第26卷第4期 数 学 研 究 与 评 论 Vol.26, No.4
2006年11月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Nov., 2006

Article ID: 1000-341X(2006)04-0649-10 **Document code**: A

Integrals of Braided Hopf Algebras

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Abstract: The faithful quasi-dual H^d and strict quasi-dual $H^{d'}$ of an infinite braided Hopf algebra H are introduced and it is proved that every strict quasi-dual $H^{d'}$ is an H-Hopf module. A connection between the integrals and the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of H^d is found. That is, $H^{d\mathrm{rat}} \cong \int_{H^d}^{l} \otimes H$ is proved. The existence and uniqueness of integrals for braided Hopf algebras in the Yetter-Drinfeld category $({}_B^B\mathcal{YD}, C)$ are given.

Key words: integral; braided Hopf algebra.
MSC(2000): 16W30, 16G10
CLC number: O153.3

1. Introduction

The integrals of Hopf algebras were introduced by Larson and Sweedler in [1]. Their connection with the maximal rational H^*-module $H^{*\mathrm{rat}}$ of H^* was given by Sweedler in [2], i.e.

$$H^{*\mathrm{rat}} \cong \int_{H^*}^{l} \otimes H \quad \text{as} \quad H \ \text{-Hopf modules.} \tag{1}$$

The uniqueness of the integrals was proved by Sullivan in [4]. The existence of the non-zero integrals was given in [5, Theorem 5.3.2]. The integrals have proved to be essential instruments in constructing invariants of surgically presented 3-manifolds or 3-dimensional topological quantum field theories [6–8].

In 1986, braided tensor categories were introduced by Joyal and Street[9]. Algebraic structures within them, especially, braided Hopf algebras or "braided groups" as well as cross products and diagrammatic techniques for such algebraic constructions were studied by Majid in [10,11]. See [12,13] for introductions. Many braided groups are known, including ones obtained by transmutation[10] from the (co)quasitriangular Hopf algebras, and the universal enveloping algebra of a Lie color algebra, the Nichols algebras[14] and the Lusztig's quantum algebras[15]. Therefore, it is interesting to extend the Hopf algebra constructions to the braided cases. For finite braided Hopf algebras (braided groups) H, i.e. braided Hopf algebra H with a left dual in braided tensor categories, Bespalov, Kerler and Lyubashenko[16], and Takeuchi[17] introduced an integral and proved that the integral is an invertible object. Moreover, Takeuchi proved that the antipode is an isomorphism and formula (1) holds.

Received date: 2004-03-17

In this paper we study integrals of infinte braided Hopf algebras in braided tensor categories. A braided Hopf algebra is called an infinite braided Hopf algebra if it has no left duals[17]. An important example of infinite braided Hopf algebras is the universal enveloping algebra of a Lie superalgebra. So the integrals of infinite braided Hopf algebras should have important applications in both mathematics and mathematical physics. We introduce the faithful quasi-dual H^d and strict quasi-dual H^d of a braided Hopf algebra H. We prove that every strict quasi-dual $H^{d'}$ is an H-Hopf module. By imitating Larson and Sweedler's Hopf module construction, we obtain the connection between the integrals and the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of H^d. That is, we prove $H^{d\mathrm{rat}} \cong \int_{H^d}^l \otimes H$. We give the existence and uniqueness of the integrals for some infinite braided Hopf algebras living in the Yetter-Drinfeld category $({}_B^B\mathcal{YD}, C)$.

This paper was organized as follows. In Section 2, since it is possible that $\mathrm{Hom}(H, I)$ is not an object in $\mathcal{C}$ for braidrd Hopf algebra H, we introduce strict (or faithful) quasi-dual $H^{d'}$, and prove that every strict quasi-dual $H^{d'}$ is an H-Hopf module. In Section 3, we concentrate on braided tensor categories consisting of some braided vector spaces. We prove $H^{d\mathrm{rat}} \cong \int_{H^d}^l \otimes H$ for an infinite braided Hopf algebra H and the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of H^d. That is, we obtain the connection between integrals and the maximal rational H^d-module $H^{d\mathrm{rat}}$ of H^d. In Section 4, we give the existence and uniqueness of the integrals for infinite braided Hopf algebras living in the Yetter-Drinfeld category $({}_B^B\mathcal{YD}, C)$. In Section 5, we show the Maschke's theorem for infinite braided Hopf algebras.

2. Strict quasi-duals and Hopf modules of braided Hopf algebras

In this section we introduce a faithful quasi-dual H^d and strict quasi-dual $H^{d'}$ of braided Hopf algebra H and show that $H^{d'}$ is an H-Hopf module. Using the fundamental theorem of Hopf modules, we show the formula similar to (1)

$$H^{d'} \cong (H^{d'})^{coH} \otimes H \quad \text{as} \quad H\text{-Hopf modules in } \mathcal{C}.$$

We first recall some notations. Let $(\mathcal{C}, \otimes, I, C)$ be a braided tensor category, where I is the identity object and C is the braiding. We also write $W \otimes f$ for $id_W \otimes f$ and $f \otimes W$ for $f \otimes id_W$. Since every braided tensor category is equivalent to a strict braided tensor category by [19, Theorem 0.1], we may view every braided tensor category as a strict braided tensor category.

Definition 2.1 *Let H be a braided Hopf algebra in $\mathcal{C}$. If there is an algebra N in $\mathcal{C}$ and a morphism $\langle , \rangle$ from $N \otimes H$ to I such that*

$$\langle , \rangle (m \otimes H) = (\langle , \rangle \otimes \langle , \rangle)(N \otimes C_{N,H} \otimes H)(N \otimes N \otimes \Delta), \quad \langle \eta, H \rangle = \varepsilon$$

then N is called a left quasi-dual of H. Moreover, if for any objects U, V and four morphisms $f : U \to V \otimes N$, $f' : U \to V \otimes N$, $g : U \to H \otimes V$, $g' : U \to H \otimes V$ in $\mathcal{C}$, $(V \otimes \langle , \rangle)(f \otimes H) = (V \otimes \langle , \rangle)(f' \otimes H)$ implies $f = f'$ and $(\langle , \rangle \otimes V)(N \otimes g) = (\langle , \rangle \otimes V)(N \otimes g')$ implies $g = g'$, then N is called a faithful quasi-dual of H under $\langle , \rangle$, written as H^d. In addition, if there are a left *ideal, written as* $H^{d'}$*, of* H^d *and two morphisms:* $\rightharpoonup: H\otimes H^d \to H^d$ *and* $\rho:\ H^{d'} \to H^{d'}\otimes H \in \mathcal{C}$ *such that*

$$\langle,\rangle(\rightharpoonup \otimes H) = \langle,\rangle(H^d\otimes m)(H^d\otimes C_{H,H})(C_{H,H^d}\otimes H)$$
$$(H^d\otimes \langle,\rangle)(C_{H^d,H^{d'}}\otimes H)(H^d\otimes \rho) = m$$

and the constraint $\rightharpoonup$ *on* $H\otimes H^{d'}$ *is a morphism to* $H^{d'}$*, then* $H^{d'}$ *is called a strict quasi-dual of* H*.*

Let $\leftharpoonup = \rightharpoonup (S\otimes H^{d'})C_{H^d,H}$. In fact, if H has a left dual H^* in $\mathcal{C}$, then H^* is a strict quasi-dual and faithful quasi-dual of H under evaluation $\langle,\rangle$.

Lemma 2.2 *Let* H^d *be a faithful quasi-dual of* H *under* $\langle,\rangle$ *and* $C_{H,H} = C^{-1}_{H,H}$*. Then* $C_{U,V} = (C_{V,U})^{-1}$*, for* $U, V = H$ *or* H^d*.*

Proof Observe that $(H\otimes\langle,\rangle)(C_{H^d,H}\otimes H) = (\langle,\rangle\otimes H)(H^d\otimes C^{-1}_{H,H}) = (\langle,\rangle\otimes H)(H^d\otimes C_{H,H}) = (H\otimes\langle,\rangle)(C^{-1}_{H^d,H}\otimes H)$. Thus $C_{H^d,H} = C^{-1}_{H^d,H}$. Note that $C_{H,H^d} = C^{-1}_{H^d,H}C_{H^d,H}C_{H,H^d} = C^{-1}_{H^d,H}C^{-1}_{H^d,H}C_{H,H^d} = C^{-1}_{H^d,H}$ and

$$\begin{aligned}(H^d\otimes\langle,\rangle)(C_{H^d,H^d}\otimes H) &= (\langle,\rangle\otimes H^d)(H^d\otimes C^{-1}_{H^d,H}) = (\langle,\rangle\otimes H^d)(H^d\otimes C_{H^d,H})\\ &= (H^d\otimes\langle,\rangle)(C^{-1}_{H^d,H^d}\otimes H).\end{aligned}$$

Thus $C_{H^d,H^d} = C^{-1}_{H^d,H^d}$. □

If $C_{H,H} = C^{-1}_{H,H}$, then we say that the braiding is symmetric on H. Throughout this section we always assume that the braiding is symmetric on H. For convenience, for $U, V = H$ or H^d we denote the braiding $C_{U,V}$ by C.

Lemma 2.3 $m(H^d\otimes \leftharpoonup) = \leftharpoonup (m\otimes H)(\rightharpoonup \otimes H^{d'}\otimes H)(C\otimes H^{d'}\otimes H)(H^d\otimes C\otimes H)(H^d\otimes H^{d'}\otimes C)(H^d\otimes H^{d'}\otimes\Delta)$.

Proof

$$\begin{aligned}&\langle,\rangle(m\otimes H)(H^d\otimes \leftharpoonup \otimes H) = (\langle,\rangle\otimes\langle,\rangle)(H^d\otimes C\otimes H)(H^d\otimes \leftharpoonup \otimes\Delta)\\ &\quad = \langle,\rangle(\langle,\rangle\otimes H^{d'}\otimes m)(H^d\otimes C\otimes C)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes S\otimes\Delta)\end{aligned}$$

and

$$\begin{aligned}&\langle,\rangle(\leftharpoonup \otimes H)(m\otimes H\otimes H)(\rightharpoonup \otimes H^{d'}\otimes H\otimes H)(C\otimes H^{d'}\otimes H\otimes H)\\ &\quad (H^d\otimes C\otimes H\otimes H)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\ &= \langle,\rangle(m\otimes H)(\rightharpoonup \otimes H^{d'}\otimes C)(H^d\otimes C\otimes S\otimes H)\\ &\quad (H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\ &= \langle,\rangle(C\otimes H)(\rightharpoonup \otimes H^{d'}\otimes m)(C\otimes H^{d'}\otimes C)\\ &\quad (H^d\otimes C\otimes S\otimes H)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\ &= \langle,\rangle(H^d\otimes m)(H^d\otimes C)(H^d\otimes H\otimes H\otimes\langle,\rangle)(H^d\otimes H\otimes C\otimes H)(H^d\otimes H\otimes H^{d'}\otimes\Delta)\\ &\quad (H^d\otimes H\otimes H^d\otimes m)(H^d\otimes C\otimes C)(H^d\otimes H^{d'}\otimes C\otimes H)\end{aligned}$$

$$
\begin{aligned}
&(H^d\otimes H^{d'}\otimes S\otimes H\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\
&=(\langle,\rangle\otimes\langle,\rangle)(H^d\otimes C\otimes H)(H^d\otimes H^{d'}\otimes m\otimes H)(H^d\otimes H^{d'}\otimes C\otimes H)\\
&\quad(H^d\otimes H^{d'}\otimes H\otimes m\otimes m)(H^d\otimes H^{d'}\otimes H\otimes H\otimes C\otimes H)\\
&\quad(H^d\otimes H^{d'}\otimes H\otimes\Delta\otimes\Delta)(H^d\otimes H^{d'}\otimes H\otimes C)\\
&\quad(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes S\otimes H\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\
&=\langle,\rangle(H^{d'}\otimes m)(H^d\otimes C\otimes\langle,\rangle)(H^d\otimes H\otimes C\otimes H)(H^d\otimes H\otimes H^{d'}\otimes m\otimes m)\\
&\quad(H^d\otimes H\otimes H^{d'}\otimes H\otimes C\otimes H)(H^d\otimes H\otimes H^{d'}\otimes\Delta\otimes S\otimes S)\\
&\quad(H^d\otimes H\otimes H^{d'}\otimes H\otimes C)(H^d\otimes H\otimes H^{d'}\otimes H\otimes\Delta)(H^d\otimes C\otimes C)\\
&\quad(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\
&=(\langle,\rangle\otimes\langle,\rangle)(H^d\otimes C\otimes H)(H^d\otimes H^{d'}\otimes m\otimes m)(H^d\otimes H^{d'}\otimes m\otimes H\otimes C)\\
&\quad(H^d\otimes H^{d'}\otimes C\otimes C\otimes H)(H^d\otimes H^{d'}\otimes H\otimes C\otimes H\otimes H)(H^d\otimes H^{d'}\otimes S\otimes S\otimes H\otimes H\otimes H)\\
&\quad(H^d\otimes H^{d'}\otimes C\otimes C\otimes H)(H^d\otimes H^{d'}\otimes\Delta\otimes H\otimes\Delta)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\
&=(\langle,\rangle\otimes\langle,\rangle)(H^d\otimes C\otimes H)(H^d\otimes H^{d'}\otimes m\otimes m)(H^d\otimes H^{d'}\otimes H\otimes m\otimes C)\\
&\quad(H^d\otimes H^{d'}\otimes C\otimes C\otimes H)(H^d\otimes H^{d'}\otimes H\otimes C\otimes H\otimes H)(H^d\otimes H^{d'}\otimes C\otimes H\otimes H\otimes H)\\
&\quad(H^d\otimes H^{d'}\otimes H\otimes S\otimes C\otimes H)(H^d\otimes H^{d'}\otimes S\otimes\Delta\otimes\Delta)(H^d\otimes H^{d'}\otimes\Delta\otimes H)\\
&=\langle,\rangle(\langle,\rangle\otimes H^{d'}m)(H^{d'}\otimes C\otimes C)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes S\otimes\Delta).
\end{aligned}
$$

Thus we complete the proof. □

Theorem 2.4 *$H^{d'}$ is an H-Hopf module.*

Proof (1) $(H^{d'},\leftharpoondown)$ is a right H-module.

$$
\begin{aligned}
&\langle,\rangle(\leftharpoondown\otimes H)(\leftharpoondown\otimes H\otimes H)=\langle,\rangle(H^{d'}\otimes m)(H^{d'}\otimes C)(\leftharpoondown\otimes S\otimes H)\\
&\quad=\langle,\rangle(H^{d'}\otimes m)(H^{d'}\otimes C)(H^{d'}\otimes H\otimes m)(H^{d'}\otimes H\otimes C)(H^{d'}\otimes S\otimes S\otimes H)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle,\rangle(\leftharpoondown\otimes H)(H^{d'}\otimes m\otimes H)=\langle,\rangle(H^{d'}\otimes C)(H^{d'}\otimes S\otimes H)(H^{d'}\otimes m\otimes H)\\
&\quad=\langle,\rangle(H^{d'}\otimes m)(H^{d'}\otimes C)(H^{d'}\otimes m\otimes H)(H^{d'}\otimes C\otimes H)(H^{d'}\otimes S\otimes S\otimes H).
\end{aligned}
$$

Thus $\leftharpoondown(\leftharpoondown\otimes H)=\leftharpoondown(H^{d'}\otimes m)$. Obviously, $\leftharpoondown(H^{d'}\otimes\eta)=\mathrm{id}_{H^{d'}}$. Therefore, $(H^{d'},\leftharpoondown)$ is a right H-module.

(2) $(H^{d'},\rho)$ is a right H-comodule.

Note that $(H^{d'}\otimes\langle,\rangle\otimes\langle,\rangle)(C\otimes C\otimes H)(H^d\otimes C\otimes H\otimes H)(H^d\otimes H^d\otimes\rho\otimes H)(H^d\otimes H^d\otimes\rho)=m(H^d\otimes m)=m(m\otimes H^d)=(H^{d'}\otimes\langle,\rangle\otimes\langle,\rangle)(C\otimes C\otimes H)(H^d\otimes C\otimes\Delta)(H^d\otimes H^d\otimes\rho)$. Thus $(\rho\otimes H)\rho=(H^{d'}\otimes\Delta)\rho$. We also have that $(\mathrm{id}\otimes\varepsilon)\rho=(H^{d'}\otimes\langle,\rangle)(C\otimes H)(\eta\otimes\rho)=m(\eta\otimes H^{d'})=\mathrm{id}$. Therefore $(H^{d'},\rho)$ is a right H-comodule.

(3) Note that $(H^{d'}\otimes\langle,\rangle)(C\otimes H)(H^d\otimes\rho)(H^d\otimes\leftharpoondown)=m(H^d\otimes\leftharpoondown)=\leftharpoondown(m\otimes H)(\rightharpoonup\otimes H^{d'}\otimes H)(C\otimes H^{d'}\otimes H)(H^d\otimes C\otimes H)(H^d\otimes H^{d'}\otimes C)(H^d\otimes H^{d'}\otimes\Delta)=(\langle,\rangle\otimes H^{d'})(\rightharpoonup

$\otimes H \otimes \leftharpoonup)(C\otimes C\otimes H)(H^{d'}\otimes C\otimes H\otimes H)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes H^{d'}\otimes H\otimes C)(H^d\otimes\rho\otimes\Delta) = (\langle,\rangle\otimes H^{d'})(H^d\otimes m\otimes\leftharpoonup)(H^d\otimes H\otimes C\otimes H)(H^d\otimes C\otimes C)(H^d\otimes\rho\otimes\Delta) = (\langle,\rangle\otimes H^{d'})(H^d\otimes C)(H^d\otimes\leftharpoonup\otimes m)(H^d\otimes H^{d'}\otimes C\otimes H)(H^d\otimes\rho\otimes\Delta)$. Thus $\rho\circ\leftharpoonup = (\leftharpoonup\otimes m)(H^{d'}\otimes C\otimes H)(\rho\otimes\Delta)$. By (1),(2) and (3), we complete the proof. □

If $\mathcal{C}$ has equalizers, then the coinvariant $(H^{d'})^{coH}$ of H in $H^{d'}$ is an object in $\mathcal{C}$. Here $(H^{d'})^{coH}$ denotes the equalizer of the diagram

$$H^{d'} \quad \overset{\rho}{\underset{\mathrm{id}\otimes\eta}{\rightrightarrows}} \quad H^{d'}\otimes H \quad .$$

Combining Theorem 2.4 and the braided Hopf module fundamental theorem [17, Theorem 3.4], we have

Theorem 2.5 *If $\mathcal{C}$ has equalizers or $(H^{d'})^{\mathrm{co}H}$ is an object in $\mathcal{C}$, then*

$$H^{d'} \cong (H^{d'})^{\mathrm{co}H}\otimes H \ \text{ as } \ H\text{-Hopf modules in } \mathcal{C}).$$

3. Connection between integrals and the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of H^d

In this section, we concentrate on braided tensor categories consisting of some braided vector spaces. We obtain $H^{d\mathrm{rat}} \cong \int^l_{H^d}\otimes H$ for an infinite braided Hopf algebra H and the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of H^d.

Throughout this section we assume the following unless otherwise stated: H is a braided Hopf algebra in $\mathcal{C}$ with $C_{H,H} = C^{-1}_{H,H}$ and $\langle,\rangle$ is the evaluation of H, and there is a faithful quasi-dual $H^d \subseteq H^*$. We also assume that k is a field and there exists a forgetful functor $F:\ \mathcal{C}\to {}_k\mathcal{M}$, which is the category of vector spaces over k such that $F(U\otimes V) = F(U)\otimes F(V)$ and $F(I) = k$.

Now we give the concept of rational H^d-modules. For H^d-module (M,α), if there is a morphism ρ from M to $M\otimes H$ in $\mathcal{C}$ such that the condition of module-comodule compatibility

$$(\mathrm{MCOM}):\quad (H\otimes\langle,\rangle)(C\otimes M)(H^d\otimes\rho) = \alpha$$

holds, then (M,α) is called rational H^d-module.

$$M^H = \{x\in M \mid h\cdot x = \varepsilon(h)x \text{ for every } h\in H\}$$

is called the invariant of H on M. In particular, if M is a regular H-module (i.e. the module operation is m), then M^H is written as $\int^l_H$. We also denote

$$\{f\in H^* \mid g*f = g(1)f \text{ for every } g\in H^*\}$$

by $\int_{H^*}^l$. Moreover, for some subset N of H^*, we also denote

$$\{f \in N \mid g * f = g(1)f \text{ for any } g \in N\}$$

by $\int_N^l$. Every element in $\int_{H^*}^l$ is called an integral on H.

Dually, if (M, ϕ) is a left H-comodule, then the set

$$M^{\mathrm{co}H} = \{x \in M \mid \phi(x) = 1 \otimes x\}$$

is called the coinvariant of H in M.

Corollary 3.1 *Assume $\mathcal{C}$ has equalizers and there exists the maximal rational H^d-submodule $H^{d\mathrm{rat}}$ of regular module H^d. If $\rightharpoonup$ is a morphism from $H \otimes H^d$ to H^d and the constraint on $H \otimes H^{d\mathrm{rat}}$ is a morphism to $H^{d\mathrm{rat}}$ in $\mathcal{C}$, then*

$$H^{d\mathrm{rat}} \cong \int_{H^d}^l \otimes H \quad (\text{as } H\text{-Hopf modules in } \mathcal{C}) .$$

Proof For convenience, let $H^\square$ denote $H^{d\mathrm{rat}}$. Obviously, $H^\square$ is a strict quasi-dual of H. By Theorem 2.5, it suffices to show $\int_{H^\square}^l = (H^\square)^{\mathrm{co}H}$. Obviously, $\int_{H^d}^l \subseteq (H^\square)^{\mathrm{co}H}$.

Conversely, we see $m = ((H^\square)^{\mathrm{co}H} \otimes \langle,\rangle)(C \otimes H)(H^d \otimes \rho) = ((H^\square)^{\mathrm{co}H} \otimes \langle,\rangle)(C \otimes \eta) = \varepsilon \otimes \mathrm{id}_{(H^\square)^{\mathrm{co}H}}$. Thus $(H^\square)^{\mathrm{co}H} \subseteq \int_{H^d}^l$.

Consequently, $\int_{H^d}^l = (H^\square)^{\mathrm{Co}H}$. $\square$

The above corollary is a generalization of Sweedler's relation (1). In fact, we have

Corollary 3.2 *If H is an ordinary Hopf algebra, then $\int_{H^{*\mathrm{rat}}}^l = \int_{H^*}^l$.*

Proof Obviously, H^* is a faithful quasi-dual of H and $H^{*\mathrm{rat}}$ is a strict quasi-dual of H. By Corollary 3.1, we complete the proof. $\square$

4. Existence and uniqueness of integrals for Yetter-Drinfeld module categories

In this section we give the existence and uniqueness of integrals for braided Hopf algebras in the Yetter-Drinfeld module category $({}_B^B\mathcal{YD}, C)$. Throughout this section, H is a braided Hopf algebra in $({}_B^B\mathcal{YD}, C)$ with finite-dimensional Hopf algebra B. Let b_B denote the coevaluation of B and $\tau : U \otimes V \to V \otimes U$ denote the flip $\tau(x \otimes y) = y \otimes x$. If (M, α) is a left B-module, we can define a left B-module structure α_{M^*} on $M^* = \mathrm{Hom}_k(M, k)$ such that $(b \cdot x^*)(x) = x^*(S(b) \cdot x)$ for any $b \in B, x \in M, x^* \in M^*$. If (M, ϕ) is a left B-comodule, we can also define a left B-comodule structure ϕ_{M^*} on M^* such that $(B \otimes \langle,\rangle)(\phi_{M^*} \otimes M) = (S^{-1} \otimes \langle,\rangle)(\tau \otimes M)(M^* \otimes \phi)$. In fact, $\phi_{M^*} = (S^{-1} \otimes \hat{\alpha})(b_B \otimes M^*)$, where $\langle,\rangle(\hat{\alpha} \otimes M) = (\langle,\rangle \otimes \langle,\rangle)(B^* \otimes \tau \otimes M)(B^* \otimes M^* \otimes \phi)$.

Lemma 4.1 *(i) If $(M, \alpha, \phi) \in ({}_B^B\mathcal{YD}, C)$, then $(M^*, \alpha_{M^*}, \phi_{M^*}) \in ({}_B^B\mathcal{YD}, C)$ and the evaluation $\langle,\rangle$ is a morphism in $({}_B^B\mathcal{YD}, C)$.*

(ii) *If (H, α, ϕ) is a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ and the antipode S of B satisfies $S = S^{-1}$, then $\rightharpoonup$ is a morphism from $H \otimes H^*$ to H^* in $({}^B_B\mathcal{YD}, C)$.*

(iii) *Let f be a k-linear map from U to V and g k-linear from V to W with U, V, W in ${}^B_B\mathcal{YD}$. If f and gf are two morphisms in ${}^B_B\mathcal{YD}$ with $\mathrm{Im}(f) = V$, then g is a morphism ${}^B_B\mathcal{YD}$.*

(iv) *Let M be an H^*-module in ${}^B_B\mathcal{YD}$, then M has the maximal H^*-submodule M^{rat} in ${}^B_B\mathcal{YD}$.*

Proof (i) It is clear that M^* is a B-module and B-comodule. For any $b \in B, h^* \in M^*, h \in M$, $\sum (b \cdot h^*)_{(-1)} \langle (b \cdot h^*)_{(0)}, h\rangle = \sum S^{-1}(h_{(-1)})\langle h^*, S(b) \cdot h_{(0)}\rangle$. On the other hand,

$$\begin{aligned}\sum b_1 h^*_{(-1)} S(b_3)\langle b_2 \cdot h^*_{(0)}, h\rangle &= \sum b_1 h^*_{(-1)} S(b_3)\langle h^*_{(0)}, S(b_2) \cdot h\rangle \\ &= \sum b_1 S^{-1}((S(b_2) \cdot h)_{(-1)}) S(b_3) \langle h^*, (S(b_2) \cdot h)_{(0)}\rangle \\ &= \sum S^{-1}(h_{(-1)}) b_2 S(b_3) \langle h^*, S(b_1) \cdot h_{(0)}\rangle \\ &= \sum S^{-1}(h_{(-1)}) \langle h^*, S(b) \cdot h_{(0)}\rangle.\end{aligned}$$

Thus M^* is a Yetter-Drinfeld B-module.

Obviously, $\langle , \rangle$ is a B-module homomorphism. In order to show that $\langle , \rangle$ is a B-comodule homomorphism, it is enough to prove that $\sum h^*_{(-1)} h_{(-1)} \langle h^*_{(0)}, h_{(0)}\rangle = 1_B \langle h^*, h\rangle$ for any $h^* \in M^*, h \in M$. Indeed, the left side $= \sum S^{-1}(h_{(-1)2}) h_{(-1)1} \langle h^*, h_{(0)}\rangle = 1_B \langle h^*, h\rangle$. This completes the proof.

(ii) For any $b \in B, h, x \in H, h^* \in H^*$, we see that

$$\langle b \cdot (h \rightharpoonup h^*), x\rangle = \langle (h \rightharpoonup h^*), S(b) \cdot x\rangle = \langle h^*, (S(b) \cdot x) h\rangle \quad \text{and}$$

$$\begin{aligned}\sum_b \langle (b_1 \cdot h) \rightharpoonup (b_2 \cdot h^*), x\rangle &= \langle h^*, S(b_2) \cdot (x(b_1 \cdot h))\rangle \\ &= \langle h^*, (S(b_2)_1 \cdot x)(S(b_2)_2 \cdot (b_1 \cdot h))\rangle \\ &= \langle h^*, (S(b_3) \cdot x)((S(b_2) b_1) \cdot h)\rangle \\ &= \langle h^*, (S(b) \cdot x) h\rangle.\end{aligned}$$

This show that $\rightharpoonup$ is a B-module homomorphism. Similarly, we can show that it is a B-comodule homomorphism.

(iii) For any $b \in B, u \in U$, since $g(b \cdot f(u)) = gf(b \cdot u) = b \cdot (gf(u))$, we have that g is a B-module homomorphism. Similarly, g is a B-comodule homomorphism.

(iv) It can be shown by usual proof (see [5, Theorem 2.2.6 and Corollary 2.1.19]) that every H^*-submodule and quotient H^*-module of rational H^*-module are rational. The direct sum of rational H^*-modules is a rational. Consequently, The maximal rational H^*-module M^{rat} is the sum of all rational H^*-modules of M. □

Every B-module category $({}_B\mathcal{M}, C^R)$ determined by quasitriangulr Hopf algebra (B, R) is a full subcategory of Yetter-Drinfeld module category $({}^B_B\mathcal{YD}, C)$. Indeed, for any B-module (V, α), define $\phi(v) = \sum R_i^{(2)} \otimes R_i^{(1)} \cdot v$ for any $v \in V$, where $R = \sum_i R_i^{(1)} \otimes R_i^{(2)}$. It is easy to check that (V, α, ϕ) is a Yetter-Drinfeld B-module. Similarly, every B-comodule category

$({}^B\mathcal{M}, C^r)$ determined by coquasitriangulr Hopf algebra (B, r) is a full subcategory of Yetter-Drinfeld module category $({}^B_B\mathcal{YD}, C)$.

Example 4.2 (Existence of integrals) Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ and the antipode S of B satisfy $S = S^{-1}$. Then

$$H^{*\mathrm{rat}} \cong \int_{H^*}^{l} \otimes H \quad (\text{as } H\text{-Hopf modules in } ({}^B_B\mathcal{YD}, C).)$$

Example 4.3 (Existence of integrals) Let H be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$. If λ is a non-zero integral of $H\#B$ with $\lambda(a \otimes b) \neq 0$ for some $a \in H, b \in B$, then $\lambda(\mathrm{id} \otimes b)$ is a non-zero integral of H, where $\lambda(\mathrm{id} \otimes b)$ denotes the k-linear map from H to k by sending h to $\lambda(h \otimes b)$ for any $h \in H$.

Proof It follows from [13, Theorem 9.4.12] and [14, p11] that the bosonization $H\#B$ of braided Hopf H is a Hopf algebra. For any $f \in H^*$ and any $x \in H$, we see

$$\begin{aligned}(f * \lambda(id \otimes b))(x) &= \sum f(x_1)\lambda(x_2 \otimes b) = ((f \otimes \varepsilon_B) * \lambda)(x \otimes b) \\ &= f(1)\lambda(x \otimes b).\end{aligned}$$

Thus $\lambda(\mathrm{id} \otimes b)$ is a non-zero integral of H. □

Remark In Example 4.3 it is possible that B is infinite-dimensional.

Example 4.4 (Existence and uniqueness of integrals)[13] If H is an ordinary coquasitriangular Hopf algebra with a non-zero integral λ, then the braided group analogue $\underline{H}$ of H has a non-zero integral λ in braided tensor category $({}^H\mathcal{M}, C^r)$. Conversely, if $\underline{H}$ has a non-zero integral, then so does H. Indeed, since the comultiplication operations of H and $\underline{H}$ are the same, we have that the multiplications of H^* and $\underline{H}^*$ are the same, so the integrals of H and $\underline{H}$ are the same. □

Example 4.5 (The uniqueness of integrals) Let (H, α, ϕ) be a braided Hopf algebra in $({}^B_B\mathcal{YD}, C)$ and B is a finite-dimensional Hopf algebra. If ϕ is trivial, then dim $\int_{H^*}^{l} = 0$ or 1.

Proof Assume that H has two linearly independent non-zero integrals u^* and w^*. Let v^* is a non-zero integral of B. By [5, Lemma 1.3.2], $(H \otimes B)^* = H^* \otimes B^*$ as vector spaces. Since the B-comodule operation of H is trivial, we have that $u^* \otimes v^*$ and $w^* \otimes v^*$ are two linear independent integrals of $H\#B$. This contradicts to the fact dim $\int_{(H\#B)^*}^{l} = 0$ or 1 (see, [5, Theorem 5.4.2]).□

5. Maschke's theorem for braided Hopf algebras

In this section we give the relation between the integrals and semisimplicity of braided Hopf algebras. Although authors in [18] gave the Maschke's theorem for rigid braided Hopf algebras , it is not known if every semisimple braided Hopf algebra is rigid or finite. Thus our research of the Maschke's theorem for infinite braided Hopf algebras is useful.

Throughout this section we assume that there exists a forgetful functor $F: \mathcal{C} \to {}_k\mathcal{M}$, such that $F(U \otimes V) = F(U) \otimes F(V)$ and $F(I) = k$, where k is a field.

Theorem 5.1 (The Maschke's theorem) *If H is a finite dimensional braided Hopf algebra living in a braided tensor category $\mathcal{C}$, then H is semisimple as ordinary algebra over field k iff $\varepsilon(\int_H^l) \neq 0$.*

Proof If H is semisimple then there is a left ideal I such that

$$H = I \oplus \ker\varepsilon.$$

For any $y \in I, h \in H$, we see that

$$\begin{aligned} hy &= ((h - \varepsilon(h)1_H) + \varepsilon(h)1_H)y = (h - \varepsilon(h)1_H)y + \varepsilon(h)y \\ &= \varepsilon(h)y \ \text{ since } (h - \varepsilon(h)1_H)y \in (\ker\varepsilon)I = 0. \end{aligned}$$

Thus $y \in \int_H^l$, and so $I \subseteq \int_H^l$, which implies $\varepsilon(\int_H^l) \neq 0$.

Conversely, if $\varepsilon(\int_H^l) \neq 0$, let $z \in \int_H^l$ with $\varepsilon(z) = 1$.

Say M is a left H-module and N is an H-submodule of M. Assume that ξ is a k-linear projection from M to N. We define

$$\mu(m) = \sum z_1 \cdot \xi(S(z_2) \cdot m)$$

for every $m \in M$. It is sufficient to show that μ is an H-module projection from M to N. Obviously, μ is a k-linear projection. Now we only need to show that it is an H-module map. We see that

$$\begin{aligned} \alpha(H \otimes \mu) &= \alpha(H \otimes \alpha)(H \otimes \mathrm{id} \otimes \xi)H \otimes \mathrm{id} \otimes \alpha)(H \otimes \mathrm{id} \otimes \mathrm{id} \otimes m \otimes m) \\ &\quad (H \otimes \mathrm{id} \otimes S \otimes S \otimes H \otimes M)(H \otimes \Delta(z) \otimes \Delta)(\Delta \otimes M) \\ &= \alpha(H \otimes \xi)(H \otimes \alpha)(H \otimes m \otimes M)(H \otimes S \otimes H \otimes M) \\ &\quad (H \otimes m \otimes H \otimes M)(m \otimes C \otimes H \otimes M)(H \otimes \Delta(z) \otimes \Delta \otimes M)(\Delta \otimes M) \\ &= \alpha(H \otimes \xi)(H \otimes \alpha)(H \otimes m \otimes M)(H \otimes S \otimes H \otimes M)(m \otimes m \otimes H \otimes M) \\ &\quad (H \otimes C \otimes id \otimes H \otimes M)(\Delta \otimes \Delta(z) \otimes H \otimes M)(\Delta \otimes M) \\ &= \alpha(H \otimes \xi)(H \otimes \alpha)(H \otimes m \otimes M)(H \otimes S \otimes H \otimes M) \\ &\quad (\Delta \otimes H \otimes M)(m \otimes H \otimes M)(H \otimes C \otimes M)(\Delta \otimes z \otimes M) \\ &= \alpha(\mathrm{id} \otimes \xi)(\mathrm{id} \otimes \alpha)(\mathrm{id} \otimes m \otimes M)(\mathrm{id} \otimes S \otimes H \otimes M)(\Delta(z) \otimes H \otimes M) \\ &= \mu \circ \alpha. \end{aligned}$$

Thus μ is an H-module morphism. □

Remark Theorem 5.1 need not $C_{H,H} = C_{H,H}^{-1}$.

It is well-known that an ordinary algebra H over a field k is called semisimple if every H-submodule N of every H-module M is a direct summand, i.e. if there is an H-submodule L such that $M = N \oplus L$. Similarly, we have the following definition. Algebra H in $\mathcal{C}$ is called semisimple with respect to $\mathcal{C}$, if every H-submodule N in $\mathcal{C}$ of every H-module M in $\mathcal{C}$ is a direct summand(i.e. there is an H-submodule L in $\mathcal{C}$ such that $M = N \oplus L$).

Theorem 5.2 *Let H be a braided Hopf algebra in $\mathcal{C}$. If H is semisimple with respect to $\mathcal{C}$ and $\ker\varepsilon \in \mathcal{C}$, then $\varepsilon(\int_H^l) \neq 0$.*

Proof It is similar to the proof of Theorem 5.1. □

Example 5.3[13,p510] Let $H = \mathbf{C}[x]$ denote the braided line algebra. It is just the usual algebra

$\mathbf{C}[x]$ of polynomials in x over complex field $\mathbf{C}$, but we regard it as a q-statistical Hopf algebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x, \quad |x^n| = n$$

and

$$C^r(x^n, x^m) = q^{nm}(x^m \otimes x^n).$$

In fact, H is a braided Hopf algebra in $({}^{\mathbf{CZ}}\mathcal{M}, C^r)$ with coquasitriangular $r(m,n) = q^{mn}$. Here $|x^n|$ denotes the degree of x^n. If $y = \sum_0^n a_i x^i \in \int_H^l$, then $a_i = 0$ for $i = 0, 1, 2, \cdots, n$ since $xy = \varepsilon(x)y = 0$. Thus $\int_H^l = 0$. It follows from Theorem 5.1 that H is not semisimple. □

Acknowledgement The work is supported by the council of education of Hunan.

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辫子 Hopf 代数的积分

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摘要: 本文引进了无限维辫子 Hopf 代数 H 的忠实拟对偶 H^d 和严格拟对偶 $H^{d'}$. 证明了每个严格拟对偶 $H^{d'}$ 是一个 H-Hopf 模. 发现了 H^d 的极大有理 H^d- 子模 $H^{d\mathrm{rat}}$ 与积分的关系，即： $H^{d\mathrm{rat}} \cong \int_{H^d}^l \otimes H$. 给出了在 Yetter-Drinfeld 范畴 $({}_B^B\mathcal{YD}, C)$ 中的辫子 Hopf 代数的积分的存在性和唯一性.

关键词: 积分；辫子 Hopf 代数.