Orthodox Semirings with Additive Idempotents Satisfying Permutation Identities

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Abstract: This paper deals with orthodox semirings whose additive idempotents satisfy permutation identities. A structure theorem for such semirings is established.

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1. Introduction and Preliminaries

Throughout this paper, we use the terminologies and notions given in [3]. A semiring $S$ is an algebraic structure $(S, +, \cdot)$ consisting of a non-empty set $S$ together with two binary operations $+$ and $\cdot$ such that $(S, +)$ and $(S, \cdot)$ are semigroups, connected by ring-like distributivity (that is, $x(y + z) = xy + xz, (y + z)x = yx + zx$, for all $x, y$ and $z$ in $S$). Usually, we write $(S, +, \cdot)$ simply as $S$, and for any $x, y \in S$, write $x \cdot y$ simply as $xy$.

An element $a$ of a semiring $S$ is called an idempotent if it satisfies $a + a = a \cdot a = a$. A semiring $S$ is an idempotent semiring if all of its elements are idempotents. An idempotent semiring $S$ is called a band semiring$^{[2]}$, if it satisfies the following conditions

$$a + ab + a = a, \quad a + ba + a = a \quad (1.1)$$

for any $a, b \in S$. A $T$ band semiring $S$ is a band semiring such that $(S, +)$ is a $T$ band$^{[2]}$. In [6], the authors proved that band semirings are always regular band semirings.

Let $D$ be a distributive lattice. For each $\alpha \in D$, let $S_\alpha$ be a semiring and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in D$ such that $\alpha \leq \beta$, let $\varphi_{\alpha, \beta} : S_\alpha \rightarrow S_\beta$ be a semiring homomorphism such that

1. $\varphi_{\alpha, \alpha} = 1_{S_\alpha}$;
2. $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma}$, if $\alpha \leq \beta \leq \gamma$;
3. $\varphi_{\alpha, \beta}$ is injective, if $\alpha \leq \beta$;
4. $S_\alpha \varphi_{\alpha, \gamma} S_{\beta} \varphi_{\beta, \gamma} \subseteq S_{\alpha \beta} \varphi_{\alpha \beta, \gamma}$, if $\alpha + \beta \leq \gamma$.

On $S = \cup_{\alpha \in D} S_\alpha$, $+$ and $\cdot$ are defined as follows: For $a \in S_\alpha$ and $b \in S_\beta$.

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With the above operations, $S$ is a semiring, and each $S_\alpha$ is a subsemiring of $S$. Write $S$ as $[D; S_\alpha, \varphi_{\alpha,\beta}]$, and call it a strong distributive lattice $D$ of semirings $S_\alpha$.

Let $S$ be a semiring, and $A$ a subset of $S$. Then $A$ is said to satisfy the permutation identity if

$$(\forall x_1, x_2, \ldots, x_n \in A) \ x_1 + x_2 \cdots + x_n = x_{p_1} + x_{p_2} + \cdots + x_{p_n},$$

where $(p_1p_2\cdots p_n)$ is a nontrivial permutation of $(12\cdots n)$. Yamada\cite{7} investigated the regular semigroups whose idempotents satisfy permutation identities and discussed the structure of such semigroups.

If $U$ is a subsemiring of $S$, the restriction of the relation $R$ on $S$ to $U$ will be denoted by $R_U$. Also, we denote the set of all additive idempotents (if there exist) of a semiring $S$ by $E$. We remark that $E$ is an ideal of the multiplicative reduct $(S, \cdot)$.

We first recall some results about band semirings.

**Theorem 1.1**\cite{5} A semiring $S$ is a rectangular band semiring if and only if $S$ is isomorphic to the direct product of a left zero band semiring and a right zero band semiring.

**Theorem 1.2**\cite{5} A semiring $S$ is a normal band semiring if and only if $S$ is a strong distributive lattice of rectangular band semirings.

The following result will be used in the sequel.

**Theorem 1.3**\cite{7} Let $S$ be a regular semigroup. Then the following statements are equivalent:

1. $E(S)$ satisfies a permutation identity;
2. $E(S)$ is a normal band.

## 2. Orthodox semirings

A semiring $S$ is *additively regular* if for each element $a$ in $S$ there exists an element $a'$ such that $a = a + a' + a$. If, in addition, the element $a'$ is unique, and satisfies $a' = a' + a + a'$, then $S$ is an additively inverse semiring. Usually, we denote the unique additive inverse of $a$ by $a^{-1}$. Additively inverse semirings were first studied by Karvellas\cite{4} in 1974, and he proved the following theorem:

**Theorem 2.1**\cite{4} Let $S$ be an additively inverse semiring. Then for $x, y$ in $S$,

$$(xy)^{-1} = x^{-1}y = xy^{-1}, xy = x^{-1}y^{-1}.$$

**Definition 2.2** An additively regular semiring $S$ is an orthodox semiring if $E$ is a band semiring.

**Definition 2.3** An additively regular semiring $S$ is called a d-inverse semiring if $E$ is a distributive lattice.
Suppose $S$ is an orthodox semiring. Then we define a relation $\sigma$ on $S$ as follows:

$$a \sigma b \iff V^+(a) = V^+(b),$$

where $V^+(x)$ is the set of all additive inverses of $x$.

**Proposition 2.4** If $S$ is an orthodox semiring, then $\sigma$ is a congruence on $S$ and $S/\sigma$ is a d-inverse semiring.

**Proof** Since $\sigma$ is an inverse semigroup congruence on $(S, +)$, we just need to prove $\sigma$ is compatible with multiplication. Let $a \sigma b$ and $c \in S$. Then by distributive laws, we can prove easily that $ca' \in V^+(ca) \cap V^+(cb)$, $a' \in V^+(ac) \cap V^+(bc)$ where $a'$ is an additive inverse of $a$. It follows that $V^+(ac) = V^+(bc)$ and $V^+(ca) = V^+(cb)$ which mean that $\sigma$ is a congruence on $S$. Now we prove $S/\sigma$ is a d-inverse semiring. Clearly, $S/\sigma$ is also an orthodox semiring and $E^+(S/\sigma) = \{ e\sigma | e \in E^+(S) \}$. Since

$$ef + efe + ef = ef(f + fe + f) = ef,$$

$$efe + ef + efe = efe(e + ef + e) = efe,$$

we have $efe \sigma efe$. Similarly, we can prove $fefe \sigma fefe$. So $e \sigma fefe$. Also, $(e + ef)\sigma(e + e + ef)\sigma(e + ef + e) = e$. Therefore, $E^+(S/\sigma)$ is a distributive lattice.

3. The quasi-spined product structure

First, we introduce the definition of quasi-spined product.

Let $T$ be a d-inverse semiring whose distributive lattice of additive idempotents is $D$, $L = [D; L_\alpha, \varphi_{\alpha, \beta}]$ a strong distributive lattice of left zero band semirings $L_\alpha$, and $R = [D; R_\alpha, \psi_{\alpha, \beta}]$ a strong distributive lattice of right zero band semirings $R_\alpha$. Let

$$M = \{ (e, \xi, f) \in L \times T \times R : \xi \in T, e \in L_{\xi + \xi^{-1}}, f \in R_{\xi^{-1} + \xi} \}.$$

We define addition “$+$” and multiplication “$.$” as follows:

$$(e, \xi, f) + (g, \eta, h) = (e + u, \xi + \eta, v + h),$$

$$(e, \xi, f) \cdot (g, \eta, h) = (eg, \xi \eta, fh),$$

where $u \in L_{\xi + \eta + (\xi + \eta)^{-1}}, v \in R_{(\xi + \eta)^{-1} + \xi + \eta}$. It is easy to see that the addition and the multiplication above are well defined respectively.

Using Theorem 2.1, we can easily prove the following lemma by simple calculation:

**Lemma 3.1** $(M, +, \cdot)$ is a semiring.

We call $(M, +, \cdot)$ the quasi-spined product of the left normal band semiring $L$, the right normal band semiring $R$ and the d-inverse semiring $T$ on the distributive lattice $D$ in this paper. We denote it by $QS(L, R, T; D)$.

**Lemma 3.2** Let $(e, \xi, f), (g, \eta, h) \in QS(L, R, T; D)$. Then $(e, \xi, f)$ is an idempotent if and only
if $\xi$ is an idempotent of $T$.

**Theorem 3.3** $S \cong QS(L, R, T; D)$ is an orthodox semiring whose additive idempotents satisfy a permutation and $S/\sigma \cong T$.

**Proof** Let $(e, \xi, f), (g, \eta, h) \in E^+(S)$. Then

$$(e, \xi, f) + (g, \eta, h) = (e, \xi, f) + (e, \xi, f) + (e, \xi, f) + (g, \eta, h) + (e, \xi, f)$$

$$= (e + u, \xi + \xi \eta, v + fh) + (e, \xi, f)$$

$$= (e, \xi, f + fh) + (e, \xi, f)$$

$$= (e, \xi, f).$$

Similarly, we can prove $(e, \xi, f) + (g, \eta, h)(e, \xi, f) + (e, \xi, f) = (e, \xi, f)$. Also,

$$(e, \xi, f) + (g, \eta, h) + (i, \tau, j) + (e, \xi, f) = (e + u, \xi + \eta + \tau + \xi, v + f)$$

$$= (e, \xi, f) + (i, \tau, j) + (g, \eta, h) + (e, \xi, f).$$

Therefore, $S$ is an orthodox semiring whose additive idempotents satisfy a permutation. Now, we define a mapping $\varphi : S \rightarrow T$ by $(e, \xi, f) \varphi = \xi$. Easily, we can prove $\varphi$ is a surjective semiring homomorphism and $\ker \varphi = \sigma$. That is, $S/\sigma \cong T$.

**Theorem 3.4** If $S$ is an orthodox semiring whose additive idempotents satisfy a permutation identity, then $S$ is isomorphic to $QS(L, R, T; D)$ where $D$ is a distributive lattice, $L = \cup_{\alpha \in D} L_{\alpha}$, $R = \cup_{\alpha \in D} R_{\alpha}$ are left and right normal band semiring respectively, and $T$ is a d-inverse semiring.

**Proof** Suppose that $S$ is an orthodox semiring whose additive idempotents satisfy a permutation identity. Then $E$ is a normal band semiring. So from Theorem 1.2, $E$ is a strong distributive lattice $[E/ \hat{D}_E; E_{\alpha}, \theta_{\alpha, \beta}]$ of rectangular band semirings $E_{\alpha}$. Let $E_{\alpha}$ be the direct product $L_{\alpha} \times R_{\alpha}$ of a left zero band semiring $L_{\alpha}$ and a right zero band semiring $R_{\alpha}$. Denote $\cup_{\alpha \in D} L_{\alpha}, \cup_{\alpha \in D} R_{\alpha}$ by $L$ and $R$ respectively. For any $\alpha, \beta \in D$ with $\alpha \leq \beta$, by Corollary IV 3.6 in [3], the additive semigroup homomorphism $\theta_{\alpha, \beta} : E_{\alpha} \rightarrow E_{\beta}$ determines additive semigroup homomorphisms $\varphi_{\alpha, \beta} : L_{\alpha} \rightarrow L_{\beta}$ and $\psi_{\alpha, \beta} : R_{\alpha} \rightarrow R_{\beta}$ such that

$$(l_{\alpha}, r_{\alpha}) \theta_{\alpha, \beta} = (l_{\alpha} \varphi_{\alpha, \beta}, r_{\alpha} \psi_{\alpha, \beta})$$

for all $(l_{\alpha}, r_{\alpha})$ in $E_{\alpha}$. Easily, we can prove that $\varphi_{\alpha, \beta}$ is a semiring homomorphism. Furthermore, since $\theta_{\alpha, \beta}$ satisfies (1)–(4), we have $\varphi_{\alpha, \beta}$ satisfies (1)–(4) accordingly. That is, $L = \cup_{\alpha \in D} L_{\alpha}$ is a strong distributive lattice $[E/ \hat{D}_E; L_{\alpha}, \varphi_{\alpha, \beta}]$ of left zero band semirings $L_{\alpha}$. Similarly, $R = \cup_{\alpha \in D} R_{\alpha}$ is a strong distributive lattice $[E/ \hat{D}_E; R_{\alpha}, \psi_{\alpha, \beta}]$ of right zero band semirings $R_{\alpha}$. Obviously, $E/ \hat{R}_E$ and $E/ \hat{L}_E$ are isomorphic to $L$ and $R$. Note that $\sigma_E = \hat{D}_E$. So, $E^+(S/\sigma)$ is isomorphic to $D$. 
For the sake of simplicity, we identity \( E/\mathcal{R}_E \), \( E/\mathcal{L}_E \) and \( E/\mathcal{D}_E \) with \( L, R \) and \( D \) respectively. Now, define a mapping \( \chi : S \rightarrow QS(L,R,T;D) \) as follows:

\[
a \chi = (\overline{a + a'}, \overline{a'}, \overline{a'} + a),
\]
where \( a' \) is an inverse of \( a \). By the properties of \( L \) and \( R \), \( \chi \) is well defined and is an additive semigroup isomorphism. Let \( a, b \in S \). Then,

\[
a \chi b \chi = (\overline{a + a'}, \overline{a'}, \overline{a'} + a)(\overline{b + b'}, \overline{b'}, \overline{b'} + b)
\]
\[
= ((a + a')(b + b'), \overline{ab}, (a' + a)(b' + b))
\]
\[
= (ab + ab', \overline{ab}, ab' + ab)
\]
\[
= (ab + (ab)', \overline{ab}, (ab) + ab)
\]
\[
= (ab) \chi.
\]

Consequently, we have proved that \( \chi \) is an isomorphism of \( S \) onto \( QS(L,R,T;D) \).

References: