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Adjacent-Vertex-Distinguishing Total Chromatic Number of $P_m \times K_n$

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Abstract: Let G be a simple graph. Let f be a mapping from $V(G)\cup E(G)$ to $\{1,2,\cdots,k\}$. Let $C_f(v)=\{f(v)\}\cup\{f(vw)|w\in V(G), vw\in E(G)\}$ for every $v\in V(G)$. If f is a k-proper-total-coloring, and if $C_f(u)\neq C_f(v)$ for $u,v\in V(G), uv\in E(G)$, then f is called k-adjacent-vertex-distinguishing total coloring of G(k-AVDTC of G for short). Let $\chi_{at}(G)=\min\{k|G$ has a k-adjacent-vertex-distinguishing total coloring$\}$. Then $\chi_{at}(G)$ is called the adjacent-vertex-distinguishing total chromatic number. The adjacent-vertex-distinguishing total chromatic number on the Cartesion product of path P_m and complete graph K_n is obtained.

Key words: graph; total coloring; adjacent-vertex-distinguishing total coloring; adjacent-vertex-distinguishing total chromatic number.
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1. Introduction

The graphs considered in this paper are connected, limited, undirected and simple graphs. A k-proper-total-coloring of a graph G is a mapping f from $V(G)\cup E(G)$ to $\{1,2,\cdots,k\}$ such that

1). $\forall u,v\in V(G)$, if $uv\in E(G)$, then $f(u)\neq f(v)$;

2). $\forall e_1,e_2\in E(G), e_1\neq e_2$, if e_1,e_2 have common end vertex, then $f(e_1)\neq f(e_2)$;

3). $\forall u\in V(G), e\in E(G)$, if u is the end vertex of e, then $f(u)\neq f(e)$.

Let f be a k-proper-total-coloring of G. Let $C_f(u)=\{f(u)\}\cup\{f(uw)|w\in V(G), uw\in E(G)\}$ (or simply denoted by $C(u)$) and $\overline{C}_f(u)=\{1,2,\cdots,k\}-C_f(u)$ (or simply denoted by $\overline{C}(u)$) for every $u\in V(G)$. $C_f(u)$ is called the color set of u's. If $\forall u,v\in V(G), uv\in E(G)$, we have $C_f(u)\neq C_f(v)$, i.e., $\overline{C}_f(u)\neq\overline{C}_f(v)$, then f is called a k-adjacent-vertex-distinguishing total coloring (k-AVDTC in short). The number $\min\{k|G$ has a k-adjacent-vertex-distinguishing total coloring $\}$ is called the adjacent-vertex-distinguishing total chromatic number and is denoted by $\chi_{at}(G)$.

The theory of vertex-distinguishing proper edge-coloring has been investigated in several papers[1−3,5]. Adjacent strong edge coloring (i.e., adjacent-vertex-distinguishing proper edge-coloring) is considered in [7] by Zhang Zhongfu et al. The concept about the adjacent-vertex-distinguishing total coloring is proposed by Zhang Zhongfu and Chen Xiang'en et al in [6]. And

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the adjacent-vertex-distinguishing total colorings of cycle, complete graph, complete bipartite graph, fan, wheel and tree are discussed in [6]. According to these results, for adjacent-vertex-distinguishing total chromatic number, a conjecture is given in [6].

Conjecture 1.1 [6] *For every connected graph G with order at least 2, we have* $\chi_{at}(G) \leq \Delta(G)+3$.

Let P_m and K_n be a path and a complete graph respectively:

$$V(P_m) = \{u_1, u_2, \cdots, u_m\}, E(P_m) = \{u_1u_2, u_2u_3, \cdots, u_{m-1}u_m\};$$

$$V(K_n) = \{v_1, v_2, \cdots, v_n\}, E(K_n) = \{v_iv_j | i, j = 1, 2, \cdots, n, i < j\}.$$

Construct a new graph $P_m \times K_n$ such that

$$V(P_m \times K_n) = \{w_{ij} | i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\},$$

$$E(P_m \times K_n) = \{w_{ij}w_{st} | i = s \text{ and } v_jv_t \in E(K_n) \text{ or } j = t \text{ and } v_iv_s \in E(P_m)\}.$$

The graph $P_m \times K_n$ is called the Cartesion product of P_m and K_n.

The adjacent-vertex-distinguishing total coloring on the Cartesion product of path P_m and complete graph K_n is studied and the corresponding chromatic number is obtained in this paper. Theorems 2.1 and 2.2 in this paper will illustrate that Conjecture 1.1 is valid for the Cartesion product of path P_m and complete graph K_n.

The following lemma is obvious.

Lemma 1.2[6] *If G does not have two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G)+1$; If G has two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G)+2$.*

For the graph-theoretic terminology the reader is referred to [4].

2. Main results

If $n = 1$, then $P_m \times K_n = P_m$. From the results in [6], we have

Theorem 2.1

$$\chi_{at}(P_m \times K_1) = \begin{cases} 3, & m = 2, 3; \\ 4, & m \geq 4. \end{cases}$$

Theorem 2.2 *If $n \geq 2$, then we have*

$$\chi_{at}(P_m \times K_n) = \begin{cases} n+2, & m = 2; \\ n+3, & m \geq 3. \end{cases}$$

Proof We distinguish 2 cases.

Case 1. $m = 2$.

In this case $\chi_{at}(P_m \times K_n) \geq n+2$ by Lemma 1.2. In order to prove $\chi_{at}(P_m \times K_n) = n+2$, we need only to prove that $P_m \times K_n$ has a $(n+2)$-AVDTC. Let $C = \{1, 2, \cdots, n+2\}$ be the set composed of all $n+2$ colors. We appoint that if some color c is less than 1 or larger than $n+2$, then we identify c with r, where $r \in \{1, 2, \cdots, n+2\}$ and $c \equiv r (\mathrm{mod}\ n+2)$. Construct a mapping f from $V(P_m \times K_n) \cup E(P_m \times K_n)$ to C as follows.

$$f(w_{1i}w_{1j}) = f(w_{2i}w_{2j}) = i + j - 2, i, j = 1, 2, \cdots, n, i \neq j.$$

$$f(w_{1i}) = n + i - 1, f(w_{2i}) = n + i, f(w_{1i}w_{2i}) = n + 2i,\ i = 1, 2, \cdots, n.$$

For the above coloring, we have that $\overline{C}(w_{11}), \overline{C}(w_{12}), \cdots, \overline{C}(w_{1n})$ are equal to $\{n+1\}, \{n+2\}, \{1\}, \{2\}, \cdots, \{n-2\}$ respectively, and $\overline{C}(w_{21}), \overline{C}(w_{22}), \cdots, \overline{C}(w_{2n})$ are equal to $\{n\}, \{n+1\}, \{n+2\}, \{1\}, \{2\}, \cdots, \{n-3\}$ respectively. Thus two adjacent vertices have different color sets. So f is a $(n+2)$-AVDTC of $P_m \times K_n$.

Case 2. $m \geq 3$.

In this case $\chi_{at}(P_m \times K_n) \geq n+3$ by Lemma 1.2. In order to prove $\chi_{at}(P_m \times K_n) = n+3$, we need only to prove that $P_m \times K_n$ has a $(n+3)$-AVDTC. Let $C = \{1, 2, \cdots, n+3\}$ be the set composed of all $n+3$ colors. We appoint that if some color c is less than 1 or larger than $n+3$, then we identify c with r, where $r \in \{1, 2, \cdots, n+3\}$ and $c \equiv r (\mathrm{mod}\ n+3)$. Construct a mapping f from $V(P_m \times K_n) \cup E(P_m \times K_n)$ to C as follows.

$$f(w_{ki}w_{kj}) = i + j - 2, k = 1, 2, \cdots, m; i, j = 1, 2, \cdots, n, i \neq j.$$

$$f(w_{ki}w_{k+1,i}) = \begin{cases} 2(i-1), & 1 \leq k \leq m-1 \text{ and } k \text{ is an odd;} \\ n+i, & 1 \leq k \leq m-1 \text{ and } k \text{ is an even.} \end{cases}$$

$$f(w_{ki}) = \begin{cases} n+i+1, & 1 \leq k \leq m-1 \text{ and } k \text{ is an odd;} \\ n+i-1, & 1 \leq k \leq m-1 \text{ and } k \text{ is an even.} \end{cases}$$

So far we have not colored the vertices $w_{m1}, w_{m2}, \cdots, w_{mn}$. Obviously, $\overline{C}(w_{1i}) = \{n+i-1, n+i\}$ and $\overline{C}(w_{11}), \overline{C}(w_{12}), \cdots, \overline{C}(w_{1n})$ are distinct. If $1 \leq k \leq m-1$ and k is an odd number, then $\overline{C}(w_{ki}) = \{n+i-1\}$ and $\overline{C}(w_{k1}), \overline{C}(w_{k2}), \cdots, \overline{C}(w_{kn})$ are distinct. If $1 \leq k \leq m-1$ and k is an even number, then $\overline{C}(w_{ki}) = \{n+i+1\}$ and $\overline{C}(w_{k1}), \overline{C}(w_{k2}), \cdots, \overline{C}(w_{kn})$ are distinct. Meanwhile arbitrary two adjacent vertices in $\{w_{2j}, w_{3j}, \cdots, w_{m-1,j}\}$ have different color sets for every $j = 1, 2, \cdots, n$.

In order to color the vertices $w_{m1}, w_{m2}, \cdots, w_{mn}$, we distinguish 4 subcases to be considered.

Case 2.1. m is an even number.

Let $f(w_{mi}) = n+i-1, i = 1, 2, \cdots, n$. Obviously, $\overline{C}(w_{mi}) = \{n+i, n+i+1\}, i = 1, 2, \cdots, n$. And $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$ are distinct. Thus f is a $(n+3)$-AVDTC of $P_m \times K_n$.

Case 2.2. m is an odd number, and $n \equiv 1, 2 (\mathrm{mod}\ 3)$.

Let $f(w_{mi}) = n+i+1, i = 1, 2, \cdots, n$. Obviously, $\overline{C}(w_{mi}) = \{n+i-1, 2(i-1)\}$, $i = 1, 2, \cdots, n$. Now we prove that $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$ are distinct. Suppose that $\overline{C}(w_{mi}) = \overline{C}(w_{mj})$, $1 \leq i, j \leq n$. Thus

$$\{n+i-1, 2(i-1)\} = \{n+j-1, 2(j-1)\}.$$

If $n+i-1 \equiv 2(j-1)$, $n+j-1 \equiv 2(i-1) \pmod{n+3}$, then $j \equiv 2(i-1)+1-n \pmod{n+3}$. So $3n \equiv 3(i-1) \pmod{n+3}$. As $n \equiv 1,2 \pmod 3$, *i.e.*, $(3,n)=1$, we have $(3,n+3)=1$ and $i \equiv n+1 \pmod{n+3}$. This is a contradiction. So we have

$$n+i-1 \equiv n+j-1, 2(i-1) \equiv 2(j-1) \pmod{n+3}.$$

Thus $i \equiv j \pmod{n+3}$. Notice that $1 \le i, j \le n$, so $i=j$. This illustrates that

$$\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$$

are distinct. So $P_m \times K_n$ has a $(n+3)$-AVDTC.

Case 2.3. m is an odd number, $n \equiv 0 \pmod 6$.

Let $f(w_{mi}) = 2(i-1), i=1,2,\cdots,n$. Obviously, $\overline{C}(w_{mi}) = \{n+i-1, n+i+1\}$, $i=1,2,\cdots,n$. And $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$ are distinct. Thus f is a $(n+3)$-AVDTC of $P_m \times K_n$.

Case 2.4. m is an odd number, $n \equiv 3 \pmod 6$.

Let $f(w_{mi}) = n+i+1, i=1,2,\cdots,n$. Obviously, $\overline{C}(w_{mi}) = \{n+i-1, 2(i-1)\}$, $i=1,2,\cdots,n$.

If $n=3$, then $\overline{C}(w_{m1}) = \{3,6\}, \overline{C}(w_{m2}) = \{4,2\}, \overline{C}(w_{m3}) = \{5,4\}$. So f is a 6-AVDTC.

Suppose that $n \ge 9$ in the following. Assume that $1 \le i < j \le n$ and $\overline{C}(w_{mi}) = \overline{C}(w_{mj})$. Then

$$n+i+1 \equiv 2(j-1),\ n+j+1 \equiv 2(i-1) \pmod{n+3}. \tag{1}$$

So $3n \equiv\ 3(i-1) \pmod{n+3}$, i.e., there exists positive integer k such that $3n-3i+3 = k(n+3)$. When $k=1$, we have $3n-3i+3 = n+3$, i.e. $i = \frac{2n}{3}$. When $k=2$, we have $3n-3i+3 = 2(n+3)$, i.e. $i = \frac{n-3}{3}$. When $k \ge 3$, we have $3n-3i+3 = k(n+3)$, i.e. $(k-3)n+3k+3i = 3$. This is impossible. Thus $i = \frac{2n}{3}$ or $\frac{n-3}{3}$. From the symmetry of i and j in Equation (1) we know that $j = \frac{2n}{3}$ or $\frac{n-3}{3}$. As $1 \le i < j \le n$, we have $i = \frac{n-3}{3}, j = \frac{2n}{3}$. This illustrates that the $n-1$ sets $\overline{C}(w_{m1}), \cdots, \overline{C}(w_{m,\frac{n-3}{3}-1}), \overline{C}(w_{m,\frac{n-3}{3}+1}), \cdots, \overline{C}(w_{mn})$ are distinct. And $\overline{C}(w_{m,\frac{n-3}{3}}) = \{n+\frac{n-3}{3}-1, \frac{2(n-6)}{3}\}$ and $\overline{C}(w_{m,\frac{2n}{3}}) = \{n+\frac{2n}{3}-1, \frac{2(2n-3)}{3}\}$ are equal.

If $n=9$, then $2 \in \overline{C}(w_{m2}) \cap \overline{C}(w_{m8})$. We redefine the color of the edge $w_{m2}w_{m8}$ such that $f(w_{m2}w_{m8}) = 2$ (Note that the original color of the edge $w_{m2}w_{m8}$ is 8). The new coloring is also a proper total coloring and, for this new coloring, $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{m9})$ are equal to $\{9,12\}$, $\{10,8\}$, $\{11,4\}$, $\{12,6\}$, $\{1,8\}$, $\{2,10\}$, $\{3,12\}$, $\{4,8\}$, $\{5,4\}$, respectively. They are distinct. So $P_m \times K_9$ has a 12-AVDTC.

Suppose that $n \ge 15$ in the following. Construct a $4 \times n$ matrix:

$$\begin{pmatrix} n & n+1 & n+2 & n+3 & 1 & \cdots & \frac{n}{3}-5 & \cdots & \frac{2(n-6)}{3} & \cdots & n-5 & n-4 \\ n+1 & n+2 & n+3 & 1 & 2 & \cdots & \frac{n}{3}-4 & \cdots & \frac{2(n-6)}{3}+1 & \cdots & n-4 & n-3 \\ n+2 & n+2 & 1 & 2 & 3 & \cdots & \frac{n}{3}-3 & \cdots & \frac{2(n-6)}{3}+2 & \cdots & n-3 & n-2 \\ n+3 & 2 & 4 & 6 & 8 & \cdots & \frac{2(n-6)}{3} & \cdots & \frac{n}{3}-5 & \cdots & 2(n-2) & 2(n-1) \end{pmatrix}$$

The above matrix is denoted by A. The entries in the first row of A are the colors of vertices w_{k1}, w_{k2}, $\cdots$, w_{kn} respectively (k is an even number); The entries in the second row of A are the colors of edges $w_{k1}w_{k+1,1}$, $w_{k2}w_{k+1,2}$, $\cdots$, $w_{kn}w_{k+1,n}$ respectively (k is an even, $1 \le k \le m-1$); The entries in the third row of A are the colors of vertices w_{k1}, w_{k2}, $\cdots$, w_{kn} respectively (k is an odd number); The entries in the fourth row of A are the colors of edges $w_{k1}w_{k+1,1}$, $w_{k2}w_{k+1,2}$, $\cdots$, $w_{kn}w_{k+1,n}$ respectively (k is an odd number, $1 \le k \le m-1$).

Let $n = 6l+3, l \ge 2$. The color $\frac{n}{3}-5$ in the first row $(\frac{n-3}{3})$th column of A is the same as the color in the fourth row $(\frac{n-9}{6})$th column of A. Thus $\frac{n}{3}-5 \in \overline{C}(w_{m,\frac{n-9}{6}}) \cap \overline{C}(w_{m,\frac{n-3}{3}})$. Redefine the color of $w_{m,\frac{n-9}{6}}w_{m,\frac{n-3}{3}}$ such that

$$f(w_{m,\frac{n-9}{6}}w_{m,\frac{n-3}{3}}) = \frac{n}{3}-5.$$

Note that the original color of $w_{m,\frac{n-9}{6}}w_{m,\frac{n-3}{3}}$ is $\frac{n-9}{2}$, and $\frac{n-9}{2} \not\equiv \frac{n}{3}-5 \pmod{n+3}$. Now we will prove that $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$ are distinct under the above new coloring. In the above n sets, there are $n-3$ (if l is even) or $n-5$ (if l is odd) sets which do not contain color $\frac{n-9}{2}$ and which are distinct. So we only consider the sets contain color $\frac{n-9}{2}$.

If l is an even number, the following 3 sets which contain color $\frac{n-9}{2}$:

$$\overline{C}(w_{m,\frac{n-9}{6}}) = \{\frac{7n-15}{6}, \frac{n-9}{2}\}, \overline{C}(w_{m,\frac{n-3}{3}}) = \{\frac{n-9}{2}, 2(\frac{n-6}{3})\},$$

$$\overline{C}(w_{m,\frac{n-1}{2}}) = \{\frac{n-9}{2}, n-3\}.$$

We may easily verify that $\frac{7n-15}{6}$, $2(\frac{n-6}{3})$ and $n-3$ are not congruent each other modulo $(n+3)$.

If l is an odd number, the following 5 sets contain color $\frac{n-9}{2}$:

$$\overline{C}(w_{m,\frac{n-9}{6}}) = \{\frac{7n-15}{6}, \frac{n-9}{2}\}, \overline{C}(w_{m,\frac{n-9}{4}+1}) = \{\frac{5n-9}{4}, \frac{n-9}{2}\},$$

$$\overline{C}(w_{m,\frac{n-3}{3}}) = \{\frac{n-9}{2}, 2(\frac{n-6}{3})\}, \overline{C}(w_{m,\frac{n-1}{2}}) = \{\frac{n-9}{2}, n-3\}.$$

$$\overline{C}(w_{m,\frac{3n+1}{4}}) = \{\frac{7n-3}{4}, \frac{n-9}{2}\}.$$

We may easily verify that $\frac{7n-15}{6}$, $\frac{5n-9}{4}$, $2(\frac{n-6}{3})$, $n-3$ and $\frac{7n-3}{4}$ are not congruent each other modulo $(n+3)$.

Thus $\overline{C}(w_{m1}), \overline{C}(w_{m2}), \cdots, \overline{C}(w_{mn})$ are distinct. So $P_m \times K_n$ has a $(n+3)$-AVDTC.

The proof is completed.

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$P_m \times K_n$ 的邻点可区别全色数

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摘要: 设 G 是简单图．设 f 是一个从 $V(G)\cup E(G)$ 到 $\{1,2,\cdots,k\}$ 的映射．对每个 $v\in V(G)$，令 $C_f(v)=\{f(v)\}\cup\{f(vw)|w\in V(G),vw\in E(G)\}$．如果 f 是 k- 正常全染色，且对任意 $u,v\in V(G),uv\in E(G)$，有 $C_f(u)\neq C_f(v)$，那么称 f 为图 G 的邻点可区别全染色 (简称为 k-AVDTC). 数 $\chi_{at}(G)=\min\{k|G$ 有 k-AVDTC$\}$ 称为图 G 的邻点可区别全色数．本文给出路 P_m 和完全图 K_n 的 Cartesion 积的邻点可区别全色数.

关键词: 图；全染色；邻点可区别全染色；邻点可区别全色数.