2(x_1), \mu_2(x_2)))) \\ &\quad = T(\mu_1(e), T(T(\mu_1(f), \mu_2(x_1)), \mu_2(x_2))) \\ &\quad = T(\mu_1(e), T(T(\mu_2(x_1), \mu_1(f)), \mu_2(x_2))) \\ &\quad = T(\mu_1(e), \mu_2(x_1)), T(\mu_1(f)), \mu_2(x_2))). \end{aligned}$$

This completes the proof.

Theorem 4.5 *Let $M \cong R \times S$ be an M-semigroup. If μ_1 is a T-fuzzy M-subsemigroup of R and*

μ_2 is a T-fuzzy M-subsemigroup of S. Then $\mu = \mu_1 \times \mu_2$ defined by $\mu(a,b) = (\mu_1 \times \mu_2)(a,b) = T(\mu_1(a), \mu_2(b))$ is a T-fuzzy M-subsemigroup of M, where $a \in R$ and $b \in S$.

Proof Let $x = (e, x_1)$ and $y = (f, x_2)$ in $R \times S \cong M$, where $e, f \in R$ and $x_1, x_2 \in S$, respectively. Then

(i) $\mu(xy) = \mu(ef, x_1x_2) = (\mu_1 \times \mu_2)(ef, x_1x_2) = T(\mu_1(ef), \mu_2(x_1x_2))$.

Since μ_1 and μ_2 are T-fuzzy M-subsemigroups, by using Definition 3.1 and Lemma 4.4, we have $\mu(xy) \geq T(T(\mu_1(e), \mu(f)), T(\mu_2(x_1), \mu(x_2))) = T(T(\mu_1(e), \mu_2(x_1)), T(\mu_1(f), \mu_2(x_2))) = T((\mu_1 \times \mu_2)(e, x_1), (\mu_1 \times \mu_2)(f, x_2)) = T(\mu(x), \mu(y))$.

(ii) $\mu(e_1, e) = (\mu_1 \times \mu_2)(e_1, e)$, where e_1 is any left identity of R and e is the two sided identity of S, we have $\mu(e_1, e) = (\mu_1 \times \mu_2)(e_1, e) = T(\mu_1(e_1), \mu_2(e)) = T(1,1) = 1$. It follows that μ is a T-fuzzy M-subsemigroup of M.

Lemma 4.6 *Let μ be a T-fuzzy M-subsemigroup of $M \cong R \times S$. Then there exists a homomorphic image μ on R, which is also a T-fuzzy M-subsemigroup of R.*

Proof Let $x \in M$. Denote the unique (right) identity of x by e_x. Then we have $xe_x = x$. Consider the map $\varphi : M \to R$ such that $\varphi(x) = e_x$ for all $x \in M$. Since $(xy)e_{xy} = xy = x(ye_y) = (xy)e_y$, we have $e_{xy} = e_y$. Since e_x is a left identity of e_y, we have $e_{xy} = e_y = e_xe_y$. This shows that $\varphi(xy) = e_{xy} = e_xe_y = \varphi(x)\varphi(y)$. Clearly, φ is an onto homomorphism.

Define a fuzzy subset ν on R by $\nu(e_x) = sup_{x \in \varphi^{-1}(e_x)} \mu(x) = \mu(e_x) = 1$.

For any $e_x, e_y \in R$, by the definition of ν and that φ is a homomorphism from M onto R, we have

(i) $\nu(e_xe_y) = \varphi(\mu)(e_xe_y) = \varphi(\mu)(e_{xy}) = 1 = \mu(e_xe_y)$.

Since μ is a T-fuzzy M-subsemigroup of M, we have $\mu(e_xe_y) \geq T(\mu(e_x), \mu(e_y))$, and thus $\nu(e_xe_y) = \mu(e_xe_y) \geq T(\mu(e_x), \mu(e_y)) = T(\nu(e_x), \nu(e_y))$.

(ii) $\nu = \varphi(\mu)(e_x) = \max \mu(e_x) = 1$.

This shows that $\nu = \varphi(\mu)$ is a T-fuzzy M-subsemigroup of R.

Lemma 4.7 *Let μ be a T-fuzzy M-subsemigroup of $M \cong R \times S$. Then there exists a homomorphic image μ on S which is also a T-fuzzy M-subsemigroup of S.*

Proof Consider the map $\psi : M \to S(= Me)$ such that $\psi(x) = xe$ for all $x \in M$. Then $\psi(xy) = xye = x(ey)e = (xe)(ye) = \psi(x)\psi(y)$. Clearly, ψ is an onto homomorphism. Define a fuzzy subset ν on S by $\sigma(xe) = \sup_{x \in \psi^{-1}(xe)} \mu(x) = \mu(xe)$.

Putting $r = xe, s = ye$ in $S(= Me)$. Then by the definition of σ and that ψ is a homomorphism from M onto S, we have

(i) $\sigma(rs) = \psi(\mu)(rs) = \psi(\mu)(xeye) = \psi(\mu)(xye) = \mu(xye) = \mu(rs)$.

Since μ is a T-fuzzy M-subsemigroup of M, we have

$$\sigma(rs) = \mu(rs) \geq T(\mu(r), \mu(s)) = T(\sigma(r), \sigma(s)).$$

(ii) $\sigma(fe) = \psi(\mu)(fe) = \psi(\mu)(e) = \mu(ee) = \mu(e) = 1$.

Hence $\sigma = \psi(\mu)$ is a T-fuzzy M-subsemigroup of S.

Theorem 4.8 *Let μ be a T-fuzzy M-subsemigroup of $M \cong R \times S$. Then μ can be written as the direct product of a T-fuzzy M-subsemigroup of R and $S(= Me)$, respectively.*

Proof Since μ is a T-fuzzy M-subsemigroup of M, by Lemmas 4.6 and 4.7, there exist $\varphi(\mu)$ and $\psi(\mu)$ which are T-fuzzy M-subsemigroups of R and $S(= Me)$ respectively. Again since μ is a T-fuzzy M-subsemigroup of M, for each $x \in M$, we have

$$\begin{aligned}\mu(x) &= \mu(xe_x) = \mu(xee_x) \geq T(\mu(e_x), \mu(xe)) = T(\varphi(\mu)(e_x), \psi(\mu)(xe)) \\ &= (\varphi(\mu) \times \psi(\mu))(e_x, xe) = (\varphi(\mu) \times \psi(\mu))(x).\end{aligned}$$

Hence $\mu \supseteq \varphi(\mu) \times \psi(\mu)$. Now, let $y \in M$, then we have $y = (e_y, ye) \in R \times S$. Hence we can deduce that

$$\begin{aligned}&(\varphi(\mu) \times \psi(\mu))(y) = (\varphi(\mu) \times \psi(\mu))(e_y, ye) \\ &\quad = T(\varphi(\mu)(e_y), \psi(\mu)(ye)) = T(\mu(e_y), \mu(ye)) = T(1, \mu(ye)) \\ &\quad = \mu(ye) \geq T(\mu(y), \mu(e)) = T(\mu(y), 1) = \mu(y).\end{aligned}$$

This leads to $\varphi(\mu) \times \psi(\mu) \supseteq \mu$. Therefore, $\mu = \varphi(\mu) \times \psi(\mu)$.

This proof is completed.

Theorem 4.9 *Let μ be a T-fuzzy M-subsemigroup of $M \cong R \times S$. For $e \in R$, let the fuzzy subset μ_e^* defined on M by $\mu_e^*(x) = \mu(xe)$ for all $x \in M$. Then we have*

(i) μ_e^* is a T-fuzzy M-subsemigroup of M;

(ii) $\bigcap_{e\in R} \mu_e^*$ is also a T-fuzzy M-subsemigroup of M.

Proof (i) Since μ is a T-fuzzy M-subsemigroup of M, we have

$$\mu_e^*(xy) = \mu(xye) = \mu(xeye) \geq T(\mu(xe), \mu(ye)) = T(\mu_e^*(x), \mu_e^*(y))$$

for all $x, y \in M$; $\mu_e^*(f) = \mu(fe) = \mu(e) = 1$.

Hence μ_e^* is a T-fuzzy M-subsemigroup of M.

(ii) For any $x, y \in M$, we have

$$\begin{aligned}&\bigcap_{e\in R} \mu_e^*(xy) = \min\{\mu_e^*(xy)|e \in R\} \geq \min\{T(\mu_e^*(x), \mu_e^*(y))|e \in R\} \\ &\quad \geq T(\min\{\mu_e^*(x)|e \in R\}, \min\{\mu_e^*(y)|e \in R\}) = T(\bigcap_{e\in R} \mu_e^*(x), \bigcap_{e\in R} \mu_e^*(y)); \\ &\bigcap_{e\in R} \mu_e^*(f) = \min\{\mu_e^*(f)|e \in R\} = \min\{\mu(fe)|e \in R\} \\ &\quad = \min\{\mu(e)|e \in R\} = 1.\end{aligned}$$

This completes the proof.

Theorem 4.10 *Let μ be a T-fuzzy M-subsemigroup of $M \cong R \times S$. For $e, f \in R$, define*

$\mu^* = \mu_e^* \times \mu_f^*$ defined by $\mu^*(a,b) = (\mu_e^* \times \mu_f^*)(a,b) = T(\mu_e^*(a), \mu_f^*(b))$, for all $a, b \in M$. Then μ^* is a T-fuzzy M-subsemigroup of M.

Proof Let $x = (e', x_1)$ and $y = (f', x_2)$ in $R \times S \cong M$, where $e', f' \in R$ and $x_1, x_2 \in S$, respectively. Then we have

(i) $\mu^*(xy) = \mu^*(e'f', x_1x_2) = (\mu_e^* \times \mu_f^*)(e'f', x_1x_2) = T(\mu_e^*(e'f'), \mu_f^*(x_1x_2))$.

Now, by Theorem 4.9, we deduce the following

$$\begin{aligned}\mu^*(xy) &\geq T(T(\mu_e^*(e'), \mu_e^*(f')), T(\mu_f^*(x_1), \mu_f^*(x_2))) = T(T(\mu_e^*(e'), \mu_f^*(x_1)), T(\mu_e^*(f'), \mu_f^*(x_2))) \\ &= T((\mu_e^* \times \mu_f^*)(e', x_1), (\mu_e^* \times \mu_f^*)(f', x_2)) = T(\mu^*(x), \mu^*(y)).\end{aligned}$$

(ii) $\mu^*(e', f') = (\mu_e^* \times \mu_f^*)(e', f') = T(\mu_e^*(e'), \mu_f^*(f')) = T(\mu(e'e), \mu(f'f)) = T(\mu(e), \mu(f)) = T(1,1) = 1$.

Consequently, from (i) and (ii), we see that μ^* is a T-fuzzy M-subsemigroup of M.

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具有 t- 范数的模糊 M- 半群的性质

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摘要: 本文利用范数，引入 M- 半群的 T- 模糊 M- 子半群的概念，刻划其性质和特征. 进一步，我们给出 T- 模糊 M- 子半群 R 与 T- 模糊 M- 子半群 S 的直积是 T- 模糊 M- 子半群这一结论. 同时，我们证明了 T- 模糊 M- 子半群可分解成 T- 模糊 M- 子半群的直积.

关键词: M- 半群； t- 范数； (虚)T- 模糊 M- 子半群； T- 积.