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On the Filtration Dimensions of a Standardly Stratified Algebra and Its Polynomial Algebra

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Abstract: This paper deals with $\triangle-$good filtration dimensions of a standardly stratified algebra and $\triangle[x]$-good fitration dimensions of its polynomial algebra.

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0. Introduction

In order to investigate semi-simple complex Lie algebras and high-weight module category of algebraic groups, L. Scott[8] introduced the concept of quasi-hereditary algebras. As a generalization of quasi-hereditary algebras, properly standardly stratified algebras and standardly stratified algebras were introduced by Cline, Parshall, Scott[9] and Dlab[10]. Since then, many mathematicians have been interested in researching these algebras. For example, in 1989, Dlab and Ringel[4] proved that the semiprimary ring with global dimension 2 is a quasi-hereditary algebra; In 1996, D. Zacharia[3] caculated the Hochschild homological groups of quasi-hereditary algebras; In 2000, I. Ágoston and D. Happel[5] investigated the relationship between standardly stratified algebras and tilting modules; In 2001, in order to calculate the glabal dimensions of GL_2- and GL_3-algebras, A.E. Parker[2] introduced the concept of $\nabla-$(or $\triangle-$)good filtration dimension for a quasi-hereditary algebra. Recently, Zhu Bin and S. Caenepeel[1] investigated these dimensions for standardly stratified algebras and properly stratified algebras. The aim of this paper is to study the filtration dimensions of a standardly stratified algebra and its polynomial algebra.

1. Preliminaries

Let R be a commutative Artinian ring and A a basic Artinian algebra over R. We will consider finitely generated left A-module. The composition of maps $f : M_1 \to M_2$ and $g : M_2 \to$

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M_3 will be denoted by gf. The category of left A-modules will be denoted by A-mod. All subcategories will be considered full and closed under isomorphism.

Given a class θ of A-mod, we denote by $\mathcal{F}(\theta)$ the full subcategory of all A-modules which have a θ-filtration, that is, a filtration

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that each factor M_{i-1}/M_i $(1 \leq i \leq n)$ is isomorphic to an object of θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\theta)$ are called θ-good modules and the category $\mathcal{F}(\theta)$ is called the θ-good module category.

In the following, $(A, \leq)$ will denote the algebra A together with a fixed ordering on a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents (given by the natural ordering of indices). For $1 \leq i \leq n$, let $E(i)$ be the simple A-module which is the simple top of the indecomposable projective $P(i) = Ae_i$. The standard module $\triangle(i)$ is by definition the maximal factor module of $P(i)$ without composition factors $E(j)$ with $j > i$. $\overline{\Delta(i)}$ will be the notation for proper standard module, which is the maximal factor module of $\Delta(i)$ such that condition $[\overline{\triangle}(i) : E(i)] = 1$.

Dually, for $1 \leq i \leq n$, we have costandard modules $\nabla(i)$ and proper costandard modules $\overline{\nabla(i)}$.

Let $\triangle$ be the full subcategory consisting of all $\triangle(\lambda)$ with $\lambda \in \Lambda$ and $\Delta_{<\lambda}$ the full subcategory of all $\triangle(\delta)$ with $\delta < \lambda$. In a similar way we introduce ∇ and $\nabla_{<\lambda}$ and so on.

The pair $(A, \leq)$ is called a standardly stratified algebra if ${}_AA \in \mathcal{F}(\triangle)$. $(A, \leq)$ is called a proper standardly stratified algebra if ${}_AA \in \mathcal{F}(\triangle)$ and ${}_AA \in \mathcal{F}(\overline{\triangle})$. Note that these properties generalize the concept of quasi-hereditary algebras where we require the additional condition that the standard modules are Schur modules.

Let $(A, \leq)$ be a standardly stratified algebra. A full subcategory $\mathcal{T}$ of A-mod is called contravariantly finite in A-mod if for any A-module M there is a module $M_1 \in \mathcal{T}$ and a morphism $f : M_1 \longrightarrow M$ such that the restriction of $\mathrm{Hom}_A(-, f)$ to $\mathcal{T}$ is surjective. Such a morphism f is called a right $\mathcal{T}$-approximation of M. A right $\mathcal{T}$-approximation $f : M_1 \longrightarrow M$ of M is called a minimal if the restriction of f to any non-zero direct summand of M_1 is nonzero. The covariant finiteness of $\mathcal{T}$ and the left $\mathcal{T}$-approximation of M can be defined dually. $\mathcal{F}(\triangle)$ and $\mathcal{F}(\nabla)$ are said to be functorially finite in A-mod, if they are at the same time covariantly and contravariantly finite in A-mod.

Lemma 1.1[1] *Let $(A, \leq)$ be a standardly stratified algebra, then the following statements hold:*

(a) $\mathcal{F}(\triangle)$ is a functorially finite and resolving subcategory;

(b) $\mathcal{F}(\overline{\nabla})$ is a covariantly finite and coresolving subcategory;

(c) $\mathcal{F}(\triangle) = \{X | Ext^1(X, \mathcal{F}(\overline{\nabla})) = 0\}$;

(d) $\mathcal{F}(\overline{\nabla}) = \{Y | Ext^1((\mathcal{F}(\triangle)), Y) = 0\}$.

It follows from Lemma 1.1 that there exists a finite $\mathcal{F}(\triangle)$-resolution

$$0 \longrightarrow M_d \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0, \tag{1}$$

where $M_i \in \mathcal{F}(\triangle)$ for all $X \in A$-mod.

Definition 1.1 *Let $(A, \leq)$ be a standardly stratified algebra, and let $\triangle - \mathrm{gfd}(X)$ be the smallest number d for which we have an $\mathcal{F}(\triangle)$-resolution (1) with $M_i \in \mathcal{F}(\triangle)$.*

Lemma 1.2[1] $\triangle - \mathrm{gfd}(X) = d$ *if and only if* $\mathrm{Ext}_R^i(X, \overline{\nabla}(\lambda)) = 0$ *for all $i \geq d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that* $\mathrm{Ext}_R^d(X, \overline{\nabla}(\lambda)) \neq 0$.

We can introduce the definition of $\overline{\nabla} - \mathrm{gfd}(X)$ by duality.

Definition 1.3 *Let $(A, \leq)$ be a standardly stratified algebra.*

$$\triangle - \mathrm{gfd}(A) = \sup\{\triangle - \mathrm{gfd}(X) | X \in A\text{-mod}\}$$

is called the $\triangle$-good filtration dimension of A.

$$\overline{\nabla} - \mathrm{gfd}(A) = \sup\{\overline{\nabla} - \mathrm{gfd}(X) | X \in A\text{-mod}\}$$

is called the $\overline{\nabla}$-good filtration dimension of A.

2. On $\triangle$-gfd(A)

Firstly, we have the following lemmas which are easy to prove.

Lemma 2.1 *Let $(A, \leq)$ be a standardly stratified algebra, the following statements hold:*

(1) $C \in \mathcal{F}(\triangle)$ if and only if $\triangle - \mathrm{gfd}(C) = 0$;

(2) $C \in \mathcal{F}(\overline{\nabla})$ if and only if $\overline{\nabla} - \mathrm{gfd}(C) = 0$.

Lemma 2.2 *Let $(A, \leq)$ be a standardly stratified algebra and X, Y, Z be A-modules. If $0 \to X \to Y \to Z \to 0$ is exact and $Y \in \mathcal{F}(\triangle)$, then*

$$\mathrm{Ext}^n(X, \overline{\nabla}(i)) \cong \mathrm{Ext}^{n+1}(Z, \overline{\nabla}(i)) \quad (n \geq 1).$$

Proof We have $\mathrm{Ext}^n(Y, \overline{\nabla}(i)) = 0$ for $n \geq 1$, since $Y \in \mathcal{F}(\triangle)$. Thus we know the lemma holds from the following exact sequence

$$0 \longrightarrow \mathrm{Ext}^n(X, \overline{\nabla}(i)) \longrightarrow \mathrm{Ext}^{n+1}(Z, \overline{\nabla}(i)) \longrightarrow 0.$$

Theorem 2.1 *Let $(A, \leq)$ be a standardly stratified algebra and X, P, Y, X', P' be A-modules. If*

$$0 \longrightarrow X \xrightarrow{\eta} P \xrightarrow{\pi} Y \longrightarrow 0$$

$$0 \longrightarrow X' \xrightarrow{\eta'} P' \xrightarrow{\pi'} Y \longrightarrow 0$$

are exact sequence and π' is a right $\mathcal{F}(\triangle)$-approximation and $P \in \mathcal{F}(\triangle)$, then there is an exact sequence

$$0 \longrightarrow X \xrightarrow{\sigma} P \oplus X' \xrightarrow{\tau} Y \longrightarrow 0. \qquad (*)$$

Proof Since π' is a right $\mathcal{F}(\triangle)$-approximation, we can define f and g such that the folowing diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\eta} & P & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & g\downarrow & & f\downarrow & & I_y\downarrow & & \\ 0 & \longrightarrow & X' & \longrightarrow & P' & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Define $\sigma : X \to P \oplus X'$, $x \to (-\eta(x), g(x))$ and $\tau : P \oplus X' \to P^{'}$, $(p, x') \to f(p) + \eta'(x')$. It is routine to check that the sequence $(*)$ is exact. □

Lemma 2.3[2] *Let $(A, \leq)$ be a standardly stratified algebra and X, Y, Z belong to A-mod. If*

$$0 \to X \to Y \to Z \to 0$$

is an exact sequence, then the following statements hold:

(1) If $\triangle - \mathrm{gfd}(Y) > \triangle - \mathrm{gfd}(X)$, then $\triangle - \mathrm{gfd}(Z) = \triangle - \mathrm{gfd}(Y)$;

(2) If $\triangle - \mathrm{gfd}(Y) < \triangle - \mathrm{gfd}(X)$, then $\triangle - \mathrm{gfd}(Z) = \triangle - \mathrm{gfd}(X) + 1$;

(3) If $\triangle - \mathrm{gfd}(Y) = \triangle - \mathrm{gfd}(X)$, then $\triangle - \mathrm{gfd}(Z) \leq \triangle - \mathrm{gfd}(X) + 1$.

Lemma 2.4 $\triangle - \mathrm{gfd}(\cup_{(\lambda\in\Lambda)} X_\lambda) = \sup_{(\lambda\in\Lambda)}\{\triangle - \mathrm{gfd}(X_\lambda)\}$.

Proof The conclusion follows from the following isomorphisms and formulae

$$\mathrm{Ext}^n(\coprod X_\lambda, \nabla(i)) \simeq \prod \mathrm{Ext}^n(X_\lambda, \nabla(i))$$

$$\mathrm{Ext}^n(\coprod X_\lambda, \nabla(i)) = 0 \Longleftrightarrow \mathrm{Ext}^n(X_\lambda, \nabla(i)) = 0 \quad (\forall i, \forall \lambda)$$

$$\triangle - \mathrm{gfd}(\coprod_{(\lambda\in\Lambda)} X_\lambda) = \sup_{(\lambda\in\Lambda)}\{\triangle - \mathrm{gfd}(X_\lambda)\}$$

By duality we have

Lemma 2.5 $\overline{\nabla} - \mathrm{gfd}(\prod_{(\lambda\in\Lambda)} X_\lambda) = \sup_{(\lambda\in\Lambda)}\{\overline{\nabla} - \mathrm{gfd}(X_\lambda)\}$.

Theorem 2.2 *Suppose $(A, \leq)$ is a standardly stratified algebra and for any A-module M there exists the following resolution*

$$0 \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0 \tag{2}$$

such that $M_{i-1}/M_i \cong E(i)$ where $E(i)$ is some simple module, then we have that $\triangle - \mathrm{gfd}(A) = \sup\{\triangle - \mathrm{gfd}(E(i))|\ i = 1, 2, \cdots, t\}$.

Proof Assume that $\sup\{\triangle - \mathrm{gfd}(E(i))|i = 1, 2, \cdots t\} = n$. Let $l(M) = r$ where $l(M)$ is the composition length of M, then M has a resolution (2). We assume $r = 1$, then M is a simple module. Thus, $\triangle - \mathrm{gfd}(M) \leq n$. If $r > 1$, then $X = M/M_0$. So, X is a simple module and

$l(X) = 1$, $l(M_0) = r - 1$. By induction hypothesis, $\triangle - \mathrm{gfd}(M_0) \leq n$, we have $\triangle - \mathrm{gfd}(X) \leq n$ and the following exact sequence

$$0 \to M_0 \to M \to X \to 0.$$

From Lemma 2.3, we have $\triangle - \mathrm{gfd}(M) \leq n$. Therefore, $\triangle - \mathrm{gfd}(A) = \sup\{\triangle - \mathrm{gfd}(E(i)) | \ i = 1, 2, \cdots, t\}$. $\square$

3. On $R[x]$-modules

Let R be an algebra, A be a R-module, and x be a letter. We call the following form

$$a(x) = a_0 + a_1 x + \cdots + a_m x^m, \quad (a_i \in A, a_m \neq 0)$$

a polynomial of degree m over R. A nonzero element $a_0 \neq 0$ in A is a polynomial of degree 0, while the zero elment 0 in A is the zero-polynomial, but it is of non-degree. We define $a(x) = b(x)$ if and only if they have the same degree and the corresponding coefficients are the same, and the sum of $a(x)$ and $b(x)$ is defined canonically (i.e. amalgamation of the same terms). Thus, the set of all polynomials over A forms an additive group(commutative), denoted by $A[x]$. If

$$\beta(x) = \beta_0 + \beta_1 x + \cdots + \beta_n x^n, \ (\beta_i \in R, \beta_n \neq 0)$$

and $a(x) \in A[x]$, we define

$$\beta(x)a(x) = b_0 + b_1(x) + b_2 x + \cdots,$$

where

$$b_s = \sum_{i+j=s} \beta_i a_j.$$

Then $R[x]$ is an algebra and is called a polynomial algebra of one variable. $A[x]$ is an $R[x]$-module. Clearly, we define $x^n x^m = x^{n+m}$. Hence, x^n can be understood as the n-th power of x, which is subject to the index law.

Let $\beta(x) = \beta_0 + \beta_1 x + \cdots + \beta_n x^n$, where $\beta_i \in A$. Let $\beta' \in R$, then we have

$$\beta(x)\beta' = \beta_0\beta' + \beta_1\beta' x + \cdots + \beta_n \beta' x^n.$$

Thus, $R[x]$ is a right R-module (of course, it is also a left R-module).

One can define a formal power series algebra $R[[x]]$ and a formal power series module $A[[x]]$ where X is a letter. $R[[x]]$ is a right R-modulle (of course, it is also a left R-module).

Lemms 3.1[7] *(1) As a right R-module, the polynomial algebra $R[x]$ is flat;*

(2) $A[x] \cong R[x] \otimes_R A$.

Similarly, we have

Lemma 3.2 *Let R be a perfect and coherent commutative algebra, then*

(1) As R-modules, the formal power series algebra $R[[x]]$ is flat;

(2) $A[[x]] \cong R[[x]] \otimes_R A$.

Lemma 3.3 *Let R be a perfect and coherent commutative algebra, $R[[x]]$ be a formal power series algebra where x is a letter, and M be an $R[[x]]$-module, then we have:*

(a) If M is an injective $R[[x]]$-module, then M is an injective R-module ;

(b) If M is a flat R-module, then $R[[x]] \otimes_R M$ is a flat $R[[x]]$-module ;

(c) If M is a flat $R[[x]]$-module, then M is a flat R-module;

(d) Assume that M is an $R[[x]]$-module and that M is an injective R-module, then $\mathrm{Hom}_R(R[[x]], M)$ is an injective $R[[x]]$-module.

Lemma 3.4 *Let R be a commutative algebra, $R[x]$ be a polynomial algebra where x is a letter, and M be an $R[x]$-module. We have:*

(a) If M is an injective $R[x]$-module, then M is an injective R-module;

(b) If M is a flat R-module, then $R[x] \otimes_R M$ is a flat R-module;

(c) If M is a flat $R[x]$-module, then M is a flat R-module;

(d) Assume that M is $R[x]$-module and that M is an injective R-module, then $\mathrm{Hom}_R(R[x], M)$ is an injective $R[x]$-module.

Lemma 3.5 $\frac{A[x]}{B[x]} = \frac{A}{B}[x]$.

Proof We define a homomorphism f from $\frac{A[x]}{B[x]}$ to $\frac{A}{B}[x]$ as follows.

$$f : \overline{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \longmapsto \overline{a_n}x^n + \overline{a_{n-1}}x^{n-1} + \cdots + \overline{a_1}x + \overline{a_0}.$$

It is easy to show that f is well-defined and is an isomorphism. $\square$

Similarly, one can have the following

Lemma 3.6 $\frac{A[[x]]}{B[[x]]} = \frac{A}{B}[[x]]$.

Lemma 3.7 *Let $(R, \leq)$ be a standardly stratified algebra. We define*

(1) $\triangle(i)[x] = R[x] \otimes_R \triangle(i)$;

(2) $\overline{\triangle}(i)[x] = R[x] \otimes_R \overline{\triangle}(i)$;

(3) $\nabla(i)[x] = R[x] \otimes_R \nabla(i)$;

(4) $\overline{\nabla}(i)[x] = R[x] \otimes_R \overline{\nabla}(i)$,

then the following conditions hold:

(a) If $M \in \mathcal{F}_R(\triangle)$, then $M[x] \in \mathcal{F}_{R[x]}(\triangle[x])$;

(b) If $M \in \mathcal{F}_R(\overline{\triangle})$, then $M[x] \in \mathcal{F}_{R[x]}(\overline{\triangle}[x])$;

(c) If $M \in \mathcal{F}_R(\nabla)$, then $M[x] \in \mathcal{F}_{R[x]}(\nabla[x])$;

(d) If $M \in \mathcal{F}_R(\overline{\nabla})$ then $M[x] \in \mathcal{F}_{R[x]}(\overline{\nabla}[x])$.

Proof (a) If $M \in \mathcal{F}_R(\triangle)$, then there exists a filtration chain

$$0 = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $\frac{M_i}{M_{i+1}} \cong \triangle(j)$ for some j in $\{1, 2, \cdots, n\}$.

Since $R[x]$ is a flat R-module, we have

$$0 = R[x] \otimes_R M_n \subseteq R[x] \otimes_R M_{n-1} \subseteq \cdots R[x] \otimes_R M_0 = R[x] \otimes_R M,$$

i.e.

$$0 = M_n[x] \subseteq M_{n-1}[x] \subseteq \cdots \subseteq M_1[x] \subseteq M_0[x] = M[x]$$

as required. By Lemma 3.5, we have $\frac{M_i[x]}{M_{i+1}[x]} \cong \frac{M_i}{M_{i+1}}[x] \cong \triangle(j)[x]$ for some $j \in \{1, 2, \cdots, n\}$, $(i = 1, 2, \cdots, n)$. So, $M[x] \in \mathcal{F}_{R[x]}(\triangle[x])$. The proofs of (b), (c) and (d) are similar to the proof of (a). □

Lemma 3.8 Let R *be a perfect and coherent commutative algebra and* $(R, \leq)$ *be a standardly stratified algebra. Define*

(1) $\triangle(i)[[x]] = R[[x]] \otimes_R \triangle(i)$; *(2)* $\overline{\triangle}(i)[[x]] = R[[x]] \otimes_R \overline{\triangle}(i)$;

(3) $\nabla(i)[[x]] = R[[x]] \otimes_R \nabla(i)$; *(4)* $\overline{\nabla}(i)[[x]] = R[[x]] \otimes_R \overline{\nabla}(i)$,

then we have

(a) If $M \in \mathcal{F}_R(\triangle)$, *then* $M[[x]] \in \mathcal{F}_{R[[x]]}(\triangle[[x]])$;

(b) If $M \in \mathcal{F}_R(\overline{\triangle})$, *then* $M[[x]] \in \mathcal{F}_{R[[x]]}(\overline{\triangle}[[x]])$;

(c) If $M \in \mathcal{F}_R(\nabla)$, *then* $M[[x]] \in \mathcal{F}_{R[[x]]}(\nabla[[x]])$;

(d) If $M \in \mathcal{F}_R(\overline{\nabla})$, *then* $M[[x]] \in \mathcal{F}_{R[[x]]}(\overline{\nabla}[[x]])$.

The proof is similar to that of Lemma 3.7.

Lemma 3.9 *If* $B[x] \in \mathcal{F}_{R[x]}(\triangle[x])$, *then* $B \in \mathcal{F}_R(\triangle)$.

Proof If $B[x] \in \mathcal{F}_{R[x]}(\triangle[x])$, then $B[x] \in \mathcal{F}_R(\triangle)$. Since $\mathcal{F}(\triangle)$ is closed under direct summands, we have $B \in \mathcal{F}_R(\triangle)$. □

Lemma 3.10 $\mathcal{F}_{R[x]}(\triangle[x])$ *is contravariantly finite in the subcategory consisting of modules which are of the form of* $A[x]$.

Proof Since $\mathcal{F}_R(\triangle)$ is contravariantly finite, there exists a morphism $f : C \longrightarrow A$ such that it is a right $\mathcal{F}_R(\triangle)-$approximation of A for all R-module A. Thus, there is an exact sequence

$$\mathrm{Hom}_R(B, C) \xrightarrow{\mathrm{Hom}(\ ,f)} \mathrm{Hom}_R(B, A) \longrightarrow 0 \ ,$$

for all $B \in \mathcal{F}_R(\triangle)$. In the following we prove that $1_{R[x]} \otimes f : R[x] \otimes_R C \simeq C[x] \longrightarrow R[x] \otimes_R A \simeq A[x]$ is a right $\mathcal{F}_{R[x]}(\triangle[x])-$ approximation of $A[x]$, that is to say, we need to prove that there exists an exact sequence

$$\mathrm{Hom}_{R[x]}(B[x], C[x]) \xrightarrow{\mathrm{Hom}_{R[x]}(\ ,1_{R[x]}\otimes f)} \mathrm{Hom}_{R[x]}(B[x], A[x]) \longrightarrow 0 \ ,$$

for all $B[x] \in \mathcal{F}_{R[x]}(\triangle[x])$. Since we have

$$\mathrm{Hom}_{R[x]}(B[x], A[x]) \simeq \mathrm{Hom}_{R[x]}(R[x] \otimes_R B, A[x])$$

$$\simeq \mathrm{Hom}_R(B, \mathrm{Hom}_{R[x]}(R[x], A[x]))$$
$$\simeq \mathrm{Hom}_R(B, A[x]),$$

and $\mathrm{Hom}_{R[x]}(B[x], C[x]) \simeq \mathrm{Hom}_R(B, C[x])$, we only need to prove that

$$\mathrm{Hom}_R(B, C[x]) \xrightarrow{\mathrm{Hom}_R(\ ,1_{R[x]}\otimes f)} \mathrm{Hom}_R(B, A[x]) \longrightarrow 0$$

i.e., for all R-module morphism $\xi : B \longrightarrow A[x]$, we need to prove that there exists an R-module morphism $\eta : B \longrightarrow C[x]$ such that the following diagram

$$\begin{tikzcd} C[x] \arrow[rr, "1_{R[x]}\otimes f"] & & A[x] \\ & B \arrow[ul, "\eta"] \arrow[ur, "\xi"'] & \end{tikzcd}$$

is commutative. As we have

$$C[x] \simeq R[x] \otimes_R C \simeq \coprod_{i=0}^{\infty} R \otimes_R C \simeq \coprod_{i=0}^{\infty} C$$

and

$$A[x] \simeq R[x] \otimes_R C \simeq \coprod_{i=0}^{\infty} R \otimes_R A \simeq \coprod_{i=0}^{\infty} A,$$

the R-module morphism $1_{R[x]} \otimes f : R[x] \otimes_R C \longrightarrow R[x] \otimes_R C$ can be regarded as R-module morphism $\coprod f : \coprod C \longrightarrow \coprod A$. Taking an injection $g : C \longrightarrow \coprod C$ and a projection $h : \coprod A \longrightarrow A$, we have $h(\coprod f)g = f$. Since $f : C \longrightarrow A$ is a right $\mathcal{F}_R(\triangle)$-approximation of A and $B \in \mathcal{F}_R(\triangle)$, there is a morphism τ such that the following diagram is commutative.

$$\begin{tikzcd} C \arrow[r, "g"] & \coprod C \arrow[r, "\coprod f"] & \coprod A \arrow[r, "h"] & A \\ & B \arrow[ul, "\tau"] \arrow[urr, "\xi"'] & & \end{tikzcd}$$

So, it is enough to take $\eta = g\tau$. Therefore, $1_{R[x]} \otimes f : R[x] \otimes_R C \simeq C[x] \longrightarrow R[x] \otimes_R A = A[x]$ is a right $\mathcal{F}_{R[[x]]}(\triangle[x])$-approximation of $A[x]$. $\mathcal{F}_{R[x]}(\triangle[x])$ is contravariantly finite in the subcategory consisting of modules which have the form of $A[x]$. $\square$

Similarly, we have

Lemma 3.11 *$\mathcal{F}_{R[[x]]}(\overline{\nabla}[[x]])$ is covariantly finite in the subcategory consisting of modules which are of the form of $A[[x]]$.*

4. On $\triangle[x] - \mathrm{gfd}$ of $R[x]$

For all $A[x]$, it follows from Lemma 3.10 that there is a finite $\mathcal{F}_{R[x]}(\triangle[x])$-resolution

$$0 \longrightarrow M_d[x] \longrightarrow \cdots \longrightarrow M_0[x] \longrightarrow A[x] \longrightarrow 0, \tag{3}$$

with $M_i[x] \in \mathcal{F}_{R[x]}(\triangle[x])$ for all $A[x]$.

Definition 4.1 *$\triangle[x] - \mathrm{gfd}(A[x])$ is the smallest number d for which we have $\mathcal{F}_{R[x]}(\triangle[x])$-resolution in (3) with $M_i[x] \in \mathcal{F}_{R[x]}(\triangle[x])$.*

$$\triangle[x] - \mathrm{gfd}(R[x]) = -sup\{\triangle[x] - \mathrm{gfd}(A[x]) | A \in \mathrm{mod}R\}.$$

Similarly, we can introduce the notions $\overline{\nabla}[[x]] - \mathrm{gfd}(A[[x]])$ and $\overline{\nabla}[[x]] - \mathrm{gfd}(R[[x]])$.

Theorem 4.1 *Let A be an R-module, then $\triangle - \mathrm{gfd}(A) = \triangle[x] - \mathrm{gfd}(A[x])$.*

Proof Since A has a finite $\mathcal{F}_R(\triangle)$-resolution

$$\cdots \longrightarrow M_n \longrightarrow M_{n-1} \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow A \longrightarrow 0,$$

$A[x]$ has an $\mathcal{F}_{R[x]}(\triangle[x])$-resolution as follows

$$\cdots \longrightarrow R[x] \otimes_R M_n \longrightarrow R[x] \otimes_R M_{n-1} \cdots \longrightarrow R[x] \otimes_R M_1 \longrightarrow R[x] \otimes_R M_0 \longrightarrow R[x] \otimes_R A \longrightarrow 0.$$

Thus, $\triangle[x] - \mathrm{gfd}(A[x]) \leq \triangle - \mathrm{gfd}(A)$. Suppose $\triangle[x] - \mathrm{gfd}(A[x]) = n$, then $A[x]$ has an $\mathcal{F}_{R[x]}(\triangle[x])$-resolution as follows

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow A[x] \longrightarrow 0,$$

where $Q_i \in \mathcal{F}_{R[x]}(\triangle[x])$ $(i = 1, 2, \cdots, n)$. As an R-module, $Q_i \in \mathcal{F}_R(\triangle)$. Since $A[x]$ is isomorphic to a direct sum of countably many A and $\triangle - \mathrm{gfd}(\bigsqcup A) = \triangle[x] = n$, we have $\triangle - \mathrm{gfd}(A) \leq n = \triangle[x] - \mathrm{gfd}(A[x])$. Therefore, $\triangle - \mathrm{gfd}(A) = \triangle[x] - \mathrm{gfd}(A[x])$. $\square$

Similarly, we have

Theorem 4.2 *Let R be a perfect and coherent commutative algebra and $(R, \leq)$ be a standardly stratified algebra. If A is an R-module, then $\overline{\nabla}[[x]] - \mathrm{gfd}(A[[x]]) = \overline{\nabla} - \mathrm{gfd}(A)$.*

Theorem 4.3 *$\triangle[x] - \mathrm{gfd}(A[x]) = d$ if and only if $\mathrm{Ext}^i_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) = 0$ for all $i \geq d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \neq 0$.*

Proof As an R-module, $\overline{\nabla}(\lambda)[x] \simeq \bigsqcup_{i=0}^{\infty} \overline{\nabla}(\lambda)$. So, we have that $\triangle[x] - \mathrm{gfd}(A[x]) = d \Longleftrightarrow \triangle - \mathrm{gfd}(A) = d \Longleftrightarrow \mathrm{Ext}^i_R(A, \overline{\nabla}(\lambda)) = 0$ for all $i \geq d$ and $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_R(A, \overline{\nabla}(\lambda)) \neq 0$. $\Longleftrightarrow \mathrm{Ext}^i_R(A, \bigsqcup \overline{\nabla}(\lambda)) = 0$ for all $i \geq d$ and $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_R(A, \bigsqcup \overline{\nabla}(\lambda)) \neq 0$. Since

$$\begin{aligned} &\mathrm{Ext}^i_{R[x]}(R[x]_R \otimes A, \overline{\nabla}(\lambda)[x]) \simeq \mathrm{Ext}^i_R(A, \mathrm{Hom}_{R[x]}(R[x], \overline{\nabla}(\lambda)[x])) \\ &\simeq \mathrm{Ext}^i_R(A, \overline{\nabla}(\lambda)[x]) \simeq \mathrm{Ext}^i_R(A, \bigsqcup \overline{\nabla}(\lambda)) \simeq \bigsqcup \mathrm{Ext}^i_R(A, \overline{\nabla}(\lambda)) \end{aligned}$$

$\triangle[x] - \mathrm{gfd}(A[x]) = d$ if and only if $\mathrm{Ext}^i_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) = 0$ for all $i \geq d$ and $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\mathrm{Ext}^d_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \neq 0$. $\square$

Similarly, we can obtain

Theorem 4.4 *Let R be a perfect and coherent commutive algebra and $(R, \leq)$ be a standardly stratified algebra. If A is an R-module, then $\overline{\nabla}[[x]] - \mathrm{gfd}(A[[x]]) = d$ if and only if $\mathrm{Ext}^i_{R[[x]]}(\triangle(\lambda)[[x]], A[[x]]) = 0$ for all $i \geq d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that*

$$\mathrm{Ext}^d_{R[[x]]}(\triangle(\lambda)[[x]], A[[x]]) \neq 0.$$

Theorem 4.5 *Let A, B, C be R-module. If*

$$0 \to A[x] \to B[x] \to C[x] \to 0$$

is exact, then we have

(1) If $\triangle[x] - \mathrm{gfd}(B[x]) > \triangle[x] - \mathrm{gfd}(A[x])$, then

$$\triangle[x] - \mathrm{gfd}(C[x]) = \triangle[x] - \mathrm{gfd}(B[x]);$$

(2) If $\triangle[x] - \mathrm{gfd}(B[x]) < \triangle[x] - \mathrm{gfd}(A[x])$, then

$$\triangle[x] - \mathrm{gfd}(C[x]) = \triangle[x] - \mathrm{gfd}(A[x]) + 1;$$

(3) If $\triangle[x] - \mathrm{gfd}(B[x]) = \triangle - \mathrm{gfd}(A[x])$, then

$$\triangle[x] - \mathrm{gfd}(C[x]) \leq \triangle[x] - \mathrm{gfd}(A[x]) + 1.$$

Proof There is a long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{Ext}^n_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]) \to \mathrm{Ext}^n_{R[x]}(B[x], \overline{\nabla}(\lambda)[x]) \to \mathrm{Ext}^n_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \\ &\to \mathrm{Ext}^{n+1}_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]) \to \mathrm{Ext}^{n+1}_{R[x]}(B[x], \overline{\nabla}(\lambda)[x]) \to \mathrm{Ext}^{n+1}_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \to \cdots, \end{aligned} \tag{4}$$

for all $\overline{\nabla}(\lambda)[x]$ and n. Suppose $\triangle[x] - \mathrm{gfd}(B[x]) = m$, $\triangle[x] - \mathrm{gfd}(A[x]) = n$. We have

1) If $m > n$, then $\mathrm{Ext}^m_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) = 0$, but there exists a λ such that

$$\mathrm{Ext}^m_{R[x]}(B[x], \overline{\nabla}(\lambda)[x]) \neq 0.$$

By the long exact sequence (4) we have that $\mathrm{Ext}^m_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]) \neq 0$,

$$\mathrm{Ext}^{m+j}_{R[x]}(B[x], \overline{\nabla}(\lambda)[x]) \simeq \mathrm{Ext}^{m+j}_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]), \quad j > 0, \quad \lambda \in \Lambda.$$

Therefore, we have $\triangle[x] - \mathrm{gfd}(C[x]) = \triangle[x] - \mathrm{gfd}(B[x])$ from Theorem 4.3.

2) If $m < n$, then $\mathrm{Ext}^n_{R[x]}(B[x], \overline{\nabla}(\lambda)[x]) = 0$, but there exists a λ such that

$$\mathrm{Ext}^n_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \neq 0.$$

By the long exact sequence (4) we have that $\mathrm{Ext}^{n+1}_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]) \neq 0$,

$$\mathrm{Ext}^{n+j}_{R[x]}(C[x], \overline{\nabla}(\lambda)[x]) \simeq \mathrm{Ext}^{n+j-1}_{R[x]}(A[x], \overline{\nabla}(\lambda)[x]) \quad j > 0, \quad \lambda \in \Lambda.$$

Therefore, we have $\triangle[x]-\mathrm{gfd}(C[x])=\triangle[x]-\mathrm{gfd}(A[x])+1$ from Theorem 4.3.

3) If $m=n$, then $\mathrm{Ext}^{n+1}_{R[x]}(B[x],\overline{\nabla}(\lambda)[x])\simeq \mathrm{Ext}^{n+1}_{R[x]}(A[x],\overline{\nabla}(\lambda)[x])=0$, but there is a $\lambda\in\Lambda$ such that $\mathrm{Ext}^{n}_{R[x]}(A[x],\overline{\nabla}(\lambda)[x])\neq 0$.

By the long exact sequence (4) we have that $\mathrm{Ext}^{n+2}_{R[x]}(C[x],\overline{\nabla}(\lambda)[x])\neq 0$.

Therefore, we have $\triangle[x]-\mathrm{gfd}(C[x]\leq \triangle[x]-\mathrm{gfd}(A[x])+1$ from Theorem 4.3. □

Similarly, we have

Theorem 4.6 *Let A,B,C be R-modules, if*

$$0\to A[[x]]\to B[[x]]\to C[[x]]\to 0$$

is exact, we have

(1) If $\overline{\nabla}[[x]]-\mathrm{gfd}(B[[x]])>\overline{\nabla}[[x]]-\mathrm{gfd}(C[[x]])$, then

$$\overline{\nabla}[[x]]-\mathrm{gfd}(A[[x]])=\overline{\nabla}[[x]]-\mathrm{gfd}(C[[x]]);$$

(2) If $\overline{\nabla}[[x]]-\mathrm{gfd}(B[[x]])<\overline{\nabla}[[x]]-\mathrm{gfd}(C[[x]])$, then

$$\overline{\nabla}[[x]]-\mathrm{gfd}(A[[x]])=\overline{\nabla}[[x]]-\mathrm{gfd}(C[[x]])+1;$$

(3) If $\overline{\nabla}[[x]]-\mathrm{gfd}(B[[x]])=\overline{\nabla}[[x]]-\mathrm{gfd}(C[[x]])$, then

$$\overline{\nabla}-\mathrm{gfd}(A[[x]])\leq\overline{\nabla}-\mathrm{gfd}(C[[x]])+1.$$

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关于标准分层代数与它的多项式代数上的滤链维数

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摘要: 在本文中，我们研究了标准分层代数的 $\triangle$- 好模滤链维数与它的多项式代数的 $\triangle[x]$- 好模滤链维数，并得到了一些有趣的结果.

关键词: 标准分层代数；滤链维数；多项式代数；拟遗传代数.