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Oscillation Criteria for Second-Order Semi-Linear Neutral Difference Equations

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Abstract: Consider the second-order semi-linear neutral difference equation

$$\Delta[a_n|\Delta(x_n+p_nx_{n-\tau})|^{\alpha-1}\Delta(x_n+p_nx_{n-\tau})]+\sum_{i=1}^{k}q_i(n)|x_{n-\sigma_i}|^{\alpha-1}x_{n-\sigma_i}=0. \tag{1}$$

The sufficient conditions are established for oscillation of the solutions of (1). These results generalize and improve some known results about both neutral and delay difference equation.

Key words: Semi-linear; neutral difference equation; oscillation.
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1. Introduction

In the paper, we consider the semi-linear second-order neutral difference equation

$$\Delta[a_n|\Delta(x_n+p_nx_{n-\tau})|^{\alpha-1}\Delta(x_n+p_nx_{n-\tau})]+\sum_{i=1}^{k}q_i(n)|x_{n-\sigma_i}|^{\alpha-1}x_{n-\sigma_i}=0, \tag{1}$$

where $n=1,2,3,\cdots,\alpha$ is a positive constant, and τ and $\{\sigma_i\}_{i=1}^{k}$ are nonnegative integers. Δ is the usual forward difference operator. Throughout this paper, we assume that

(h_1) $\alpha\geq 1$, $0\leq p_n<1$ for $n=0,1,2,\cdots$.

(h_2) $\{q_n\}$ is a nonnegative sequence with infinitely many positive terms.

(h_3) $a_n>0$, $n=0,1,2,\cdots$, and $\sum^{\infty}1/a_n^{1/\alpha}=\infty$.

A solution $\{x_n\}$ of (1) is defined for $n\geq -\max\{\tau,\sigma_i,i=1,2,\cdots,k\}$ and satisfies (1) for $n=1,2,3,\cdots$. A solution $\{x_n\}$ of (1) is said to be oscillatory if for every $N>0$, there exists an $n\geq N$ such that $x_nx_{n+1}\leq 0$. Otherwise, it is nonoscillatory.

Most of the previous studies on the oscillation theory of (1) have been restricted to the case in which $\alpha=1$, $p_n=0$ and $a_n=1$[1−4].

We note that the following equation is related to the continuous version of (1)

$$[a(t)|(x(t)+p(t)x(t-\tau))'|^{\alpha-1}(x(t)+p(t)x(t-\tau))']'+q(t)|x(t-\sigma)|^{\alpha-1}x(t-\sigma)=0.$$

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where $a(t) > 0$, $q(t) > 0$ has been the subject matter of many recent investigations, e.g.[5]. Our results not only extend the known theorems for semi-linear differential equation to a discrete case, but also include and improve several other known criteria discussed in [1].

Throughout this paper, unless otherwise specified, we always follow a convention that all the difference inequalities hold for all sufficiently large positive integers n, and for convenience we adopt the notation $z_n = x_n + p_n x_{n-\tau}$.

2. Lemmas and main results

In order to prove our theorems, we need the following lemmas.

Lemma 1 *Assume that* $(h_1) - (h_3)$ *hold. If* $\{x_n\}$ *is a nonoscillatory solution of (1), then*

$$\Delta(a_n|\Delta z_n|^{\alpha-1}\Delta z_n) \le 0, \quad \Delta z_n \ge 0, \quad z_n > 0, \quad \text{and} \quad z_n \ge x_n > 0,$$

or

$$\Delta(a_n|\Delta z_n|^{\alpha-1}\Delta z_n) \ge 0, \quad \Delta z_n \le 0, \quad z_n < 0, \quad \text{and} \quad z_n \le x_n < 0.$$

Proof Let $\{x_n\}$ be a nonoscillatory solution of (1). Without lost of generality, we assume that $x_n > 0$, $x_{n-\tau} > 0$, $x_{n-\sigma} > 0$ for $n \ge n_0 \in N$. It follows from (h_1) and (h_2) that $z_n \ge x_n > 0$ for $n \ge n_0$ and

$$\Delta(a_n|\Delta z_n|^{\alpha-1}\Delta z_n) \le 0, \quad \text{for} \quad n \ge n_0. \tag{2}$$

Hence, $\{a_n|\Delta z_n|^{\alpha-1}\Delta z_n\}$ is a decreasing sequence. We claim that $\Delta z_n \ge 0$ for $n \ge n_0$. Otherwise there is an $n_1 \ge n_0$ such that $\Delta z_{n_1} < 0$. It follows from (2) and (h_3) that

$$z_n \le z_{n_1} - \sum_{s=n_1}^{n-1} (-\zeta/a_s)^{1/\alpha} \to -\infty,$$

which contradicts the fact that $z_n > 0$ for all $n \ge n_0$. This completes the proof.

Lemma 2 *Assume that* (h_1)–(h_3) *hold and* $\{x_n\}$ *is a nonoscillatory solution of (1). Let* $\sigma = \max_{1\le i\le k}\{\sigma_i\}$, *then*

$$w_n = a_n \frac{|\Delta z_n|^{\alpha-1}\Delta z_n}{|z_{n-\sigma}|^{\alpha-1} z_{n-\sigma}} \tag{4}$$

satisfies the following Riccati inequality:

$$\Delta w_n + \frac{\alpha}{a_{n-\sigma}^{1/\alpha}} w_{n+1}^{1+1/\alpha} + \sum_{i=1}^{k} q_i(n)(1 - p_{n-\sigma_i})^{\alpha} \le 0. \tag{4}$$

Proof Without lost of generality, let $\{x_n\}$ be an eventually positive solution of (1), then there exists n_1 sufficiently large such that $x_{n-\tau} > 0$, $x_{n-\sigma_i} > 0$, $1 \le i \le k$. By (1) and Lemma 1, we obtain

$$\Delta(a_n(\Delta z_n)^{\alpha}) + \sum_{i=1}^{k} q_i(n)(z_{n-\sigma_i} - p_{n-\sigma_i} x_{n-\tau-\sigma_i})^{\alpha} = 0, \tag{5}$$

which, in view of the fact that $z_n \geq x_n$ and z_n is increasing, implies

$$\Delta(a_n(\Delta z_n)^\alpha) + \sum_{i=1}^{k} q_i(n)(1-p_{n-\sigma_i})^\alpha z_{n-\sigma_i}^\alpha \leq 0. \tag{6}$$

From $\sigma = \max_{1\leq i\leq k}\sigma_i$, we know

$$\Delta(a_n(\Delta z_n)^\alpha) + \sum_{i=1}^{k} q_i(n)(1-p_{n-\sigma_i})^\alpha z_{n-\sigma}^\alpha \leq 0.$$

By (3) and the differential mean value theorem we can easily show that

$$\Delta w_n = \frac{\Delta(a_n(\Delta z_n)^\alpha)}{z_{n-\sigma}^\alpha} - \frac{\alpha a_{n+1}(\Delta z_{n+1})^\alpha \xi^{\alpha-1}\Delta z_{n-\sigma}}{z_{n-\sigma}^\alpha z_{n+1-\sigma}^\alpha}, \quad (z_{n-\sigma} \leq \xi \leq z_{n+1-\sigma}). \tag{7}$$

Using the fact that $\{a_n(\Delta z_n)^\alpha\}$ is decreasing and noting (6) and (7), we have

$$\Delta w_n \leq -\sum_{i=1}^{k} q_i(n)(1-p_{n-\sigma_i})^\alpha - \frac{\alpha a_{n+1}^{1+1/\alpha}}{a_{n-\sigma}^{1/\alpha}}\left(\frac{\Delta z_{n+1}}{z_{n+1-\sigma}}\right)^{1+\alpha}. \tag{8}$$

By (3) and (8), we get that (4) holds.

Lemma 3 *Assume that $\alpha > 0$ and $k \geq \frac{\alpha}{(1+\alpha)^{1+1/\alpha}}$. Then*

$$k(1+x)^{1+1/\alpha} \geq x \quad \text{for} \quad x \geq -1,$$

where the equality holds if and only if $k = \frac{\alpha}{(1+\alpha)^{1+1/\alpha}}$.

The proof of this lemma can be done by an elementary mathematical analysis.

Theorem 1 *Assume that conditions (h_1)–(h_3) hold and that*

(i) $\sum^\infty \sum_{i=1}^{k} q_i(n)(1-p_{n-\sigma_i})^\alpha = +\infty$, or

(ii) $\sum^\infty \sum_{i=1}^{k} q_i(n)(1-p_{n-\sigma_i})^\alpha < +\infty$ and there exists a positive constant $\rho > \frac{1}{(1+\alpha)^{1+1/\alpha}}$, such that

$$\sum_{s=n}^{\infty} \frac{1}{a_{s-\sigma}^{1/\alpha}}\left[\sum_{m=s+1}^{\infty}\sum_{i=1}^{k} q_i(m)(1-p_{m-\sigma_i})^\alpha\right]^{1+1/\alpha} \geq \rho \sum_{m=n+1}^{\infty}\sum_{i=1}^{k} q_i(m)(1-p_{m-\sigma_i})^\alpha. \tag{10}$$

Then every solution of (1) is oscillatory.

Proof Assume, for the sake of contradiction, that Eq.(1) has an eventually positive solution $\{x_n\}$. By Lemma 2, we obtain

$$\sum_{s=n+1}^{l} \frac{\alpha}{a_{s-\sigma}^{1/\alpha}} w_{s+1}^{1+1/\alpha} + \sum_{s=n+1}^{l}\sum_{i=1}^{k} q_i(s)(1-p_{s-\sigma_i})^\alpha \leq w_{n+1} - w_l. \tag{11}$$

If (i) holds, then $w_l \to -\infty$ as $l \to +\infty$. This contradicts the fact that $w_n > 0$. If (ii) holds, by (11), we have

$$\sum_{s=n+1}^{\infty} \frac{\alpha}{a_{s-\sigma}^{1/\alpha}} w_{s+1}^{1+1/\alpha} + \sum_{s=n+1}^{\infty}\sum_{i=1}^{k} q_i(s)(1-p_{s-\sigma_i})^\alpha \leq w_{n+1}. \tag{12}$$

Let $C_n = \sum_{s=n}^{\infty}\sum_{i=1}^{k} q_i(s)(1-p_{s-\sigma_i})^{\alpha}$. We define a sequence as follows

$$u^{(1)}(n) = \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}C_{s+1}^{1+1/\alpha},\quad u^{(2)}(n) = \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}[C_{s+1}+u^{(1)}(s)]^{1+1/\alpha},\cdots,$$

$$u^{(m+1)}(n) = \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}[C_{s+1}+u^{(m)}(s)]^{1+1/\alpha},\quad m=1,2,\cdot \tag{13}$$

It is obvious that $0 < u^{(1)}(n) \le u^{(2)}(n) \le \cdots u^{(m)}(n) \le u^{(m+1)}(n) \le \cdots$. By (12), we have

$$u^{(1)}(n) + C_{n+1} \le w_{n+1}. \tag{14}$$

Suppose that

$$u^{(m)}(n) + C_{n+1} \le w_{n+1}. \tag{15}$$

From (12), (14) and (15), we obtain

$$u^{(m+1)}(n) + C_{n+1} \le \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}w_{s+1}^{1+1/\alpha} + C_{n+1} \le w_{n+1}.$$

So by induction (15) holds for any positive integer m. It follows from Lebesgue's dominated convergence theorem that

$$\lim_{m\to\infty} u^{(m)}(n) = u(n) \ \text{ exists } \quad \text{and} \ \ u(n) \le w_{n+1}. \tag{16}$$

On the other hand, by (10) we have

$$u^{(1)}(n) = \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}C_{s+1}^{1+1/\alpha} \ge kC_{n+1},$$

where $k = \alpha\rho$ and

$$u^{(2)}(n) = \sum_{s=n+1}^{\infty}\frac{\alpha}{a_{s-\sigma}^{1/\alpha}}[C_{s+1}+u^{(1)}(s)]^{1+1/\alpha} \ge k(1+k)^{1+1/\alpha}C_{n+1}.$$

Let $d_1 = k(1+k)^{1+1/\alpha}$. By induction, we easily see that

$$u^{(m+1)}(n) \ge d_m C_{n+1}, \tag{17}$$

where

$$d_m = k(1+d_m)^{1+1/\alpha},\quad m=2,3,\cdots. \tag{18}$$

By Lemma 3, it is easy see that the sequence $\{d_m\}$ is increasing. Now we prove that

$$\lim_{m\to\infty} d_m = +\infty. \tag{19}$$

Otherwise, $\lim_{m\to\infty} d_m = c$ would imply that

$$c = k(1+c)^{1+1/\alpha}. \tag{20}$$

By Lemma 3 we know that (20) does not hold if $k = \alpha\rho > \frac{\alpha}{(\alpha+1)^{1+1/\alpha}}$. Thus the assumption that $\lim_{m\to\infty} d_m = c$ is impossible. By (17) and (19) the sequence $\{u^{(m)}(n)\}$ can not be convergent. This contradicts (16) and so the proof is completed.

Corollary 1 *Assume that conditions* (h_1)–(h_3) *and* $i = 1$ *hold and that*

(i) $\sum^{\infty} q_n(1 - p_{n-\sigma})^{\alpha} = +\infty$, *or*

(ii) $\sum^{\infty} q_n(1 - p_{n-\sigma})^{\alpha} < +\infty$ *and there exists a positive constant* $\rho > \frac{1}{(1+\alpha)^{1+1/\alpha}}$, *such that*

$$\sum_{s=n}^{\infty} \frac{1}{a_{s-\sigma}^{1/\alpha}} \Big[\sum_{m=s+1}^{\infty} q_m(1 - p_{m-\sigma})^{\alpha} \Big]^{1+1/\alpha} \geq \rho \sum_{m=n+1}^{\infty} q_m(1 - p_{m-\sigma})^{\alpha}.$$

Then every solution of (1) is oscillatory.

By taking $\alpha = 1$, $a_n = 1$ in Corollary 1 we obtain

Corollary 2 *Assume that* (h_1)–(h_3) *hold and that*

(i) $\sum^{\infty} q_s(1 - p_{s-\sigma}) = \infty$, *or*

(ii) $\sum^{\infty} q_s(1 - p_{s-\sigma}) < \infty$ *and there exists a positive constant* $\rho > 1/4$ *such that*

$$\sum_{s=n}^{\infty} [\sum_{m=s+1}^{\infty} q_m(1 - p_{m-\sigma})]^2 \geq \rho \sum_{m=n+1}^{\infty} q_m(1 - p_{m-\sigma}).$$

Then, every solution of Equation (1) is oscillatory.

Remark 1 Corollary 1 can be considered as discrete analogues of Theorem 1 given in [5] for the neutral delay equation

$$[a(t)|(x(t) + p(t)x(t-\tau))'|^{\alpha-1}(x(t) + p(t)x(t-\tau))']' + q(t)|x(t-\sigma)|^{\alpha-1}x(t-\sigma) = 0.$$

Remark 2 When $k = 1$, $\alpha = 1$ and $a_n = 1$, Equation (1) reduces to

$$\Delta^2(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} = 0. \tag{21}$$

Hence Corollary 2 is an extension of Theorem 2.5 in [1]. But we weakened the conditions $0 \leq p_n \leq p < 1$ and $\sum^{\infty} q_n = +\infty$. To author's knowledge, the results are even new for Equation (21).

4. Some applications

In this section, we indicate some applications of our results. These applications are given as examples.

Example 1 Consider the neutral difference equation

$$\Delta^2(x_n + (1 - \frac{1}{8n})x_{n-\tau}) + 2n x_{n-\sigma} = 0, \tag{22}$$

where $\alpha = 1$, $a_n = 1$, $p_n = 1 - 1/8n$, $q_n = 2n$, then all conditions of Corollary 2 are satisfied. Hence, all solutions of (22) are oscillatory.

Example 2 Consider the neutral difference equation

$$\Delta^2(x_n+px_{n-\tau})+\frac{\delta}{n^2}x_{n-\sigma}=0, \tag{23}$$

where τ, $\sigma>0$, $0\le p<1$ and $\delta>\frac{1}{3(1-p)}$. It is easy to verify that

$$\sum_{s=n+1}^{\infty}\frac{\delta}{s^2}(1-p)=\delta(1-p)\sum_{s=n+1}^{\infty}\frac{1}{s^2}\ge\sum_{s=n+1}^{\infty}\frac{\delta(1-p)}{s(s+1)}=\delta(1-p)\frac{1}{n+1},$$

$$\sum_{s=n}^{\infty}[\sum_{m=s+1}^{\infty}q_m(1-p_{m-\sigma})]^2\ge[\delta(1-p)]^2\sum_{s=n}^{\infty}\frac{1}{(s+1)^2}>C_{n+1}/3.$$

Choose a constant $\rho=(1/4,1/3)$. Then, the conditions of Corollary 2 are satisfied and therefore every solution of Equation (23) is oscillatory. But in Equation (23), the condition $\sum^{\infty}q_n=\infty$ does not justify the oscillation of Equation (23).

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二阶半线性中立型差分方程的振动性准则

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摘要: 考虑二阶半线性中立型差分方程

$$\Delta[a_n|\Delta(x_n+p_nx_{n-\tau})|^{\alpha-1}\Delta(x_n+p_nx_{n-\tau})]+\sum_{i=1}^{k}q_i(n)|x_{n-\sigma_i}|^{\alpha-1}x_{n-\sigma_i}=0. \tag{1}$$

给出了方程 (1) 的解的振动性的充分条件. 所有结果推广和改进了关于中立和时滞差分方程已有结果.

关键词: 半线性; 中立型差分方程; 振动性.