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# Positive Solutions for Second Order Impulsive Differential Equations with Dependence on First Order Derivative

*CAI Guo-lan*[1], *GE Wei-gao*[2]
(1. Dept. of Math., Central University for Nationalities, Beijing 100081, China;
2. Dept. of Appl. Math., Beijing Institute of Technology, Beijing 100081, China )
(E-mail: caiguolan@163.com)

**Abstract**: We study positive solutions for second order three-point boundary value problem:

$$\begin{cases} x''(t)+f(t,x(t),x'(t))=0, & t\neq t_i \\ \triangle x(t_i)=I_i(x(t_i),x'(t_i)), & i=1,2,\cdots,k \\ \triangle x'(t_i)=J_i(x(t_i),x'(t)), & \\ x(0)=0=x(1)-\alpha x(\eta), & \end{cases}$$

where $0<\eta<1, 0<\alpha<1$, and $f:[0,1]\times[0,\infty)\times R\to[0,\infty)$, $I_i:[0,\infty)\times R\to R, J_i:[0,\infty)\times R\to R, (i=1,2,\cdots,k)$ are continuous. Based on a new extension of Krasnoselskii fixed-point theorem (which was established by Guo Yan-ping and GE Wei-gao[5]), the existence of positive solutions for the boundary value problems is obtained. In particular, we obtain the Green function of the problem, which makes the problem simpler.

**Key words**: impulsive differential equation; fixed point theorem; Green function.
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## 1. Introduction

There is an increasing interest in the study of three-point BVPs for non-linear ordinary differential equations. For example, authors in [1–5] studied the second-order ordinary differential equations three-point BVPs. MA Ru-yun[1−3], HE Xiao-ming and GE Wei-gao[4] studied the equation

$$\begin{cases} x''(t)+\lambda h(t)f(x(t))=0, \text{ or} \\ x''(t)+f(t,x(t),x'(t))=0, \ 0<t<1, \\ x(0)=0=x(1)-\alpha x(\eta). \end{cases}$$

GUO Yan-ping and GE Wei-gao[5] studied

$$\begin{cases} x''(t)+f(t,x(t),x'(t))=0, \ 0<t<1, \\ x(0)=0=x(1)-\alpha x(\eta). \end{cases}$$

In [5], a theorem was obtained which extended the Krasnoselskii's compression-expansion theorem in cones. Based on it, positive solutions for the boundary value problems were proved.

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On the other hand, with the applications of differential equations involving impulse effects in population dynamics, ecology, biological systems, etc., many authors are interested in the study of impulsive differential equations in [6–8]. For example, Ravi P. Agarwal, and Donal O'Regan[7] studied

$$\begin{cases} y''(t)+\varphi(t)f(t,y(t))=0,\ t\neq t_k, \\ \triangle y(t_k)=I_k(y(t_k)),\ k=1,2,\cdots,m, \\ \triangle y'(t_k)=J_k(y(t_k)), \\ y(0)=y(1)=0 \end{cases}$$

with the method of Krasnoselskii's fixed point theorem.

Guo Da-jun[8,p229−240] studied the existence of solution to the following equation

$$\begin{cases} x''(t)+f(t,x(t),x'(t))=0,\ t\in J', \\ \triangle x(t_i)=I_i(x(t_i)),\ i=1,2,\cdots,k, \\ \triangle x'(t_i)=J_i(x(t_i),x'(t)), \\ ax(0)-bx'(0)=x_0, cx(1)+dx'(1)=x_0^*, \end{cases}$$

by use of the Banach contraction mapping principle and the Schauder fixed-point theorem.

However, to the best of our knowledge, existence results of positive solutions to three-point BVPs of the second-order impulsive differential equation with dependence on the first order derivative have not been found in literature.

In the paper, we are concerned with positive solutions of the following problem:

$$\begin{cases} x''(t)+f(t,x(t),x'(t))=0,\ t\in J' \\ \triangle x(t_i)=I_i(x(t_i),x'(t_i)),\ i=1,2,\cdots,k \\ \triangle x'(t_i)=J_i(x(t_i),x'(t)), \\ x(0)=0=x(1)-\alpha x(\eta). \end{cases} \tag{1.1}$$

Through this paper, we assume that

(C$_1$) $J=[0,1]$, $t_i(i=1,2,\cdots,k)$ are impulsive moments and $0<t_1<t_2<\cdots<t_k<1$, $J'=J\backslash\{t_1,t_2,\cdots,t_k\}$, $J_j=(t_j,t_{j+1}], j=1,2,\cdots,k, J_0=[0,t_1], t_{k+1}=1$. $0<\eta<1, 0<\alpha<1$.

(C$_2$) $f:J\times[0,\infty)\times R\to[0,\infty)$ is continuous, and $f(t,\cdot,\cdot)$ does not vanish identically on any subset of $[0,1]$. $I_i\in C([0,\infty)\times R,R), J_i\in C([0,\infty)\times R,R), i=1,2,\cdots,k$.

## 2. Preliminary lemmas

In order to prove our main result, we present in this section a series of useful lemmas.

**Lemma 2.1**[5] *Let $X$ be a Banach space and $K\subset X$ a cone. Assume $\alpha,\beta:X\to R^+$ are two nonnegative continuous convex functionals such that, for each $x\in X$,*

$$\alpha(\lambda x)=|\lambda|\alpha(x),\beta(\lambda x)=|\lambda|\beta(x),x\in X,\lambda\in R,$$

$$\|x\|\leq M\max\{\alpha(x),\beta(x)\},x\in X,$$

*where $M>0$ is a constant, and*

$$\alpha(x)\leq\alpha(y),x,y\in K,x\leq y,$$

*Assume that there exist constants $r_2 > r_1 > 0, L > 0$ and*

$$\Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\}, i = 1, 2$$

*are two open bounded sets in X. Let*

$$D_i = \{x \in X : \alpha(x) = r_i\}.$$

*And suppose that $T : K \to K$ is a completely continuous operator such that*

*($A_1$) $\alpha(Tu) < r_1, u \in D_1 \cap K; \alpha(Tu) > r_2, u \in D_2 \cap K;$*

*($A_2$) $\beta(Tu) < L, u \in K;$*

*($A_3$) There exists a $p \in (\Omega_1 \bigcap K)\backslash\{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \geq \alpha(x), x \in K$.*

*Then $T$ has at least a fixed point $x \in (\Omega_2\backslash\overline{\Omega}_1) \cap K$.*

Consider the boundary value problem

$$\begin{cases} x'' + y(t) = 0, \ t \in J', \\ \Delta x(t_i) = a_i, \ i = 1, , 2 \cdots k, \\ \Delta x'(t_i) = b_i, \\ x(0) = 0 = x(1) - \alpha x(\eta), \end{cases} \tag{2.1}$$

where $a_i$ and $b_i$ are constants, $i = 1, 2, \cdots k$.

**Lemma 2.2**[9] *Let $\alpha\eta \neq 1$, then for $y \in PC[0, 1]$, problem (2.1) has a unique solution*

$$x(t) = \int_0^1 \frac{t(1-s)}{1-\alpha\eta} y(s)\mathrm{d}s - \int_0^\eta \frac{\alpha(\eta - s)t}{1-\alpha\eta} y(s)\mathrm{d}s - \frac{t}{1-\alpha\eta} \sum_{t_i<1} b_i(1 - t_i) + \frac{\alpha t}{1-\alpha\eta} \sum_{t_i<\eta} b_i(\eta - t_i) -$$
$$\sum_{0<t_i<1} \frac{t}{1-\alpha\eta} a_i + \sum_{0<t_i<\eta} \frac{\alpha t}{1-\alpha\eta} a_i - \int_0^t (t-s)y(s)\mathrm{d}s + \sum_{t_i<t} b_i(t - t_i) + \sum_{t_i<t} a_i. \tag{2.2}$$

Consider BVP

$$\begin{cases} x'' + f(t, y(t), y'(t)) = 0, & t \in J' \\ \Delta x(t_i) = I_i(y(t_i), y'(t_i)), & i = 1, 2, \cdots, k \\ \Delta x'(t_i) = J_i(y(t_i), y'(t_i)), \\ x(0) = 0 = x(1) - \alpha x(\eta). \end{cases} \tag{2.3}$$

**Lemma 2.3**[9] *Let $\alpha\eta \neq 1$, then BVP (2.3) has a unique solution*

$$x(t) = \int_0^1 G(t, s)f(s, y(s), y'(s))\mathrm{d}s + \sum_{i=1}^k H_i(t)J_i(y(t_i), y'(t_i)) +$$
$$\sum_{i=1}^k W_i(t)[I_i(y(t_i), y'(t_i)) - J_i(y(t_i), y'(t_i))t_i], \tag{2.4}$$

*where*

$$G(t, s) = \begin{cases} \frac{s[(1-t)+\alpha(t-\eta)]}{1-\alpha\eta}, & s \leq \min\{t, \eta\} \\ \frac{t[(1-s)+\alpha(s-\eta)]}{1-\alpha\eta}, & t \leq s < \eta \\ \frac{s(1-t)+\alpha\eta(t-s)}{1-\alpha\eta}, & \eta \leq s \leq t \\ \frac{t(1-s)}{1-\alpha\eta}, & s \geq \max\{t, \eta\} \end{cases} \tag{2.5}$$

$$W_i(t)=\begin{cases}\frac{(1-t)+\alpha(t-\eta)}{1-\alpha\eta}, & t_i\le\min\{t,\eta\}\\ \frac{t(\alpha-1)}{1-\alpha\eta}, & t\le t_i<\eta\\ \frac{1-t-\alpha\eta}{1-\alpha\eta}, & \eta<t_i\le t\\ \frac{-t}{1-\alpha\eta}, & t_i\ge\max\{t,\eta\}\end{cases} \tag{2.6}$$

$$H_i(t)=\begin{cases}0, & t_i\le\min\{t,\eta\}\\ -t, & t\le t_i<\eta\\ -\frac{\alpha\eta t}{1-\alpha\eta}, & \eta<t_i\le t\\ \frac{-t}{1-\alpha\eta}, & t_i\ge\max\{t,\eta\}\end{cases} \tag{2.7}$$

From Lemma 2.3, solving BVP (1.1) is equivalent to finding a solution $x(t)\in PC[0,1]$ which satisfies the following integral equation

$$x(t)=\int_0^1 G(t,s)f(s,x(s),x'(s))\mathrm{d}s+\sum_{i=1}^k H_i(t)J_i(x(t_i),x'(t_i))+$$
$$\sum_{i=1}^k W_i(t)[I_i(x(t_i),x'(t_i))-J_i(x(t_i),x'(t_i))t_i]. \tag{2.8}$$

**Lemma 2.4** *Suppose that in (1.1), $\alpha\in(0,1)$, $f(t,x(t),x'(t))\ge 0$ and*

$$\frac{I_i(x,x')}{x}\ge -1,\ x>0, \tag{2.9}$$

$$J_i(x,x')\le 0,\ i=1,2,\cdots,k. \tag{2.10}$$

*Then for each solution $x(t)$ of the problem (1.1), it holds that $x(t)\ge 0, t\in[0,1]$.*

**Proof** Let

$$I_i^*(x,x')=\begin{cases}I_i(x,x'), & x\ge 0,\\ (\alpha^{\frac{1}{k}}-1)x, & x<0.\end{cases} \tag{2.11}$$

Let $j=\min\{l\in\{1,2,\cdots,k+1\}$, then there is $s\in(t_{l-1},t_l]$ such that $x(s)<0$ $\}$ and $t_{k+1}=1$. Then

$$x(t)\ge 0,\quad t\le t_{j-1}.$$

We claim that $x(t_{j-1}^+)\ge 0$. Otherwise

$$0>x(t_{j-1}^+)=x(t_{j-1})+I_{j-1}(x(t_{j-1}),x'(t_{j-1}))\ge x(t_{j-1})-x(t_{j-1})=0,$$

a contradiction. Then there is $\xi\in[t_{j-1},s]$ such that

$$x(\xi)=0, x(t)<0, t\in(\xi,s].$$

This yields in turn $x'(\xi)\le 0$. By (2.10) and the nonnegativity of $f$, one has

$$x'(t)\le 0, x(t)<0, t>\xi. \tag{2.12}$$

Furthermore, at the same time it holds that

$$x(t)\ge 0, t\in[0,\xi] \tag{2.13}$$

i) If $\eta \in (0, \xi]$, then (2.12) and (2.13) imply $0 > x(1) = \alpha x(\eta) \geq 0$, a contradiction.

ii) If $\eta \in (\xi, 1)$, from (2.11) $x(t) < 0, t > \xi$, one has $x(\eta) < 0$. Without loss of generality, we suppose $\eta \in (t_{l-1}, t_l]$ then from the denotative definition of $I_i^*$ (2.10) and $x'(t) \leq 0, t > \eta, x(\eta) < 0$, we have

$$\begin{aligned} x(1) &\leq x(t_k^+) \leq \alpha^{\frac{1}{k}} x(t_k) < \alpha^{\frac{1}{k}} x(t_{k-1}^+) \leq \alpha^{\frac{2}{k}} x(t_{k-1}) < \cdots \\ &\leq \alpha^{\frac{k-l+1}{k}} x(t_l) \leq \alpha^{\frac{k-l+1}{k}} x(\eta) \leq \alpha x(\eta). \end{aligned}$$

This contradicts the boundary condition $x(1) = \alpha x(\eta)$.

Therefore, for each solution $x(t)$ of the problem (2.3), it holds that $x(t) \geq 0, t \in [0, 1]$. □

**Lemma 2.5** *Let* $\frac{1}{2} < \eta < 1$, $\frac{1-\eta}{\eta} \leq \alpha < 1$, *and suppose that (2.6), (2.7) hold,* $J_i(u(t_i)) \leq 0, (i = 1, 2, \cdots, k)$ *and* $I_i(u(t_i))$ *satisfy the follow conditions*

$$\begin{array}{ll} I_i(u(t_i)) \leq 0, & \eta < t_i < 1 \\ \max\{-u(t_i), t_i J_i(u(t_i))\} \leq I_i(u(t_i)), & 0 < t_i < 1. \end{array} \tag{2.13}$$

*Then*

$$W_i(t) I_i(u(t_i)) \geq 0, \quad \text{for} \ \ \eta \leq t, t_i \leq 1,$$

$$[t_i W_i(t) - H_i(t)] J_i(u(t_i)) \leq 0, \quad \text{for} \ \ 0 \leq t, t_i \leq 1.$$

**Proof** First, we will show $W_i(t) I_i(u(t_i)) \geq 0$, for $\eta \leq t, t_i \leq 1$.

From (2.6), for $\eta \leq t, t_i \leq 1$, one has

$$W_i(t) = \begin{cases} \frac{1-t-\alpha\eta}{1-\alpha\eta}, & \eta < t_i \leq t \\ \frac{-t}{1-\alpha\eta}, & t_i \geq t \geq \eta. \end{cases}$$

Hence $W_i'(t) = \frac{-1}{1-\alpha\eta} < 0$, and, for $\eta \leq t \leq 1$

$$W_i(t) \leq W_i(\eta) \leq \frac{1-\eta-\alpha\eta}{1-\alpha\eta}.$$

When $\eta < t_i < 1$, $I_i(u(t_i)) \leq 0$, so

$$W_i(t) I_i(u(t_i)) \geq \frac{1-\eta-\alpha\eta}{1-\alpha\eta} I_i(u(t_i)),$$

from $\alpha \geq \frac{1-\eta}{\eta}$, $1 - \eta - \alpha\eta \leq 0$. So, for $\eta \leq t, t_i \leq 1$,

$$W_i(t) I_i(u(t_i)) \geq \frac{1-\eta-\alpha\eta}{1-\alpha\eta} I_i(u(t_i)) \geq 0.$$

Next, we will show

$$[t_i W_i(t) - H_i(t)] J_i(u(t_i)) \leq 0, \quad \text{for} \ \ 0 \leq t, t_i \leq 1.$$

From (2.6) and (2.7), we have

$$t_i W_i(t) - H_i(t) = \begin{cases} \frac{t_i[1-t+\alpha(t-\eta)]}{1-\alpha\eta}, & t_i \leq \min\{t, \eta\}, \\ \frac{t(1-t_i)+\alpha t(t_i-\eta)}{1-\alpha\eta}, & t \leq t_i \leq \eta, \\ \frac{t_i(1-t)+\alpha\eta(t-t_i)}{1-\alpha\eta}, & \eta < t_i \leq t, \\ \frac{t(1-t_i)}{1-\alpha\eta}, & t_i \geq \max\{t, \eta\}. \end{cases}$$

From the above form, we know

$$t_iW_i(t) - H_i(t) \geq 0, i = 1, 2, \cdots, k.$$

So, $[t_iW_i(t) - H_i(t)]J_i(u(t_i)) \leq 0$. □

## 3. Main results

In this section, we shall apply the lemmas to the boundary value problem (1.1) to obtain the existence theorem of positive solutions.

Let $E = PC^1(J, R) = \{x : J \to R \mid x(t)$ is continuously differentiable, for $t \in J'$; there exist $x'(t_i^+), x'(t_i^-)$ and $x(t_i) = x(t_i^-)$,$x'(t_i) = x'(t_i^-)$, $i = 1, 2, \cdots k\}$.

Let $\|x\| = \sup_{t\in J}[x^2(t) + (x'(t))^2]^{\frac{1}{2}}$, then obviously, $E$ is a Banach space.

Let

$$K = \{x \in E : x(t) \geq 0, x(t) \text{is concave}, t \in [0, 1]\}.$$

A function $x \in PC^1[J, R] \bigcap C^2[J', R]$ is called the solution of the BVP (1.1), if it satisfies the Equation (1.1).

$\forall x \in E$, define two functionals

$$\alpha(x) = \sup_{t\in[0,1]} |x(t)|, \tag{3.1}$$

$$\beta(x) = \sup_{t\in[0,1]} |x'(t)|, \tag{3.2}$$

then $\|x\| \leq \sqrt{2} \max\{\alpha(x), \beta(x)\}$ and

$$\alpha(\lambda x) = |\lambda|\alpha(x), \quad \beta(\lambda x) = |\lambda|\beta(x), x \in E, \lambda \in R$$

$$\alpha(x) \leq \alpha(y), \qquad \forall x, y \in K, x \leq y.$$

If ($C_1$) holds, then Green function $G(t, s) \geq 0$. Let $y(t) = 1$, then we have

$$\int_0^1 G(t, s)\mathrm{d}s = -\frac{1}{2}t^2 + \frac{t(1 - \alpha\eta^2)}{2(1 - \alpha\eta)}. \tag{3.3}$$

By a simple calculation, we know that

$$|\frac{\partial G(t, s)}{\partial t}| < \frac{1}{1 - \alpha\eta}$$

and $|H_i'(t)| < |W_i'(t)| < \frac{1}{1-\alpha\eta}$.

Let

$$M := \frac{(1 - \alpha\eta^2)^2}{8(1 - \alpha\eta)^2} + \frac{k(3 - 2\alpha\eta)}{1 - \alpha\eta} \tag{3.4}$$

$$m := \max_{t\in[0,1]} \int_\eta^1 G(t, s)\mathrm{d}s \tag{3.5}$$

$$Q := \frac{1+2k}{1-\alpha\eta} \tag{3.6}$$

Let

$$f^*(t,u,v) = \begin{cases} f(t,u,v), & (t,u,v) \in J' \times [0,b] \times (-\infty,\infty), \\ f(t,b,v), & (t,u,v) \in J' \times (b,\infty) \times (-\infty,\infty), \end{cases}$$

$$I_i^*(u,v) = \begin{cases} I_i(u,v), & (u,v) \in [0,b] \times (-\infty,\infty), \\ I_i(b,v), & (u,v) \in (b,\infty) \times (-\infty,\infty), \end{cases}$$

$$J_i^*(u,v) = \begin{cases} J_i(u,v), & (u,v) \in [0,b] \times (-\infty,\infty), \\ J_i(b,v), & (u,v) \in (b,\infty) \times (-\infty,\infty), \end{cases}$$

and

$$f^{**}(t,u,v) = \begin{cases} f^*(t,u,-L), & (t,u,v) \in J' \times [0,b] \times (-\infty,-L], \\ f^*(t,u,v), & (t,u,v) \in J' \times [0,b] \times [-L,L], \\ f^*(t,u,L), & (t,u,v) \in J' \times [0,b] \times [L,\infty), \end{cases}$$

$$I_i^{**}(u,v) = \begin{cases} I_i^*(u,-L), & (u,v) \in [0,b] \times (-\infty,-L], \\ I_i^*(u,v), & (u,v) \in [0,b] \times [-L,L], \\ I_i^*(u,L), & (u,v) \in [0,b] \times [L,\infty), \end{cases}$$

$$J_i^{**}(u,v) = \begin{cases} J_i^*(u,-L), & (u,v) \in [0,b] \times (-\infty,-L], \\ J_i^*(u,v), & (u,v) \in [0,b] \times [-L,L], \\ J_i^*(u,L), & (u,v) \in [0,b] \times [L,\infty). \end{cases}$$

So $f^{**}(t,u,v) \in C(J' \times [0,\infty) \times R, R^+)$, $I_i^{**}(u,v), J_i^{**}(u,v) \in C([0,\infty) \times R, R)$.

Define an operator

$$(Tx)(t) = \int_0^1 G(t,s) f^{**}(s,x(s),x'(s))ds + \sum_{i=1}^k H_i(t) J_i^{**}(x(t_i),x'(t_i)) + \sum_{i=1}^k W_i(t)[I_i^{**}(x(t_i),x'(t_i)) - J_i^{**}(x(t_i),x'(t_i))t_i]. \tag{3.7}$$

**Theorem 3.1** *Suppose that ($C_1$), ($C_2$) and Lemma 2.5 hold, and that there exist constants $L > b > c > 0$, such that*

($H_1$) $f(t,u,v) < \frac{c}{M}$, $(t,u,v) \in J' \times [0,c] \times [-L,L]$, $I_i(u,v) < \frac{c}{M}$, $(u,v) \in [0,c] \times [-L,L]$, $J_i(u,v) > -\frac{c}{M}$, $(u,v) \in [0,c] \times [-L,L]$;

($H_2$) $f(t,u,v) > \frac{b}{m}$, $(t,u,v) \in J' \times (c,b] \times [-L,L]$;

($H_3$) $f(t,u,v) > \frac{L}{Q}$, $(t,u,v) \in J' \times [0,b] \times [-L,L]$, $|I_i(u,v)| > \frac{L}{Q}$, $(u,v) \in [0,b] \times [-L,L]$, $J_i(u,v) < -\frac{L}{Q}$, $(u,v) \in [0,b] \times [-L,L]$.

*Then BVP (1.1) has at least one positive solution $x(t)$ such that $c < \alpha(x) < b$, $|x'(t)| < L$.*

**Proof** Let

$$\Omega_1 = \{x \in E : \alpha(x) = \sup_{t\in[0,1]} |x(t)| < c, \beta(x) = \sup_{t\in[0,1]} |x'(t)| < L\},$$

$$\Omega_2 = \{x \in E : \alpha(x) = \sup_{t\in[0,1]} |x(t)| < b, \beta(x) = \sup_{t\in[0,1]} |x'(t)| < L\}$$

be two open bounded sets in $E$, and let

$$D_1 = \{x \in E, \alpha(x) = \sup_{t\in[0,1]} |x(t)| = c\},$$

$$D_2=\{x\in E, c<x(t)\le b, \alpha(x)=\sup_{t\in[0,1]}|x(t)|=b\}.$$

From (3.7), one knows that $T:K\to K$ is completely continuous. And takes $p\in(0,c)\backslash\{0\}$ such that $\forall x\in K$, and $\lambda\ge 0$,

$$\alpha(x+\lambda p)=\sup_{t\in[0,1]}|x(t)+\lambda p|=\lambda p+\sup_{t\in[0,1]}|x(t)|\ge\alpha(x)$$

If $x\in D_1\bigcap K$, then $0<x(t)\le c$. From ($\mathrm{H}_1$), $W_i(t)\le 1, -H_i(t)\le\frac{1}{1-\alpha\eta}$ and (2.14), one has

$$\begin{aligned}\alpha(Tx)=&\sup_{t\in[0,1]}|\int_0^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s+\sum_{i=1}^k H_i(t)J_i^{**}(x(t_i),x'(t_i))+\\&\sum_{i=1}^k W_i(t)[I_i^{**}(x(t_i),x'(t_i))-J_i^{**}(x(t_i),x'(t_i))t_i]|\\ \le&\sup_{t\in[0,1]}[\int_0^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s-\sum_{i=1}^k\frac{1}{1-\alpha\eta}J_i^{**}(x(t_i),x'(t_i))+\\&\sum_{i=1}^k[I_i^{**}(x(t_i),x'(t_i))-J_i^{**}(x(t_i),x'(t_i))t_i]]\\ \le&\sup_{t\in[0,1]}[\int_0^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s+\sum_{i=1}^k I_i^{**}(x(t_i),x'(t_i))-\\&\frac{2-\alpha\eta}{1-\alpha\eta}\sum_{i=1}^k J_i^{**}(x(t_i),x'(t_i))]\\ <&\frac{c}{M}[\sup_{t\in[0,1]}\int_0^1 G(t,s)\mathrm{d}s+\frac{k(3-2\alpha\eta)}{1-\alpha\eta}]\\ =&\frac{c}{M}[\frac{(1-\alpha\eta^2)^2}{8(1-\alpha\eta)^2}+\frac{k(3-2\alpha\eta)}{1-\alpha\eta}]\\ =&c.\end{aligned}$$

Next, if $x\in D_2\cap K$, then $c<x(t)\le b$. From $(H_2)$ and Lemma 2.5, one has

$$\begin{aligned}\alpha(Tx)=&\sup_{t\in[0,1]}|\int_0^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s+\sum_{i=1}^k H_i(t)J_i^{**}(x(t_i),x'(t_i))+\\&\sum_{i=1}^k W_i(t)[I_i^{**}(x(t_i),x'(t_i))-J_i^{**}(x(t_i),x'(t_i))t_i]|\\ \ge&\sup_{t\in[0,1]}[\int_\eta^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s+\sum_{\eta\le t_i,s\le 1\}}W_i(t)I_i^{**}(x(t_i),x'(t_i))+\\&\sum_{\eta\le t_i,s\le 1\}}[t_iW_i(t)-H_i(t)](-J_i^{**}(x(t_i),x'(t_i)))]\\ \ge&\sup_{t\in[0,1]}\int_\eta^1 G(t,s)f^{**}(s,x(s),x'(s))\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&> \frac{b}{m}\sup_{t\in[0,1]}\int_\eta^1 G(t,s)\mathrm{d}s\\
&= b.
\end{aligned}
$$

For $x \in K$, From ($H_3$) and (2.14), one has

$$
\begin{aligned}
\beta(Tx) &= \sup_{t\in[0,1]} |(Tx)'(t)|\\
&= \sup_{t\in[0,1]} |\int_0^1 \frac{\partial G(t,s)}{\partial t} f^{**}(s,x(s),x'(s))\mathrm{d}s + \sum_{i=1}^k H_i'(t)J_i^{**}(x(t_i),x'(t_i)) +\\
&\quad \sum_{i=1}^k W_i'(t)I_i^{**}(x(t_i),x'(t_i))|\\
&\le \sup_{t\in[0,1]} [\int_0^1 \frac{\partial G(t,s)}{\partial t} f^{**}(s,x(s),x'(s))\mathrm{d}s - \sum_{i=1}^k |H_i'(t)|J_i^{**}(x(t_i),x'(t_i)) +\\
&\quad \sum_{i=1}^k |W_i'(t)||I_i^{**}(x(t_i),x'(t_i))|]\\
&\le \frac{L}{Q}[\frac{1}{1-\alpha\eta} + \sum_{i=1}^k \frac{2}{1-\alpha\eta}]\\
&= \frac{L}{Q}\frac{1+2k}{1-\alpha\eta}\\
&= L.
\end{aligned}
$$

From Lemma 2.1, there exists $x(t) \in (\Omega_2\backslash\overline{\Omega}_1)\cap K$ such that $Tx = x$, i.e., BVP (1.1) has at least one positive solution $x(t)$ such that $c < \alpha(x) < b$, $|x'(t)| < L$. □

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# 依赖于一阶导数的二阶脉冲微分方程边值问题的正解

蔡果兰[1]， 葛渭高[2]

(1. 中央民族大学数学与计算机学院，北京 100081; 2. 北京理工大学数学系，北京 100081 )

**摘要**: 本文研究一类二阶脉冲微分方程：

$$\begin{cases} x''(t)+f(t,x(t),x'(t))=0, & t\neq t_i \\ \triangle x(t_i)=I_i(x(t_i),x'(t_i)), & i=1,2,\cdots,k \\ \triangle x'(t_i)=J_i(x(t_i),x'(t)) & \\ x(0)=0=x(1)-\alpha x(\eta) & \end{cases}$$

的正解存在性. 其中，$0<\eta<1, 0<\alpha<1, f:[0,1]\times[0,\infty)\times R\to[0,\infty), I_i:[0,\infty)\times R\to R, J_i:[0,\infty)\times R\to R, (i=1,2,\cdots,k)$ 均为连续函数．本文所用方法是文献 [5] 推广的 Krasnoselskii 不动点定理，此定理为解决依赖于一阶导数的边值问题提供了理论依据．基于此定理，获得了问题正解存在性定理．特别地，我们获得此类问题的 Green 函数，使问题的解决更直观和简单.

**关键词**: 脉冲微分方程；不动点定理； Green 函数.