Article ID: 1000-341X(2006)04-0685-09 **Document code**: A

Some Properties on Baer PP and PS Rings

ZHANG Mian-mian
(Dept. of Math., Zhejiang University, Hangzhou 310027, China)
(E-mail: zhmm126@yahoo.com.cn)

Abstract: This paper gives some results on Strong-Armendariz rings and the Ore-extensions $R[x,x^{-1};\alpha]$ of Bare, PP and PS rings. And the main two results are: (1) R is a Bear (PP) ring if and only if $R[[x]]$ is a Baer (PP) ring; (2) If R is an α-rigid ring, then R is a Baer (PP, PS) ring if and only if $R[x,x^{-1};\alpha]$ is a Baer (PP, PS) ring.

Key words: Baer ring; PP ring; PS ring; Strong-Armendariz ring; Ore extension.
MSC(2000): 13B02
CLC number: O153

1. Introduction

Through this paper, all rings are associative with identity. This paper is composed of three parts. The first part concerns the relationship between Strong-Armendariz rings and reduced rings, being motivated by [1,2]. Armendariz rings which was initiated by Armendariz[1] and Rege and Chhawchharia[3] is related to polynomial rings, while Strong-Armendariz rings are related to formal polynomial rings. A ring R is called an Armendariz ring if whenever $f(x)=\sum_{i=0}^{m}a_ix^i, g(x)=\sum_{j=0}^{n}b_jx^j\in R[x]$ satisfy $f(x)g(x)=0$, we have $a_ib_j=0$ for each i,j. A ring R is called a Strong-Armendariz ring if whenever $f(x)=\sum_{i=0}^{\infty}a_ix^i, g(x)=\sum_{j=0}^{\infty}b_jx^j\in R[x]$ satisfy $f(x)g(x)=0$, we have $a_ib_j=0$ for each $i,j\geq 0$. A ring is called reduced if it has nonzero nilpotent elements. We will also show in this paper that the properties of Baer, PP and PS of R are closed under formal polynomial extension when R is a Strong-Armendariz ring. By Kaplansky[4], a ring R is called Baer if the right annihilator of every nonempty subset of R is generated by an idempotent. PP rings are closely related to these rings. A ring is called a right PP ring if each principle right ideal of R is projective, or equivalently, if the right annihilator of each element of R is generated by an idempotent. A ring R is called a PP ring if it is both a right and a left PP ring. Baer rings are clearly right PP rings. A ring R is called a left PS ring if $\mathrm{Soc}(_RR)$ is projective.

We denote the right annihilator over a ring R by $r_R(-)$ and the left annihilator $l_R(-)$. The second part of this paper concerns the generalization of McCoy's theorem. McCoy[5] proved that if R is a commutative ring, then whenever $g(x)$ is a zero-divisor in $R[x]$ there exists a nonzero element $c\in R$ such that $cg(x)=0$. Yasuyuki Hirano[6] generalized this result as follows: Let

Received date: 2004-11-01
Foundation item: the Program for New Century Excellent Talents in University (04-0522), and the National Natural Science Foundation of China (10571153)

$f(x)$ be an element of the polynomial ring $R[x]$ over a (not necessarily commutative) ring R. If $r_{R[x]}(f(x)R[x]) \neq 0$, then $\Psi(r_{R[x]}(f(x)R[x])) = r_{R[x]}(f(x)R[x]) \cap R \neq 0$. We will show in this paper that it is still right when $R[x, x^{-1}]$ instead of $R[x]$. The definition is stated later. The last part concerns the Skew Laurent polynomial $R[x, x^{-1}; \alpha]$. Recall the Skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$ with a ring automorphism $\alpha : R \to R$ and relation: $x^{-1}a = a^{-1}x^{-1}; xa = \alpha(a)x; xx^{-1} = u; x^{-1}x = \alpha^{-1}(u)$, u is a central entire, $a, u \in R$. When $\alpha = 1$, we write $R[x, x^{-1}]$ instead of $R[x, x^{-1}; \alpha]$. we will show in this paper that the properties of Baer, PP, PS, weakly PP and PS are closed under Ore extension. A ring R is called weakly PP if every principle left ideal ${}_eRer$ is projective for each $r \in R$ and each primitive idempotent $e \in R$. We give an example in this paper about $R[x, x^{-1}; \alpha]$ which is quasi-Baer, but R is not quasi-Baer. A ring is called quasi-Baer if the left annihilator of every ideal is generated, as a left ideal, by an idempotent.

2. Strong-Armendariz ring

Lemma 2.1 *If R is a reduced ring, then R is a Strong-Armendariz ring.*

Proof Let $f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots$, $f(x) = b_0 + b_1x + \cdots + b_nx^n + \cdots \in R[[x]]$ with $0 = f(x)g(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + \cdots + (a_nb_0 + a_{n-1}b_1 + a_0b_n)x^n + \cdots$. So we have the following system of equations:

(0) $a_0b_0 = 0$;

(1) $a_1b_0 + a_0b_1 = 0$;

$\cdots$;

(n) $a_nb_0 + a_{n-1}b_1 + \cdots + a_0b_n = 0$;

$\cdots$;

Multiply Eq.(1) on the right side by a_1b_0, we get $a_1b_0a_1b_0 + a_0b_1a_1b_0 = 0$. But $a_0b_1a_1b_0 = 0$ since R is reduced and $a_0b_0 = 0$. Hence $(a_1b_0)^2 = a_1b_0a_1b_0 = 0$, that is $a_1b_0 = 0$. So Eq.(1) becomes $a_0b_1 = 0$. Now assume that k is a positive integer such that $a_ib_j = 0$ for all $i + j \leq k$. Multiply Eq.$(k + 1)$ on the right side by $a_{k+1}b_0$, we get $a_{k+1}b_0 = 0$. Eq.$(k + 1)$ becomes $a_kb_1 + \cdots + a_0b_{k+1} = 0$. Continue the method we get $a_ib_j = 0$, for all $i + j = k + 1$. Therefore $a_ib_j = 0$, for all $i, j \geq 0$. Then R is Strong-Armendariz.

A ring is called abelian if every idempotent of it is central.

Lemma 2.2 *If R is a Strong-Armendariz ring, then R is an abelian ring.*

Proof Let $f(x) = (ere - er) + ex, g(x) = (ere - er) + (e - 1)x \in R[[x]]$, for any $e, r \in R, e^2 = e$. $ere - er = e(ere - er) = 0$, since R is Strong-Armendariz and $f(x)g(x) = 0$. Let $f(x) = (ere - re) + (e - 1)x, g(x) = (ere - re) + (e - 1)x \in R[[x]]$, for any $e, r \in R, e^2 = e$. We get $ere - re = (e - 1)(ere - re) = 0$ by the same reason. Hence $er = re$, for any $e, r \in R, e^2 = e$. Then R is abelian.

Lemma 2.3[1] *Suppose that a ring R is abelian, then we have the following:*

(1) Every idempotent of $R[x]$ is in R and $R[x]$ is abelian.

(2) Every idempotent of $R[[x]]$ is in R and $R[[x]]$ is abelian.

Lemma 2.4 *If R is a Strong-Armendariz ring, then for any idempotent $e \in R, eRe$ is a Strong-Armendariz ring.*

Proof Let $f(x)=\sum_{i=0}^{\infty} a_i x^i, g(x)=\sum_{j=0}^{\infty} b_j x^j \in eRe[[x]]$ be polynomials satisfy $f(x)g(x)=0$. Obviously $eRe \subseteq R$, then $f(x), g(x) \in R[[x]]$. Since R is Strong-Armendariz, then $a_i b_j=0$ for any $i, j \geq 0$. Then eRe is Strong-Armendariz.

One may suspect that if eRe is a Strong-Armendariz ring for any nonidentity idempotent e of R, then R is a Strong-Armendariz ring. However, it is not true in general by the following example.

Example 2.1 Let Z_2 be the ring of integers modulo 2 and consider the ring $R=\left(\begin{smallmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{smallmatrix}\right)$. Then by Example 1 in [1], R is not Armendariz, so is not Strong-Armendariz. Notice that the only nontrivial nonidentity idempotents of R are

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right),\left(\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right),$$

and that $eRe \cong Z_2$ is a Strong-Armendariz ring for any nontrivial nonidentity idempotent e in R.

Lemma 2.5 *Let R be a Strong-Armendariz ring. Then R is a Baer ring if and only if $R[[x]]$ is a Baer ring.*

Proof Assume that R is Baer. Let A be a nonempty subset of $R[[x]]$, and R^* be the set of all coefficients of elements of A. Then R^* is a nonempty subset of R, and so $r_R(A^*)=eR$ for some idempotent $e \in R$. Since $e \in r_{R[[x]]}(A)$, we get $eR[[x]] \subseteq r_{R[[x]]}(A)$. Now, let $g=b_0+b_1 x+\cdots+b_n x^n+\cdots \in r_{R[[x]]}(A)$. Then $Ag=0$ and hence $fg=0$ for any $f \in A$. Thus $b_0, b_1, \cdots, b_n, \cdots \in r_R(A^*)=eR$ since R is a Strong-Armendariz ring. Hence there exists $c_0, c_1, \cdots, c_n, \cdots \in R$, such that $g=ec_0+ec_1+\cdots+ec_n+\cdots=e(c_0+c_1+\cdots+c_n+\cdots) \in eR[[x]]$. Therefore, $R[[x]]$ is Baer.

Conversely, assume that $R[[x]]$ is a Baer ring. Let B be a nonempty subset of R. Then $r_{R[[x]]}(B)=eR[[x]]$ for some idempotent $e \in R$ by Lemma 2.3. Hence $r_R(B)=eR$ and R is a Baer ring.

Theorem 2.6 *Let R be a Strong-Armendariz ring. Then R is a PP ring if and only if $R[[x]]$ is a PP ring.*

Proof Assume that R is a PP ring. Let $p=a_0+a_1+\cdots+a_n x^n+\cdots \in R[[x]]$. There exists $e_i^2=e_i \in R$ such that $r_R(a_i)=e_i R$, for $i=0,1, \cdots, n, \cdots$. Let $e=e_0 e_1 \cdots e_n \cdots$. Then by Lemma 2.2, $e^2=e \in R$ and $eR=\bigcap_{i=0}^{\infty} r_R(a_i)$. So $pe=a_0 e+a_1 ex+\cdots+a_n ex^n+\cdots=0$. Hence $eR[[x]] \subseteq r_{R[[x]]}(p)$. Let $q=b_0+b_1+\cdots+b_n x^n+\cdots \in r_{R[[x]]}(p)$. Since $pq=0$ and R is a Strong-Armendariz, $a_i b_j=0$ for all $i, j \geq 0$. Then $b_j \in eR$ for all $j=0,1, \cdots, n, \cdots$. Hence $q \in eR[[x]]$. Consequently, $eR[[x]]=r_{R[[x]]}(p)$ and $R[[x]]$ is a PP ring.

Conversely, assume that $R[[x]]$ is a PP ring. Let $a \in R$. By Lemma 2.3, there exists an idempotent $e \in R$ such that $r_{R[[x]]}(a) = eR[[x]]$. Hence, $r_R(a) = r_{R[[x]]}(a) \cap R = eR$ and R is a PP ring.

Theorem 2.7 *Let R be an Armendariz ring. Then R is a PS ring if and only if $R[x]$ is a PS ring.*

Proof By the same method in the proof of Theorem 2.6.

Theorem 2.8 *Let R be a Strong-Armendariz ring. Then R is a PS ring if and only if $R[[x]]$ is a PS ring.*

Proof If R is a PS ring, then $R[[x]]$ is a PS ring according to the Theorem 3.1 in [7].

Conversely, if L is a maximal ideal of R, then $I = L[[x]]$ is a maximal ideal of $R[[x]]$. According to the fact that $R[[x]]$ is a PS ring, we have $r_{R[[X]]}(I) = eR[[x]], e^2 = e \in R$, because R is a Strong-Armendariz ring and Lemma 2.3. So $r_R(L) \supseteq eR$. Assume there exists an element $0 \neq a \in r_R(L) - eR, a \neq 0$. For any element $g = b_m x^m + b_{m+1}x^{m+1} + \cdots + b_n x^n \in I, b_m \neq 0, ga = 0$. That is $a \in r_{R[[x]]}(I) = eR[[x]]$, a contradiction. Thus $r_R(L) = eR$ as required.

Theorem 2.9 *Let R be a reduced ring. Then R is a PS ring if and only if $R[x]$ or $R[[x]]$ is a PS ring.*

Proof We prove the result for $R[[x]]$ only; The proof for $R[x]$ is similar.

The "if" part has been proved by [7]. Let us see the "only if" part. It is clear that $R[[x]] = R[[x;\alpha]]$, where $\alpha = 1$. By hypothesis R is a reduced ring, then if $r\alpha(r) = r^2 = 0$, we have $r = 0$. Thus $R[[x]]$ is a 1-rigid ring. Let L be the maximal ideal of R, then $I = L[[x]]$ is a maximal ideal of $R[[x]]$. So $r(I) = eR[[x]], e^2 = e \in R$ according to $R[[x]]$ is a 1-rigid ring. Hence $r(L) \supseteq eR$. If there exists an element $0 \neq a \in (r_R(L) - eR)$. For any element $g = b_m x^m + b_{m+1}x^{m+1} + \cdots \in I, b_m \neq 0, b_{m+1}, \cdots \in L, ga = 0$. So $a \in eR[[x]] \bigcap R = eR$, a contradiction. Thus $r(L) = eR$ as required.

3. A generalization of McCoy's theorem

First define the degree of $f(x) = \sum_{i=m}^{n} a_i x^i \in R[x, x^{-1}]$ in this way that $\deg(f(x)) = |n|$ if $n = m$; $\deg(f(x)) = n - m$, if $n \neq m$.

We begin with the following lemma.

Lemma 3.1 *Let $f(x)$ and $g(x)$ be two elements of $R[x, x^{-1}]$. Then $f(x)Rg(x) = 0$ if and only if $f(x)R[x, x^{-1}]g(x) = 0$.*

Proof Assume that $f(x)Rg(x) = 0$ and take an arbitrary element $\sum_{k=p}^{q} c_k x^k$ of $R[x, x^{-1}]$. Then $f(x)(\sum_{k=p}^{q} c_k x^k)g(x) = \sum_{k=p}^{q} f(x)c_k g(x)x^k = 0$. This implies $f(x)R[x, x^{-1}]g(x) = 0$. The "only if part" is clear.

Theorem 3.2 *Let $f(x)$ be an element of $R[x,x^{-1}]$. If $r_{R[x,x^{-1}]}(f(x)R[x,x^{-1}]) \neq 0$, then $r_{R[x,x^{-1}]}(f(x)R[x,x^{-1}]) \cap R \neq 0$.*

Proof We freely use Lemma 3.1 without mention it. Let $f(x) = \sum_{i=m}^{n} a_i x^i \in R[x,x^{-1}]$. If $\deg(f(x)) = 0$ or $f = 0$, the assertion is clear. So let $\deg(f(x)) = n - m > 0$. Assume contrary, let $0 \neq g(x) = \sum_{j=s}^{t} x^j \in R[x,x^{-1}] \in r_{R[x,x^{-1}]}(f(x)R[x,x^{-1}])$ with minimal degree. Since

$$(\sum_{i=m}^{n} a_i x^i)R[x,x^{-1}](\sum_{j=s}^{t} b_j x^j) = 0,$$

$$(\sum_{i=m}^{n} a_i x^i)R(\sum_{j=s}^{t} b_j x^j) = 0,$$

then $a_n R b_t = 0$. Hence

$$a_n R[x,x^{-1}]g(x) = a_n R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s)$$

and

$$(f(x)R[x,x^{-1}]a_n)R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s) = (f(x)R[x,x^{-1}]a_n)R[x,x^{-1}]g(x) = 0.$$

Since $g(x)$ is of minimal degree, we have

$$a_n R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s) = 0.$$

Therefore,

$$a_n \in l_R(R[x,x^{-1}]b_t x^t + R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s)).$$

Hence,

$$(a_{n-1}x^{n-1} + \cdots + a_m x^m)R[x,x^{-1}](b_t x^t + \cdots + b_s x^s) = 0 \quad \text{and} \quad a_{n-1}Rb_t = 0.$$

Thus we obtain

$$f(x)R[x,x^{-1}](a_{n-1}R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s)) = f(x)(R[x,x^{-1}]a_{n-1}R[x,x^{-1}])g(x) = 0.$$

Since $g(x)$ is of minimal degree, we obtain $a_{n-1}R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s) = 0$. Therefore,

$$a_n, a_{n-1} \in l_R(R[x,x^{-1}]b_t x^t + R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s)).$$

Repeating, we obtain

$$a_n, a_{n-1}, \cdots, a_m \in l_R(R[x,x^{-1}]b_t x^t + R[x,x^{-1}](b_{t-1}x^{t-1} + \cdots + b_s x^s)).$$

This implies that

$$b_s, b_{s-1}, \cdots, b_t \in r_{R[x,x^{-1}]}(f(x)R[x,x^{-1}]).$$

Contradicted.

Corollary 3.3 *Let R be a semi-commutative ring. If $f(x)$ is a zero-divisor in $R[x]$, then there exists a nonzero element $c \in R$ such that $f(x)c = 0$.*

4. Ore extension of $R[x, x^{-1}; \alpha]$

Lemma 4.1[9] *Let R be an α-rigid ring, α is a ring automorphism, $a, b \in R$, then we have:*

(1) If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$, for any $n \in Z$.

(2) If $a\alpha^k(b) = \alpha^k(a)b = 0$, for some $k \in Z$, then $ab = 0$.

(3) If a is central entire, then $\alpha(a)$ is still a central entire in R.

Lemma 4.2 *Ore extension $R[x, x^{-1}; \alpha]$ is reduced if and only if R is an α-rigid ring. In this case, $\alpha(e) = e$, for some $e^2 = e \in R$.*

Proof Suppose that R is α-rigid. Assume to the contrary that $R[x, x^{-1}; \alpha]$ is not reduced. Then there exists $0 \neq f \in R[x, x^{-1}; \alpha]$ such that $f^2 = 0$. Since R is reduced, $f \notin R$. Thus we put $f = \sum_{i=n}^{m} a_i x^i$, where $a_i \in R$, for $n \leq i \leq m$ and $a_n \neq 0, a_m \neq 0$. Since $f^2 = 0$, we have $a_m \alpha^m(a_m) = 0, a_n \alpha^n(a_n) = 0$. By Lemma 4.1, $a_m^2 = 0, a_n^2 = 0$ and so $a_m = 0, a_n = 0$, which is a contradiction. Therefore, $R[x, x^{-1}; \alpha]$ is reduced.

Conversely, suppose that $R[x, x^{-1}; \alpha]$ is reduced. Clearly, R is reduced as a subring. If $a\alpha(a) = 0$ and $\alpha(a)xa = 0$. Thus $0 = \alpha(a)\alpha(a)x = (\alpha(a))^2 x$, and so $\alpha(a) = 0$. Since α is an automorphism, we have $a = 0$. Therefore, R is α-rigid.

Next, let e be an idempotent in R. Then e is central, and so $ex = xe = \alpha ex$. This implies that $\alpha e = e$.

Lemma 4.3[9] *Let R be an α-rigid ring. If $p = \sum_{i=n}^{m} a_i x^i, q = \sum_{j=s}^{t} b_j x^j \in R[x, x^{-1}; \alpha], m, n, s, t$ are integers, then $pq = 0$ if $a_i b_j = 0$, for any $n \leq i \leq m, s \leq j \leq t$.*

Lemma 4.4 *Let R be an α-rigid ring. If $e^2 = e \in R[x, x^{-1}; \alpha], e = e_n x^n + \cdots + e_0 + \cdots + e_m x^m$, then $e = e_0$.*

Proof Since $1 - e = (1 - e_0) - \sum_{i=n}^{-1} - \sum_{j=1}^{m} e_j x_j$, we get $e_0(1 - e_0) = 0$ and $e_i^2 = 0$ for all $n \leq i \leq -1, 1 \leq i \leq m$ by Lemma 4.3. Thus $e_i = 0$ for all $n \leq i \leq m, i \neq 0$, and so $e = e_0 = e_0^2 \in R$.

Birkermeier proved if R is a quasi-Baer ring, then $R[x, x^{-1}; \alpha]$ is a quasi-Baer ring. However the following example shows that there exists $R[x, x^{-1}; \alpha]$ which is quasi-Baer, but R is not quasi-Baer.

Example 4.1 Let Z be the ring of integers and consider the ring $Z \oplus Z$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in Z \oplus Z \mid a \equiv b (\mathrm{mod} 2)\}$ of $Z \oplus Z$ is a commutative reduced ring. Note that only idempotents of R are $(0, 0)$ and $(1, 1)$. In fact, if $(a, b)^2 = (a, b)$, then $(a^2, b^2) = (a, b)$ and so $a^2 = a, b^2 = b$. Since $a \equiv b (\mathrm{mod} 2)$, then $(a, b) = (0, 0)$ or $(a, b) = (1, 1)$. Now we claim that R is not quasi-Baer. For $(2, 0) \in R$, we note that $r_R((2, 0)) = \{(0, 2n) \mid n \in Z\}$. So we can see that $r_R((2, 0))$ does not contain a nonzero idempotent of R. Hence R is not a quasi-Baer ring.

Now let $\alpha : R \to R$ be defined by $\alpha((a,b)) = (b,a)$. Then α is an automorphism of R. Note that R is not α-rigid. We claim that $R[x,x^{-1};\alpha]$ is quasi-Baer. Let I be a nonzero right ideal of $R[x,x^{-1};\alpha]$ and $p \in I$, put $p = (a_i,b_i)x^i + \cdots + (a_m,b_m) \neq 0$. Then for some positive integer $2k - i > |i| + |m| + |j| + |n|, j, n$ is the integer in q, it will be stated later in the following. $p(1,1)x^{2k-i} = (a_i,b_i)\prod_{h=0}^{-\min\{0,j\}} a^{-h}(u)x^{2k} + \cdots + (a_m,b_m)\prod_{h=0}^{-\min\{0,m\}} a^{-h}(u)x^{2k+m-i} \in I$ and $p(1,1)x^{2k+1-i} = (a_i,b_i)\prod_{h=0}^{-\min\{0,j\}} a^{-h}(u)x^{2k+1} + \cdots + (a_m,b_m)\prod_{h=0}^{-\min\{0,m\}} a^{-h}(u)x^{2k+m+1-i} \in I$ (where $a^{-2}(u) = \alpha^{-1}(\alpha^{-1}(u)), h = 2$). Suppose that $0 \neq q \in r_{R[x,x^{-1};\alpha]}(I)$ and put $q = (u_j,v_j)x^j + \cdots + (u_n,v_n)x^n x^n$, where $n - j$ is the smallest integer such that $(a_i,b_i) \neq 0, (a_m,b_m) \neq 0$. Then $p(1,1)x^{2k-i}q = 0$ and $p(1,1)x^{2k+1-i}q = 0$. So we have

$$(a_i,b_i)\prod_{h=0}^{-\min\{0,i\}} \alpha^{-h}(u)x^{2k}(u_j,v_j)x^j + \cdots = (a_i,b_i)(u_j,v_j)\prod_{h=0}^{-\min\{0,i\}} \alpha^{-h}(u)u^{-\min\{0,j\}}x^{2k+j} + \cdots$$

and

$$(a_i,b_i)\prod_{h=0}^{-\min\{0,i\}} \alpha^{-h}(u)x^{2k+1}(u_j,v_j)x^j + \cdots = (a_i,b_i)(u_j,v_j)\prod_{h=0}^{-\min\{0,i\}} \alpha^{-h}(u)u^{-\min\{0,j\}}x^{2k+1+j} + \cdots.$$

Hence $(a_iu_j, b_iv_j) = (0,0)$ and $(a_iv_j, b_iu_j) = (0,0)$. This implies that $a_iu_j = b_iv_j = 0$ and $a_iv_j = b_iu_j = 0$. Since $(a_i,b_j) \neq 0, a_i$ or b_i is nonzero. Then we have $(u_j,v_j) = (0,0)$, which is a contradiction. So $r_{R[x,x^{-1}]}(I) = (0,0)$ and hence $R[x,x^{-1}]$ is quasi-Baer.

Lemma 4.5 *Let R be an α-rigid ring. Then R is a PP ring if and only if $R[x,x^{-1};\alpha]$ is a PP ring.*

Proof Assume that R is a PP ring. Let $p = a_nx^n + \cdots + a_mx^m \in R[x,x^{-1};\alpha]$. There exists an idempotent $e_i \in R$ such that $r_R(a_i) = e_iR$ for $i = n, \cdots, m$. Let $e = e_n \cdots e_m$. Then $e^2 = e \in R, eR = \bigcap_{i=n}^{m} r_R(a_i)$. So by Lemma 4.2, $pe = a_n\alpha^n(e)x^n + \cdots + a_m\alpha^m(e)x^m = a_nex^n + \cdots + a_mex^m = 0$. Hence $eR[x,x^{-1};\alpha] \subseteq r_{R[x,x^{-1};\alpha]}(p)$. Let $q = b_sx^s + \cdots + b_tx^t \in r_{R[x,x^{-1};\alpha]}(p)$. Since $pq = 0, a_ib_j = 0$ for all $n \leq i \leq m, s \leq j \leq t$. Then $b_j \in eR$ for $s \leq j \leq t$, and so $q \in R[x,x^{-1};\alpha]$. Consequently, $eR[x,x^{-1};\alpha] = r_{R[x,x^{-1};\alpha]}(p)$. Thus $R[x,x^{-1};\alpha]$ is a PP ring.

Conversely, assume that $R[x,x^{-1};\alpha]$ is a PP ring. Let $a \in R$ by Lemma 4.4, there exists an idempotent $einR$ such that $r_{R[x,x^{-1};\alpha]}(a) = eR[x,x^{-1};\alpha]$. Hence $r_R(a) = eR$. Therefore, R is a PP ring.

According to [3,Lemma 1]. Let R be a reduced ring. Then the following statement are equivalent:

(1) R is a PP ring; (2) R is a p.q-Baer ring.

Then we have the following corollary:

Corollary 4.6 *Let R be an α-rigid ring. Then R is a p.q-Baer ring if and only if $R[x,x^{-1};\alpha]$ is a p.q-Baer ring.*

Theorem 4.7 *Let R be an α-rigid ring. Then R is a weakly PP ring if and only if $R[x,x^{-1};\alpha]$*

is a weakly PP ring.

Proof For every $f=\sum_{i=n}^m b_ix^i\in R[x,x^{-1};\alpha]$ and every primitive idempotent $e\in R[x,x^{-1};\alpha]$, by Lemma 4.4, $e\in R$. If $ef=0$, then $R[x,x^{-1};\alpha]ef$ is projective. Suppose that $ef\neq 0$. Then there exists an integer $n\leq k\leq m$ such that $eb_k\neq 0$. It is obvious that $R[x,x^{-1};\alpha](1-e)\subseteq l_{R[x,x^{-1};\alpha]}(ef)$. Conversely, for any $g\in l_{R[x,x^{-1};\alpha]}(ef), g=\sum_{j=s}^t a_jx^j, gef=0$. Then by Lemma 4.3, $a_jeb_i=0$. From $eb_k\neq 0$ and R is a weakly PP ring, we get $l_R(eb_k)=R(1-e)$, thus $a_j\in R(1-e)$. By Lemma 4.2, $g\in R[x,x^{-1};\alpha](1-e)$. Hence $R[x,x^{-1};\alpha](1-e)\subseteq l_{R[x,x^{-1};\alpha]}(ef)$. Hence $R[x,x^{-1};\alpha]$ is a weakly PP ring.

Conversely, if $R[x,x^{-1};\alpha]$ is a weakly PP ring, then for every $r\in R$ and every primitive idempotent $e\in R$, we have $l_{R[x,x^{-1};\alpha]}(ef)=R[x,x^{-1};\alpha]f, f^2=f\in R[x,x^{-1};\alpha]$. Hence R is a weakly PP ring.

Theorem 4.8 *Let R be an α-rigid ring. Then R is a PS ring if and only if $R[x,x^{-1};\alpha]$ is a PS ring.*

Proof If L is a maximal ideal of $R[x,x^{-1};\alpha]$ we show that $r_{R[x,x^{-1};\alpha]}(L)=eR[x,x^{-1};\alpha]$ for an idempotent $e^2=e\in R[x,x^{-1};\alpha]$. Let I denote the set of all constant coefficients of polynomials in L. Let J be the left ideal of R which is generated by I. If $J=R$, then there exists $s_1,\cdots,s_n\in I, r_1,\cdots,r_n\in R$, such that $1=r_1s_1+\cdots+r_ns_n$. Assume $h\in r_{R[x,x^{-1};\alpha]}(L), h_0\neq 0, h_0$ is the constant coefficient of h for any $f=\sum_{i=s}^t f_ix^i\in L, f_0h_0=0$, for the arbitrary of f, we get $s_ih_0=0, 1\leq i\leq n$, so $h_0=0$, a contradiction. Hence $r_{R[x,x^{-1};\alpha]}(L)=0$.

Now assume $J\neq R$ we show that J is a maximal left ideal of R. Let $r\in(R-J)$, obviously $r\in R[x,x^{-1};\alpha]$, if $r\in L$, then $r\in J$, a contradiction. So $r\notin L$. Then $R[x,x^{-1};\alpha]=L+R[x,x^{-1};\alpha]r$. Hence $1=f+gr=f_0+g_0r, g\in R[x,x^{-1};\alpha]$. If $f_0=0$, then $1\in Rr, R=J+Rr$. If $f_0\neq 0$, then $f_0\in J, R=J+Rr$. Hence J is a maximal left ideal of R.

Because R is a PS ring, then there exists an idempotent $e\in R$, such that $r_R(J)=eR$. So $Le=0$. Hence $r_{R[x,x^{-1};\alpha]}(L)\supseteq eR[x,x^{-1};\alpha]$. Conversely, let $g=\sum_{j=k}^m b_jx^j\in r_{R[x,x^{-1};\alpha]}(L), b_k\neq 0$, for any $f=\sum_{i=t}^n a_ix^i\in L, a_t\neq 0, fg=0$. By Lemma 4.3, $a_ib_j=0, t\leq i\leq n, k\leq j\leq m$. Particularly, $a_0b_j=0$, where a_0 is the constant coefficient of f. For the arbitrary of $f, b_j\in r_R(J)=eR$. So $g\in eR[x,x^{-1};\alpha]$. Hence $r_{R[x,x^{-1};\alpha]}(L)\subseteq eR[x,x^{-1};\alpha]$. Thus $r_{R[x,x^{-1};\alpha]}(L)=eR[x,x^{-1};\alpha]$. $R[x,x^{-1};\alpha]$ is a PS ring.

Conversely, if L is a maximal left ideal of R, then $I=L[x,x^{-1};\alpha]$ is a maximal left ideal of $R[x,x^{-1};\alpha]$. By hypothesis and Lemma 4.4, $r(I)=eR[x,x^{-1};\alpha], e^2=e\in R$. Hence $r(L)\supseteq eR$. Assume there exists an element $0\neq a\in r(L)-eR$. For any $g=\sum_{i=s}^t g_it^i\in I, ga=0$. By using Lemma 4.1, then $a\in r(I)=eR[x,x^{-1};\alpha]\bigcap R=eR$, a contradiction. Hence $r(L)=eR$.

At last we prove the following two Theorems.

Recall that for a ring R with a ring endomorphism $\alpha:R\to R$ and an α-derivation $\delta:R\to R$, the Ore extension $R[x;\alpha,\delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication

$$xr=\alpha(r)x+\delta(r)$$

for all $r \in R$. If we have $\delta = 0$, we write $R[x;\alpha,0]$ is stead of $R[x;\alpha,0]$ and $R[x,\alpha]$ is called Ore extension of endomorphism type (also called a skew polynomial ring). While $R[[x;\alpha]]$ is called a skew power series ring.

Theorem 4.9 *Let R be an α-rigid ring. Then $R[x;\alpha,\delta]$ is a PS ring if and only if R is a PS ring.*

Proof The "if" part has been proved in [9]. Let us see the "only if" part. Let L be the maximal left ideal of R. Then $L[x]$ is a maximal left ideal of $R[x;\alpha,\delta]$ is a PS ring. So $r_{R[x;\alpha,\delta]}(L[x]) = eR[x;\alpha,\delta], e^2 = e \in R$. Hence $r_R(L) \supseteq eR$. If there exists an element $0 \neq a \in r_R(L) - eR$, for any element $g = b_0 + b_1 + \cdots + b_m x^m \in L[x], ga = 0$. Hence $a \in r_{R[x;\alpha,\delta]}(L[x]) \cap R = eR$, a contradiction. Thus $r_R(L) = eR$.

Theorem 4.10 *Let R be an R be an α-rigid ring. Then $R[[x;\alpha]]$ is a PS ring if and only if R is a PS ring.*

Proof The proof of this theorem is similar to the proof of Theorem 4.9.

References:

[1] ARMENDARIZ E P. *A note on extensions of Baer and P.P.-rings* [J]. J. Austral. Math. Soc., 1974, **18**: 470–473.
[2] KIM N K, LWW Y. *Armendariz rings and Reduced rings* [J]. J. Algebra, 2000, **223**: 477–488.
[3] REGE M B, ROBSON S C. *Armendariz rings* [J]. Proc. Japan Acad. Ser. A Math. Sci., 1997, **73**: 14–17.
[4] KAPLANSKY I. *Rings of Operators* [M]. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
[5] MCCOY N H. *Remarks on divisors of zero* [J]. Amer. Math. Monthly, 1942, **49**: 286–295.
[6] HIRANO Y. *On annihilator ideals of a polynomial ring over a noncommutative ring* [J]. J. Pure Algebra, 2002, **168**: 45–52.
[7] NICHOLSON W K, WATTERS J F. *Rings with projective socles* [C]. Projective socles, Proceedings of the America Mathematical society, V102,No 3, March 1988;
[8] HONG C Y, KIM N K, KWAK T K. *Ore extension of Baer and PP ring* [J]. J. Pure Algebra, 2000, **151**: 215–226.
[9] SONG Jun-quan. *Ore extensions of PP and PS rings* [J]. J. Zhejiang Univ. Sci. Ed., 2004, **31**(1): 10–13. (in Chinese)

关于 Baer PP 和 PS 环的一些性质

张棉棉
(浙江大学数学系，浙江 杭州 310027)

摘要: 在此文中，我们对 Strong-Armendariz 环和 Baer PP 及 PS 环 Ore- 扩张 $R[x,x^{-1};\alpha]$ 的一些性质进行了讨论研究，并得到了一些结果．主要证明了 R 是 Baer (PP) 环当且仅当 $R[[x]]$ 是 Baer(PP) 环及 R 是 α-rigid 环时， R 是 Baer(PP,PS) 环当且仅当 $R[[x]]$ 是 Baer (PP,PS) 环.

关键词: Bare 环; PP 环; PS 环; Strong-Armendariz 环; Ore- 扩张.