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On Monoids over which All Strongly Flat Right S-Acts Are Regular

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Abstract: This paper investigates the characterizations of monoids over which all strongly flat right S-acts are regular. It is shown that all strongly flat right S-acts are regular if and only if S is a right PSF monoid and every left collapsible submonoid of S contains a left zero. This result gives a new answer to the problem in Kilp and Knauer[7].

Key words: strong flat S-acts; regular S-acts.
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1. Introduction

Let S be a monoid. The definitions of right S-act A and left S-act B can be found in [1]. Strong flatness and regular property of right acts over a monoid S have been the topic of many papers on homological classification of monoids during last decades (see for example [2] and the references cited there). In [3], Stenström called a right S-act A strongly flat, if the functor $A\otimes$- (from the category of left S-acts to the category of sets) preserves pullbacks and equalizers. In fact the author has shown that A is strongly flat if and only if it satisfies the following condition (P) and (E):

$$(\forall a, a' \in A)(\forall s, s' \in S)(as = a's' \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs')). \quad ((\mathrm{P}))$$

$$(\forall a \in A)(\forall s, s' \in S)(as = as' \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \wedge us = us')). \quad ((\mathrm{E}))$$

According to Knauer[4], a right S-act A is projective if and only if $A \simeq \amalg e_i S$ for $e_i^2 = e_i \in S$. As one would expect, projective $\Rightarrow$ strongly flat $\Rightarrow$ property (E). It is also known that these three concepts are distinct[5,6].

Regular right S-acts were introduced by Tran Lam Hach in [8] and were investigated by many authors. A right S-act A is called regular if for any $a \in A$, there exists a homomorphism $f : aS \to S$ such that $af(a) = a$[7,8]. It is known that a von Neumann regular monoid S is an example of regular right S-act. It was proved in [8] that A is regular if and only if all cyclic subacts of A are projective.

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A monoid S is said to be right reversible if for any $p, q \in S$, there exist $u, v \in S$ such that $up = vq$. A monoid S is said to be left collapsible if for any $p, q \in S$, there exists $r \in S$ such that $rp = rq$.

A monoid S is called a right PP monoid if and only if every principal right ideal of S is projective. It is well-known that S is right PP if and only if for any $s \in S$, there exists an idempotent $e \in S$, such that, $se = s$ and for all $x, y \in S$, $sx = sy$ implies $ex = ey$. Right PP monoids were investigated by Fountain in [8] and Kilp in [9].

According to Knauer[6], an element $u \in S$ is called left semi-cancellable if $us = ut$ $(s, t \in S, u \in S)$ implies that there exists $r \in S$ such that $u = ur$ and $rs = rt$. In [10], a monoid S is called a right PSF monoid if every principal right ideal of S is strongly flat. It was discussed in [10] that, S is a right PSF monoid if and only if every element of S is left semi-cancellable. It is clear that a right PP monoid must be a right PSF monoid.

As in [12], S is called a right P(P) monoid if and only if every principal right ideal of S satisfies condition (P), and if and only if $us = ut$ $(s, t \in S, u \in S)$ implies that there exists $p, q \in S$ such that $u = up = uq$ and $ps = qt$.

In [7], it was shown that if all strongly flat left S-acts are regular, then S is a left PP monoid such that all cyclic strongly flat left S-acts are projective. And if S is a left PP monoid such that all cyclic strongly flat left S-acts are projective, then all finitely generated strongly flat left S-acts are regular. But the problem of characterizations of monoids over which all strongly flat left S-acts are regular remains open. In [1] the authors proved the following result:

Theorem 1.1 *The following conditions on S are equivalent:*

(1) All strongly flat left S-acts are regular.

(2) All left S-acts having the property (E) are regular.

(3) S is a left PP monoid and satisfies (FP_2).

According to Knauer[6], a monoid S is said to have property (FP_2) if for any set M of idempotents of S with the property: for $e_1, \cdots, e_n, f_1, \cdots, f_m \in M$, there exists $f \in M$ such that $e_1 \cdots e_n f = f_1 \cdots f_m f$, then the subsemigroup of S generated by M contains a right zero.

In this paper, we provide some new characterizations of monoids over which all strongly flat right S-acts are regular. It is shown that all strongly flat right S-acts are regular if and only if S is a right PSF monoid and every left collapsible submonoid of S contains a left zero, and if and only if S is a right PP monoid and every left collapsible submonoid of S contains a left zero.

2. Main results

Lemma 2.1[7] *The following conditions on S are equivalent:*

(1) All projective right S-acts are regular.

(2) S is a right PP monoid.

Lemma 2.2[9] *Let A be a right S-act, the following conditions for $a \in A$ are equivalent:*

(1) aS is a projective right S-act.

(2) There exists $e^2 = e \in S$ such that $ae = a$ and $as = at$ implies $es = et$ where $s, t \in S$.

Lemma 2.3[2] *The following conditions on S are equivalent:*

(1) All strongly flat cyclic right S-acts are projective.

(2) If $P \subseteq S$ is a left collapsible submonoid, then P contains a left zero.

Theorem 2.4 *The following conditions on S are equivalent:*

(1) All strongly flat right S-acts are regular.

(2) All right S-acts having the property (E) are regular.

(3) S is a right PSF monoid and if $P \subseteq S$ is a left collapsible submonoid, then P contains a left zero.

Proof (2)$\Rightarrow$ (1). is clear.

(1)$\Rightarrow$ (3). Suppose all strongly flat right S-acts are regular, then all projective right S-acts are regular. By Lemma 2.1 S is a right PP monoid. Obviously S is a right PSF monoid. Again applying (1), we get that all cyclic strongly flat right S-acts are projective. By Lemma 2.3, if $P \subseteq S$ is a left collapsible submonoid, then P contains a left zero.

(3)$\Rightarrow$ (2). Assume A is an arbitrary right S-act having the property (E) and a an arbitrary element of A. Since $a1 = a$ for every $a \in A$, it follows that the set P_a=$\{s \in S \mid a = as\}$ is non-empty, and P_a is a submonoid of S.

First, we show that P_a contains a left zero. For any $x, y \in P_a$, we have $a = ax = ay$. Because A satisfies condition (E), there exist $a_1 \in A$ and $u \in S$ such that $a = a_1u$ and $ux = uy$. Since S is a right PSF monoid, every element of S is left semi-cancellable. For $ux = uy$, there exists $r \in S$ such that $u = ur$ and $rx = ry$. Then $ar = a_1ur = a_1u = a$. Clearly $r \in P_a$. Thus P_a is a left collapsible submonoid of S. By our assumption, it contains a left zero e, and certainly $ae = a$.

Now $as = at$, where $a \in A$ and $s, t \in S$. Condition (E) implies the existence of $a' \in A$ and $x \in S$ such that $a = a'x$ and $xs = xt$. Since S is a right PSF monoid, there exists $p \in S$ such that $x = xp$ and $ps = pt$. Hence $ap = a'xp = a'x = a$, and this means that $p \in P_a$. But e is a left zero of P_a, so we get $ep = e$. Consequently $ps = pt$ implies $es = (ep)s = e(ps) = e(pt) = (ep)t = et$. Now by Lemma 2.2, A is regular.

By [11] a left S-act A is called a CSF S-act if all cyclic subacts of A are strongly flat. Let A be a left S-act and $a \in A$. The cyclic subact Sa is strongly flat if and only if for $sa = ta (s, t \in S)$ there exists $r \in S$ such that $ra = a$ and $sr = tr$.

Corollary 2.5[11] *The following conditions on S are equivalent:*

(1) All free left S-acts are CSF S-acts.

(2) All projective left S-acts are CSF S-acts.

(3) All strongly flat left S-acts are CSF S-acts.

(4) S is a left PSF monoid.

Corollary 2.6 *The following conditions on S are equivalent:*

(1) All strongly flat right S-acts are regular.

(2) All right S-acts having the property (E) are regular.

(3) S is a right PP monoid and if $P \subseteq S$ is a left collapsible submonoid, then P contains a left zero.

Proposition 2.7 *If S is a right P(P) monoid and every right reversible submonoid of S contains a left zero, then all strongly flat right S-acts are regular.*

Proof Suppose A is an arbitrary right S-act having the property (E). For every $a \in A$, we shall show that aS is a projective right S-act. By Lemma 2.2, we only need to show that there exists $e(=e^2) \in S$ such that $ae = a$ and $as = at$ implies $es = et$ where $s, t \in S$.

Let $P_a = \{s \in S \mid a = as\}$. It is clear that $P_a \neq \emptyset$, and P_a is a submonoid of S. For any $x, y \in P_a$, we get $a = ax = ay$. Because A satisfies condition (E), for $ax = ay$, there exist $a_1 \in A$ and $u \in S$ such that $a = a_1u$ and $ux = uy$. Since S is a right P(P) monoid, there exist $p, q \in S$ such that $u = up = uq$ and $px = qy$. Hence $ap = a_1up = a_1u = a$ and $aq = a_1uq = a_1u = a$, which implies $p, q \in P_a$. Thus P_a is a right reversible submonoid of S. By the assumption, it contains a left zero e, and it is clear that $ae = a$.

Now $as = at$, where $a \in A$ and $s, t \in S$, since A satisfies condition (E) and there exist $a' \in A$ and $x \in S$ such that $a = a'x$, and $xs = xt$. Since S is a right P(P) monoid, there exist $p_1, q_1 \in S$ such that $x = xp_1 = xq_1$ and $p_1s = q_1t$. Then $ap_1 = a'xp_1 = a'x = a$ and $aq_1 = a'xq_1 = a'x = a$. This means $p_1, q_1 \in P_a$. But e is a left zero of P_a, so we have $ep_1 = eq_1 = e$. Finally $p_1s = q_1t$ implies $es = (ep_1)s = e(p_1s) = e(q_1t) = (eq_1)t = et$. Thus the result follows.

By [12] a right S-act A is called a right C(P) act if all cyclic subacts of A satisfy condition (P). Let A be a right S-act and $a \in A$, the cyclic subact aS satisfies condition (P) if and only if for $as = at (s, t \in S)$ there exists $p, q \in S$ such that $a = ap = aq$ and $ps = qt$. Then we have the following:

Corollary 2.8[12] *The following conditions on S are equivalent:*

(1) All free right S-acts are C(P) S-acts.

(2) All projective right S-acts are C(P) S-acts.

(3) All strongly flat right S-acts are C(P) S-acts.

(4) All strongly flat cyclic right S-acts are C(P) S-acts.

(5) S is a right P(P) monoid.

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关于所有强平坦右 S- 系是正则系的幺半群

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摘要: 本论文考虑了所有强平坦右 S- 系是正则系的幺半群的刻画，证明了所有强平坦右 S- 系是正则 S- 系当且仅当 S 是右 PSF 幺半群并且 S 的每一个左 collpasible 子幺半群包含左零元. 该结果对 Kilp 和 Knauer 在文献 [7] 中的问题给出了一个新的回答. .

关键词: 强平坦 S- 系；正则 S- 系.