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具有离散参数齐次随机场的线性预测与马氏性 (II)

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摘 要: 本文提出和研究 3 个新型的具有离散参数齐次随机场 $\{x(m,n)\}$ 的马氏性问题，求出了它们具有马氏性的充分必要条件.

关键词: 线性预测；预测值；预测误差；马氏性.
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1 引言与定义

设 $\{x(m,n)\}$ 是含有两个取整数值 m,n 的齐次随机场， $EX(m,n)=0$, 它的线性预测问题的一般提法是：设 T 及 T' 是平面上格子点 (m,n) 的两个集合，当 $(m,n)\in T$ 时， $\{x(m,n)\}$ 值已观察到，而当点 $(m',n')\in T'$ 时， $\{x(m',n')\}$ 值未知，现在要以已观察到的值的线性组合及均方意义下的极限去预测未观察到的值 $\{x(m',n'),(m',n')\in T'\}$, 使均方误差最小. 熟知，其预测值是 $P_{H_x(T)}X(m',n')$, 其中 P 是正交投影算子， $H_X(T)$ 是由 $\{X(m,n),(m,n)\in T\}$ 所张成的线性闭包.

马氏性问题在理论与应用上都是感兴趣的问题, 具有离散参数齐次随机场的马氏性问题的较一般性提法是 [4]: 设 T 与 T_0 是平面上两个格子点的集合, $T_0\subset T,(m',n')\overline{\in}T$, 若 $P_{H_X(T)}X(m',n')=P_{H_X(T_0)}X(m',n')$, 则我们称：对 $X(m',n')$, 空间 $H_X(T)$ 关于 $H_X(T_0)$ 具有马氏性. 江泽培 [1] 首先提出和研究了：

$$T=\{(m,n),-\infty<m<\infty,n\le t-1\};$$

$$T_0=\{(m,t-1),-\infty<m<\infty\},\quad X(m',n')=X(m',t).$$

不久，钱敏平 [2] 推广为

$$T=\{(m,n),-\infty<m<\infty,n\le t-1\};T_0=\{(m,n),-\infty<m<\infty,t-1\ge n\ge t-N\},$$

$$X(m',n')=X(m',t).$$

黄大威 [3] 提出和研究了：

$$T=\{(s+m,t+n),-\infty<m<\infty,n<0\ \text{或}\ -\infty<m\le -1,n=0\};$$

$$T_0=\{(s+m,t),m=-1,-2,\cdots,-M\}\cup\{(s+m,t+n),m=0,\pm1,\pm2,\cdots,\pm M;n=-1,-2,\cdots,-N\},$$

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$$X(m',n')=X(s,t),$$

后来，本人也提出和研究了 [4]:

1). $T=\{(m,n),-\infty<m<\infty,n\le t-1\}$; $T_0=\{(m,t-1),-\infty<m<\infty\}$, $X(m',n')=X(m',t+h)(h\ge 0)$ (这是 [1] 的推广，当 $h=0$ 时，即是 [1]).

2). $T=\{(m,n),-\infty<m<\infty,n<0$ 或 $-\infty<m\le -1,n=0\}$;

$T_0=\{(m,0),-\infty<m<0\}\cup\{(m,-1),-\infty<m<\infty\}$, $X(m',n')=X(m',0),(m'\ge 0)$.

3). $T=\{(m,n),-\infty<m,n<\infty,n\ne 0\}$; $T_0=\{(m,n),-\infty<m<\infty,n=\pm 1\}$, $X(m',n')=X(m',0)$.

江泽培 [5] 又研究了：

1). $T=\{(s+j,t+k),j<0$ 或 $k<0\}$;

$T_0=\{(s+j,t+k),-M\le j\le 0,-N\le k\le 0,(j,k)\ne(0,0)\}$; $X(m',n')=X(s,t)$. (D.Tjstheim[6] 也研究过这问题，但不严格).

2). $T=\{(s+j,t+k),-\infty<j<\infty,k\le 0,(j,k)\ne(0,0)\}$;

$T_0=\{(s+j,t+k),j=\pm 1,\pm 2,\cdots,\pm M,k=0,$ 或 $j=0,\pm 1,\pm 2,\cdots,\pm M,k=-1,-2,\cdots,-N\}$, $X(m',n')=X(s,t)$ (R.L. Kashyap[7] 也研究过这问题，但不完全).

3). $T=\{(s+j,t+k),-\infty<j,k<\infty,(j,k)\ne(0,0)\}$;

$T_0=\{(s+j,t+k),-N\le j,k\le N,(j,k)\ne(0,0)\}$; $X(m',n')=X(s,t)$.

本文目的是提出和研究下列 3 个马氏型问题:

1). $T=\{(m,n),-\infty<m<\infty,n\le 0,(m,n)\ne(0,0)\}\triangleq T_{6(0)}$;

$T_0=\{(m,-1),-\infty<m<\infty\}\cup\{(m,0),-\infty<m<\infty,m\ne 0\}\triangleq T_{11}(1,0)$, $X(m',n')=X(0,0)$.

2). $T=\{(m,n),-\infty<m,n<\infty,n\ne 0\}\cup\{(m,0),-\infty<m\le -1\}\triangleq T_{5(1,0)}$;

$T_0=\{(m,n),-\infty<m<\infty,n=\pm 1\}\cup\{(m,0),-\infty<m\le -1\}\triangleq T_{9(1,1)}$, $X(m',n')=X(0,0)$.

3). $T=\{(m,n),-\infty<m,n<\infty,(m,n)\ne(0,0)\}\triangleq T_{3(1,0)}$;

$T_0=\{(m,n),-\infty<m<\infty,n=\pm 1\}\cup\{(m,0),-\infty<m<\infty,m\ne 0\}\triangleq T_{11(1,1)}$, $X(m',n')=X(0,0)$.

2 引理与定理

引理 1[1,8] 设齐次随机场 $\{x(m,n)\}$ 具有谱密度 $f(\lambda,\mu)$ 且为半平面正则，即

$$\int_{B_+-\delta_0}\mathrm{d}\lambda=0,B_+=\{\lambda:g_X(\lambda)=\int_{-\pi}^{\pi}f(\lambda,\mu)\mathrm{d}\mu>0\},$$

$$\delta_0=\{\lambda:|\int_{-\pi}^{\pi}\log f(\lambda,\mu)\mathrm{d}\mu|<\infty\}),$$

$\mathrm{d}Z_X(\lambda,\mu)$ 是 $\{x(m,n)\}$ 的随机谱测度，则

$$P_{H_X(-1)}X(m,n_0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}e^{i(m\lambda+n_0\mu)}\frac{\sum_{k=n_0+1}^{\infty}\varphi_k(\lambda)e^{-ik\mu}}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-ik\mu}}\mathrm{d}Z_X(\lambda,\mu),\tag{1}$$

其中 $H_X(-1)$ 是由 $\{X(m,n),-\infty<m<\infty,n\le -1\}$ 所张成的线性闭包，$\{\varphi_k(\lambda)\}$ 由下列式决定：

$$\sum_{k=0}^{\infty}\varphi_k(\lambda)Z^k=\exp\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{-it}+Z}{e^{-iu}-Z}\log\frac{f(\lambda,u)}{g_X(\lambda)}\mathrm{d}\mu\}=\exp\{\frac{A_0(\lambda)}{2}+\sum_{k=1}^{\infty}A_k(\lambda)Z^k\},\tag{2}$$

$$A_k(\lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{iku}\log\frac{f(\lambda,u)}{g_X(\lambda)}\mathrm{d}u\quad(k=0,1,2,\cdots).\tag{3}$$

记 $\delta^2(n_0+1)$ 是预测误差，则

$$\delta^2(n_0+1)=2\pi\int_{-\pi}^{\pi}\sum_{k=0}^{n_0}|\varphi_k(\lambda)|^2g_X(\lambda)\mathrm{d}\lambda,\quad E|\frac{\mathrm{d}Z_x(\lambda,\mu)}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-ik\mu}}|^2=g_X(\lambda)\mathrm{d}\lambda\mathrm{d}\mu.\tag{4}$$

引理 2[9] 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,\mu)$ 为正，则预测问题 $9(N,M)$(即 $T=\{(m,n),-\infty<m<\infty,n=n_i<0,i=1,2,\cdots,N\}$. $\cup\{(m,n),-\infty<m<\infty,n=l_j>0,j=1,2,\cdots,M$ (当 $M=0$, 定义这集合是空集)}.

$\cup\{(m,0),m\le -1\}\triangleq T_{9(N,M)};T'=\{(m',0),m'\ge 0\}$) 线性预测 $X(m',0)(m'\ge 0)$ 的预测值为

$$P_{H_X(T_{9(N,M)})}X(m',0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_{9(N,M)}^{(m')}(\lambda,u)\mathrm{d}Z_X(\lambda,u),\tag{5}$$

$$C_{9(N,M)}^{(m')}(\lambda,\mu)=\begin{cases}e^{im'\lambda}\sum\limits_{k=n,\cdots n_N,e_1,e_M}\Phi_k(\lambda)e^{ik\mu}+e^{im'\lambda}\Big[\dfrac{\sum_{k=m'+1}^{\infty}h_ke^{-ik\lambda}}{\sum_{k=0}^{\infty}h_ke^{-ik\lambda}}\Big]\\ \quad\times\big[1-\sum\limits_{k=n_1,\cdots n_N,\ell_1,\cdots,\ell_M}\Phi_k(\lambda)e^{ik\mu}\big],\quad 当\ |\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda|<\infty;\\ e^{im'\lambda},\qquad\qquad\qquad\qquad\qquad\qquad 当\ |\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda|=\infty\end{cases}\tag{6}$$

$$f_y(\lambda)=\int_{-\pi}^{\pi}|1-\sum_{k=n_1,\cdots n_N,\ell_1,\cdots\ell_M}\Phi_k(\lambda)e^{ik\mu}|^2f(\lambda,\mu)\mathrm{d}\mu,\tag{7}$$

其中 $\{h_k\}$ 由下列式决定：

$$\exp\{\tfrac{D_0}{2}+\sum_{k=1}^{\infty}D_kZ^k\}=\sum_{k=0}^{\infty}h_kZ^k,\quad D_k=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda,(k=0,1,2,\cdots)\tag{8}$$

$\Phi_k(\lambda)$ 可由下列 $N+M$ 阶线性方程组解得

$$\begin{aligned}&\sum_{k=n_1,\cdots n_N,\ell_1,\cdots,\ell_M}\Phi_k(\lambda)\int_{-\pi}^{\pi}e^{i(k-q)\mu}f(\lambda,\mu)\mathrm{d}\mu\\&=\int_{-\pi}^{\pi}e^{-iq\mu}f(\lambda,\mu)\mathrm{d}\mu(q=n_1,\cdots n_N,\ell_1,\cdots\ell_M),\end{aligned}\tag{9}$$

且预测问题 9(N, M) 线性预测 $X(m',0)(m'\geq 0)$ 的预测误差为

$$\delta^2_{9(N,M)}(m')=\begin{cases}2\pi\sum\limits_{k=0}^{m'}|h_k|^2,\ \text{当}\ |\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda|<\infty,\\ 0,\ \text{当}\ |\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda|=\infty.\end{cases}\tag{10}$$

引理 3[9] 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,\mu)$ 为正，则预测问题 $11(N,M)$ 即

$$\begin{aligned}T=&\{(m,n),-\infty<m<\infty,n=n_i<0,i=1,2,\cdots,N\}\\ &\cup\{(m,n),-\infty<m<\infty,n=\ell_j>0,j=1,2,\cdots,M(\text{当 } M=0,\text{ 定义这集合是空集})\}\\ &\cup\{(m,0),-\infty<m<\infty,m\neq 0\}\triangleq T_{11(N,M)};\end{aligned}$$

$T'=\{(0,0)\}$) 线性预测 $X(0,0)$ 的预测值为

$$P_{H_X(T_{11(N,M)})}X(0,0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C^{(0)}_{11(N,M)}(\lambda,\mu)\mathrm{d}Z_X(\lambda,\mu)\tag{11}$$

$$C^{(0)}_{11(N,M)}(\lambda,u)=\begin{cases}\sum\limits_{k=n_1,\cdots,n_N,\ell_1,\cdots,\ell_M}\Phi_k(\lambda)e^{iku}+[1-\dfrac{2\pi}{f_y(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_y(\lambda)}}]\\ \quad\times\big[1-\sum\limits_{k=n_1,\cdots,n_N,\ell_1,\cdots,\ell_M}\Phi_k(\lambda)e^{iku}\big],\ \text{当}\ \int_{-\pi}^{\pi}\dfrac{\mathrm{d}\lambda}{f_y(\lambda)}<\infty\\ 1,\qquad\qquad\qquad\qquad\qquad\qquad\text{当}\ \int_{-\pi}^{\pi}\dfrac{\mathrm{d}\lambda}{f_y(\lambda)}=\infty\end{cases}\tag{12}$$

且预测问题 $11(N,M)$ 线性预测 $X(0,0)$ 的预测误差为

$$\delta^2_{11(N,M)}=\begin{cases}4\pi^2\Big/\int_{-\pi}^{\pi}\dfrac{\mathrm{d}\lambda}{f_y(\lambda)},\quad\text{当}\ \int_{-\pi}^{\pi}\dfrac{\mathrm{d}\lambda}{f_y(\lambda)}<\infty\\ 0,\qquad\qquad\qquad\text{当}\ \int_{-\pi}^{\pi}\dfrac{\mathrm{d}\lambda}{f_y(\lambda)}=\infty.\end{cases}\tag{13}$$

引理 4[10,11] 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,u)$ 为正，则预测问题 $5(1,0)$(即 $T=T_{5(1,0)},T'=\{(m',0),m'\geq 0\}$) 线性预测 $X(0,0)$ 的预测误差为

$$\delta^2_{5(1,0)}=\begin{cases}2\pi\exp\Big\{\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\log\big[4\pi^2/\int_{-\pi}^{\pi}\dfrac{\mathrm{d}u}{f(\lambda,u)}\big]\mathrm{d}\lambda\Big\}>0,\quad\text{当}\ \int_{-\pi}^{\pi}\log\big[\int_{-\pi}^{\pi}\dfrac{\mathrm{d}u}{f(\lambda,u)}\big]\mathrm{d}\lambda<\infty\\ 0,\qquad\qquad\qquad\qquad\qquad\qquad\text{当}\ \int_{-\pi}^{\pi}\log\big[\int_{-\pi}^{\pi}\dfrac{\mathrm{d}u}{f(\lambda,u)}\big]\mathrm{d}\lambda=\infty.\end{cases}\tag{14}$$

当 $\delta^2_{5(1,0)}>0$ 时，其预测值为

$$P_{H_X(T_{5(1,0)})}X(0,0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big[\big(1-\frac{C_0}{\sum_{k=0}^{\infty}C_ke^{-ik\lambda}}\big)\frac{b_0^{(0)}(\lambda)}{f(\lambda,u)}+1-\frac{b_0^{(0)}(\lambda)}{f(\lambda,u)}\Big]\mathrm{d}Z_X(\lambda,u),\tag{15}$$

$b_0^{(0)}(\lambda)=2\pi\big/\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}$, $\{C_k\}$ 由下列式决定：

$$\sum_{k=0}^{\infty}C_kZ^k=\exp\big\{\frac{\tilde{C}_0}{2}+\sum_{k=1}^{\infty}\tilde{C}_kZ^k\big\},\quad \tilde{C}_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ik\lambda}\log f_Z^{(0)}(\lambda)\mathrm{d}\lambda\quad(k=0,1,2,\cdots),\tag{16}$$

$$f_Z^{(0)}(\lambda)=\int_{-\pi}^{\pi}\frac{|b_0^{(0)}(\lambda)|^2}{f(\lambda,u)}\mathrm{d}u=4\pi^2\Big/\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}.\tag{17}$$

引理 5[11] 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,\mu)$ 为正，且满足 $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(\lambda,\mu)]^{-1}\mathrm{d}\lambda\mathrm{d}\mu<\infty$, 则预测问题 $3(N_0,n_0)$(即 $T=\{(m,n),-\infty<m<\infty,n\le-1$ 或 $n\ge N_0\}\cup\{(m,n_0),m=\pm1,\pm2,\cdots\}\triangleq T_{3(N_0,n_0)}$, $T'=\{(0,n_0)\}$, 其中 N_0, n_0 是固定的非负整数， $N_0>n_0\ge0$) 线性预测 $X(0,n_0)$ 的预测误差为：

$$0<\delta_{3(N_0,n_0)}^2=4\pi^2\Big/\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_Z(\lambda)},\quad f_Z(\lambda)=\int_{-\pi}^{\pi}\frac{|\sum_{k=0}^{N_0-1}b_k^{(0)}(\lambda)e^{iku}|^2}{f(\lambda,u)}\mathrm{d}u\tag{18}$$

预测值为 $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_{3(N_0,n_0)}(\lambda,u)\mathrm{d}Z_X(\lambda,u)$, 而

$$C_{3(N_0,n_0)}(\lambda,u)=\Big[1-\frac{2\pi}{f_Z(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_Z(\lambda)}}\Big]\Big[\frac{\sum_{k=0}^{N_0-1}b_k^{(0)}(\lambda)e^{iku}}{f(\lambda,u)}\Big]+e^{in_0u}-\frac{\sum_{k=0}^{N_0-1}b_k^{(0)}(\lambda)e^{iku}}{f(\lambda,u)}\tag{19}$$

$\{b_k^{(0)}(\lambda)\}$ 由下列线性方程组解得：

$$\sum_{k=0}^{N_0-1}b_k^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{e^{i(k-\ell)u}}{f(\lambda,u)}\mathrm{d}u=\int_{-\pi}^{\pi}e^{i(n_0-\ell)u}\mathrm{d}u,\quad \ell=0,1,2,\cdots,N_0-1.\tag{20}$$

引理 6[11] 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,u)$ 为正，且满足

$$\Big|\int_{-\pi}^{\pi}\log f(\lambda,u)\mathrm{d}u\Big|<\infty,\quad \text{a.e. }\mathrm{d}\lambda\tag{21}$$

则预测问题 $6(n_0)$(即 $T=\{(m,n),-\infty<m<\infty,n\le-1\}\cup\{(m,n_0),m=\pm1,\pm2,\cdots\}\triangleq T_{6(n_0)}$; $T'=\{(0,n_0)\}$, 其中 n_0 是非负固定的整数), 线性预测 $X(0,n_0)$ 的预测误差为

$$\delta_{6(n_0)}^2=\begin{cases}4\pi^2\Big/\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(n_0)}(\lambda)}, & \text{当 }\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(n_0)}(\lambda)}<\infty\\ 0, & \text{当 }\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(n_0)}(\lambda)}=\infty\end{cases}\tag{22}$$

当 $\delta_{6(n_0)}^2>0$ 时，预测值为 $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_{6(n_0)}(\lambda,u)\mathrm{d}Z_X(\lambda,u)$, 而

$$\begin{aligned}C_{6(n_0)}(\lambda,u)=&e^{in_0\lambda}\Big(1-\frac{2\pi}{f_\delta^{(n_0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(n_0)}(\lambda)}}\Big)\Big(\frac{\sum_{k=0}^{n_0}\varphi_k(\lambda)e^{-iku}}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}}\Big)+\\&e^{in_0\lambda}\frac{\sum_{k=n_0+1}^{\infty}\varphi_k(\lambda)e^{-iku}}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}},\end{aligned}\tag{23}$$

其中 $f_\delta^{(n_0)}(\lambda)=2\pi\sum_{k=0}^{n_0}|\varphi_k(\lambda)|^2 g_X(\lambda)$, $\{\varphi_k(\lambda)\}$ 由 (2), (3) 决定.

定理 1 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,u)$ 为正，且满足

$$\int_{-\pi}^{\pi}\exp\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\lambda,u)\mathrm{d}u\}\mathrm{d}\lambda<\infty, \tag{24}$$

则对 $X(0,0)$, $H_X(T_{6(0)})$ 关于 $H_X(T_{11(1,0)})$ 具有马氏性的充分必要条件是

$$f(\lambda,u)=\frac{h(\lambda)}{|1-\Phi(\lambda)e^{-iu}|^2},\quad |\Phi(\lambda)|<1,\quad \text{a.e.} \tag{25}$$

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{h(\lambda)}<\infty,\quad \int_{-\pi}^{\pi}\frac{h(\lambda)\mathrm{d}\lambda}{1-|\Phi(\lambda)|^2}<\infty \tag{26}$$

且当充分条件满足时，预测误差为

$$\delta_{11(1,0)}^2=\delta_{6(0)}^2=\frac{8\pi^3}{\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{h(\lambda)}}>0. \tag{27}$$

预测值为

$$\begin{aligned}&P_{H_X(T_{11(1,0)})}X(0,0)=P_{H_X(T_{6(0)})}X(0,0)\\&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\Phi(\lambda)e^{-iu}+\Big[1-\frac{2\pi}{h(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{h(\lambda)}}\Big][1-\Phi(\lambda)e^{-iu}]\right)\mathrm{d}Z_X(\lambda,u).\end{aligned} \tag{28}$$

证明 *充分性*. 根据定义及正交投影的唯一性，只需验证预测误差 $\delta_{6(0)}^2$ 与 $\delta_{11(1,0)}^2$ 相等，由引理 3, 引理 6 及 (24) 得

$$f_\delta^{(0)}(\lambda)=2\pi\,|\varphi_0(\lambda)|^2\,g_X(\lambda)=2\pi\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\lambda,u)\mathrm{d}u\}, \tag{29}$$

$$0<\delta_{6(0)}^2=4\pi^2\Big[\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{2\pi\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\lambda,u)\mathrm{d}u\}}\Big]^{-1}, \tag{30}$$

$$0<\delta_{11(1,0)}^2=4\pi^2\Big[\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_y^{(0)}(\lambda)}\Big]^{-1}. \tag{31}$$

用围道积分及留数定理，通过计算，由 (25) 得 $\Phi(\lambda)=\int_{-\pi}^{\pi}e^{-iu}f(\lambda,u)\mathrm{d}u/g_X(\lambda)$, 又由 (9), 解得 $\Phi_{-1}(\lambda)=\Phi(\lambda)$, 故

$$f_y^{(0)}(\lambda)=\int_{-\pi}^{\pi}\left|1-\Phi_{-1}(\lambda)e^{-iu}\right|^2 f(\lambda,u)\mathrm{d}u=2\pi h(\lambda). \tag{32}$$

由上面式子就可验证 $\delta_{6(0)}^2=\delta_{11(1,0)}^2$. 充分性证毕.

必要性. 由引理 3 及引理 6 得

$$P_{H_X(T_{6(0)})}X(0,0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\Big[[1-\frac{2\pi}{f_\delta^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(0)}(\lambda)}}\Big]\Big[\frac{\varphi_0(\lambda)}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}}\Big]+\right.$$

$$\Big[1-\frac{\varphi_0(\lambda)}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}}\Big]\Big\}\mathrm{d}Z_X(\lambda,u), \tag{33}$$

$$\begin{aligned}&P_{H_X(T_{11(1,0)})}X(0,0)\\&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{\Phi_{-1}(\lambda)e^{-iu}+\Big[1-\frac{2\pi}{f_y^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_y^{(0)}(\lambda)}}\Big][1-\Phi_{-1}(\lambda)e^{-iu}]\mathrm{d}Z_X(\lambda,u)\Big\},\end{aligned} \tag{34}$$

其中 $\Phi_{-1}(\lambda)=\int_{-\pi}^{\pi}e^{-iu}f(\lambda,u)\mathrm{d}u/g_X(\lambda)$, 易知 $|\Phi_{-1}(\lambda)|<1$. 由于对 $X(0,0)$, $H_X(T_{6(0)})$ 关于 $H_X(T_{11(1,0)})$ 具有马氏性及正交投影的唯一性，结合 (33), (34) 得

$$\begin{aligned}&\Big[1-\frac{2\pi}{f_\delta^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(0)}(\lambda)}}\Big]\Big[\frac{\varphi_0(\lambda)}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}}\Big]+1-\frac{\varphi_0(\lambda)}{\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}}\\&=\Phi_{-1}(\lambda)e^{-iu}+\Big[1-\frac{2\pi}{f_y^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_y^{(0)}(\lambda)}}\Big][1-\Phi_{-1}(\lambda)e^{-iu}].\end{aligned} \tag{35}$$

由引理 1, 知

$$f(\lambda,u)=g_X(\lambda)|\sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}|^2,\quad \sum_{k=0}^{\infty}\varphi_k(\lambda)e^{-iku}=(g_X(\lambda))^{-1/2}\sqrt{f(\lambda,u)}e^{i\theta(\lambda,u)}, \tag{36}$$

记， $\Phi(\lambda)=\Phi_{-1}(\lambda)$, $g_1(\lambda)=1-2\pi/f_\delta^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_\delta^{(0)}(\lambda)}$, $g_2(\lambda)=2\pi/f_y^{(0)}(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{f_y^{(0)}(\lambda)}>0$, $h(\lambda)=g_X(\lambda)|\varphi_0(\lambda)|^2|g_1(\lambda)-1|^2/(g_2(\lambda))^2$, 于是 (35) 可简化为 (25). 又由于 (24) 得

$$\infty>\int_{-\pi}^{\pi}\exp\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\lambda,u)\mathrm{d}u\}\mathrm{d}\lambda=\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{h(\lambda)}$$

$$\infty>\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(\lambda,u)\mathrm{d}\lambda\mathrm{d}u=\int_{-\pi}^{\pi}\frac{h(\lambda)\mathrm{d}\lambda}{1-|\Phi(\lambda)|^2}.$$

即得 (26), 必要性证毕. 定理 1 证毕.

定理 2 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,u)$ 为正，且满足

$$\int_{-\pi}^{\pi}\log\big[\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}\big]\mathrm{d}\lambda<\infty, \tag{37}$$

则对 $X(0,0)$, $H_X(T_{5(1,0)})$ 关于 $H_X(T_{9(1,1)})$ 具有马氏性的充分必要条件是

$$f(\lambda,u)=\frac{q(\lambda)}{1-\bar{\psi}(\lambda)e^{-iu}-\psi(\lambda)e^{iu}},\quad |\psi(\lambda)|<\frac{1}{2},\quad \text{a.e.} \tag{38}$$

$$\int_{-\pi}^{\pi}\log q(\lambda)\mathrm{d}\lambda>-\infty,\quad \int_{-\pi}^{\pi}\frac{q(\lambda)\mathrm{d}\lambda}{\sqrt{1-4|\psi(\lambda)|^2}}<\infty. \tag{39}$$

当充分条件满足时，预测误差为

$$0<\delta_{9(1,1)}^2(0)=\sigma_{5(1,0)}^2=4\pi^2\exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log q(\lambda)\mathrm{d}\lambda\}. \tag{40}$$

预测值为

$$\begin{aligned}&P_{H_X(T_{5(1,0)})}X(0,0)=P_{H_X(T_{9(1,1)})}X(0,0)\\&=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big\{\bar{\psi}(\lambda)e^{-iu}+\psi(\lambda)e^{iu}+\Big[1-\frac{h_0}{\sum_{k=0}^{\infty}h_ke^{-ik\lambda}}\Big]\\&\qquad[1-\bar{\psi}(\lambda)e^{-iu}-\psi(\lambda)e^{iu}]\Big\}\mathrm{d}Z_X(\lambda,u),\end{aligned}\tag{41}$$

其中 $\{h_k\}$ 由下列式决定

$$\exp\{\frac{D_0}{2}+\sum_{k=1}^{\infty}D_kZ^k\}=\sum_{k=0}^{\infty}h_kZ^k,\quad D_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log 2\pi q(\lambda)e^{ik\lambda}\mathrm{d}\lambda\quad(k=0,1,2,\cdots).\tag{42}$$

证明 *充分性*. 只需验证预测误差 $\sigma_{5(1,0)}^2$ 与 $\delta_{9(1,1)}^2(0)$ 相等.

由 (37) 及引理 4 得

$$0<\sigma_{5(1,0)}^2=2\pi\exp\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{4\pi^2}{\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}}\mathrm{d}\lambda\Big\}.\tag{43}$$

在引理 2 中取 $N=1$, $M=1$, $n_1=-1, l_1=1$, 由 (9) 得方程组为

$$\Phi_{-1}(\lambda)\int_{-\pi}^{\pi}f(\lambda,u)\mathrm{d}u+\Phi_1(\lambda)\int_{-\pi}^{\pi}e^{2iu}f(\lambda,u)\mathrm{d}u=\int_{-\pi}^{\pi}e^{iu}f(\lambda,u)\mathrm{d}u,\tag{44}$$

$$\Phi_{-1}(\lambda)\int_{-\pi}^{\pi}e^{-2iu}f(\lambda,u)\mathrm{d}u+\Phi_1(\lambda)\int_{-\pi}^{\pi}f(\lambda,u)\mathrm{d}u=\int_{-\pi}^{\pi}e^{-iu}f(\lambda,u)\mathrm{d}u.\tag{45}$$

解之得

$$\Phi_{-1}(\lambda)=\bar{\Phi}_1(\lambda)=\frac{g_X(\lambda)\int_{-\pi}^{\pi}e^{iu}f(\lambda,u)\mathrm{d}u-\int_{-\pi}^{\pi}e^{2iu}f(\lambda,u)\mathrm{d}u\int_{-\pi}^{\pi}e^{-iu}f(\lambda,u)\mathrm{d}u}{|g_X(\lambda)|^2-\big|\int_{-\pi}^{\pi}e^{2iu}f(\lambda,u)\mathrm{d}u\big|^2}.\tag{46}$$

由 (38), 通过计算可验证 $\bar{\psi}(\lambda),\psi(\lambda)$ 也满足方程组 (44), (45), 故 $\bar{\psi}(\lambda)=\Phi_{-1}(\lambda)$, $\psi(\lambda)=\Phi_1(\lambda)$. 这时

$$\begin{aligned}&\delta_{9(1,1)}^2(0)=2\pi|h_0|^2,\quad h_0=\exp\{\frac{D_0}{2}\},\quad D_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f_y(\lambda)\mathrm{d}\lambda,\\&f_y(\lambda)=\int_{-\pi}^{\pi}|1-\Phi_{-1}(\lambda)e^{-iu}-\Phi_1(\lambda)e^{iu}|^2f(\lambda,u)\mathrm{d}u=2\pi q(\lambda).\end{aligned}\tag{47}$$

从上面式子, 通过计算就可验证 $\sigma_{5(1,0)}^2=\delta_{9(1,1)}^2(0)$, 充分性证毕.

必要性. 若 $P_{H_X(T_{9(1,1)})}X(0,0)=P_{H_X(T_{5(1,0)})}X(0,0)$, 由引理 2, 引理 4 得

$$\begin{aligned}&\Phi_{-1}(\lambda)e^{-iu}+\Phi_1(\lambda)e^{iu}+\Big[\frac{\sum_{k=1}^{\infty}h_ke^{-ik\lambda}}{\sum_{k=0}^{\infty}h_ke^{-ik\lambda}}\Big][1-\Phi_{-1}(\lambda)e^{iu}-\Phi_1(\lambda)e^{iu}]\\&=\frac{\sum_{k=1}^{\infty}C_ke^{-ik\lambda}}{\sum_{k=0}^{\infty}C_ke^{-ik\lambda}}\Big[\frac{2\pi}{f(\lambda,u)\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}}\Big]+1-\frac{2\pi}{f(\lambda,u)\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}}\end{aligned}\tag{48}$$

记 $g_3(\lambda)=\sum_{k=1}^{\infty}h_ke^{-ik\lambda}/\sum_{k=0}^{\infty}h_ke^{-ik\lambda}$, $g_4(\lambda)=\sum_{k=1}^{\infty}C_ke^{-ik\lambda}/\sum_{k=0}^{\infty}C_ke^{-ik\lambda}$, 则

$$1-g_3(\lambda)=\frac{h_0}{\sum_{k=0}^{\infty}h_ke^{-ik\lambda}}\neq 0,\quad 1-g_4(\lambda)=\frac{C_0}{\sum_{k=0}^{\infty}C_ke^{-ik\lambda}}\neq 0,$$

$$f(\lambda,u)[1-\Phi_{-1}(\lambda)e^{-iu}-\Phi_1(\lambda)e^{iu}][1-g_3(\lambda)]=(1-g_4(\lambda))\frac{2\pi}{\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}}\neq 0. \tag{49}$$

故若记 $q(\lambda)=(1-g_4(\lambda))2\pi/(1-g_3(\lambda))\int_{-\pi}^{\pi}\frac{\mathrm{d}u}{f(\lambda,u)}$, $\Phi_1(\lambda)=\psi(\lambda)$, 则即得 (38) 式. 下证 $|\Phi_1(\lambda)|<\frac{1}{2}$, a. e., 用反证法. 若入在一个正测度集合 A 上处处有 $|\Phi_1(\lambda)|\geq\frac{1}{2}$, 则把 (49) 改写为

$$\Phi_1(\lambda)e^{-iu}\Big[\frac{\bar{\Phi}_1(\lambda)}{\Phi_1(\lambda)}+e^{Ziu}-\frac{e^{iu}}{\Phi_1(\lambda)}\Big]f(\lambda,u)=-q(\lambda). \tag{50}$$

注意二次方程 $Z^2-\frac{1}{\Phi_1(\lambda)}Z+\frac{\bar{\Phi}_1(\lambda)}{\Phi_1(\lambda)}=0$ 的二个根为 $Z_1=\frac{1+\sqrt{1-4|\Phi_1(\lambda)|^2}}{2\Phi_1(\lambda)}$, $Z_2=\frac{1-\sqrt{1-4|\Phi_1(\lambda)|^2}}{2\Phi_1(\lambda)}$. 当 $\lambda\in A$ 时， $|\Phi_1(\lambda)|\geq\frac{1}{2}$, 这时容易验证 $|Z_1|=|Z_2|=1$. 记 $Z_1=e^{i\theta_1(\lambda)}$, $Z_2=e^{i\theta_2(\lambda)}$, 则这时 (50) 又可改写为

$$\Phi_1(\lambda)e^{-iu}(e^{iu}-e^{i\theta_1(\lambda)})(e^{iu}-e^{i\theta_2(\lambda)})f(\lambda,u)=-q(\lambda),(\lambda,u)\in A\otimes[-\pi,\pi]. \tag{51}$$

当 $u=\theta_1(\lambda)$ 时，就使 $q(\lambda)=0$, 这与 (49) 矛盾，故证得 $|\Phi_1(\lambda)|<\frac{1}{2}$, a.e. 又由于

$$\infty>\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(\lambda,u)\mathrm{d}\lambda\mathrm{d}u=2\pi\int_{-\pi}^{\pi}\frac{q(\lambda)\mathrm{d}\lambda}{\sqrt{1-4|\Phi(\lambda)|^2}};$$

$$0<\delta_{9(1,1)}^2(0)=2\pi h_0=4\pi^2\exp\{\int_{-\pi}^{\pi}\log q(\lambda)\mathrm{d}\lambda\},$$

即可知 $\int_{-\pi}^{\pi}\log q(\lambda)\mathrm{d}\lambda>-\infty$, 定理 2 证毕.

定理 3 设齐次随机场 $\{X(m,n)\}$ 的谱函数绝对连续，谱密度 $f(\lambda,u)$ 为正，且满足

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(\lambda,u)]^{-1}\mathrm{d}\lambda\mathrm{d}u<\infty,$$

则对 $X(0,0)$, $H_X(T_{3(1,0)})$ 关于 $H_X(T_{11(1,1)})$ 具有马氏性的充分必要条件是

$$f(\lambda,u)=\frac{r(\lambda)}{1-\bar{J}(\lambda)e^{-iu}-J(\lambda)e^{iu}},\quad |J(\lambda)|<\frac{1}{2},\ \text{a.e.}$$

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{r(\lambda)}<\infty,\int_{-\pi}^{\pi}\frac{r(\lambda)\mathrm{d}\lambda}{\sqrt{1-4|J(\lambda)|^2}}<\infty.$$

当充分条件满足时，预测误差为

$$\delta_{11(1,1)}^2=\sigma_{3(1,0)}^2=8\pi^3\Big/\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{r(\lambda)}>0.$$

预测值为

$$P_{H_X(T_{11(1,1)})}X(0,0)=P_{H_X(T_{3(1,0)})}X(0,0)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}C_3(\lambda,u)\mathrm{d}Z_X(\lambda,u),$$

$$C_3(\lambda,u)=\bar{J}(\lambda)e^{-iu}+J(\lambda)e^{iu}+\Big[1-\frac{2\pi}{r(\lambda)\int_{-\pi}^{\pi}\frac{\mathrm{d}\lambda}{r(\lambda)}}\Big][1-\bar{J}(\lambda)e^{-iu}-J(\lambda)e^{iu}].$$

证明 在引理 3 中取 $N=1$, $M=1$, $n_1=-1$, $l_1=1$ 及在引理 5 中取 $N_0=1$, $n_0=0$, 用定理 2 的证明方法即可证明定理 3. 故从略.

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Linear Prediction Theory and Markov Models of a Homogeneous Random Field with Discrete Parameters (II)

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Abstract: Generally, the definations of the Markov type for homogeneous random field as follows:

Let $H_X(T)$ be the closed linear manifold spanned by all $X(m,n),(m,n)\in T,T_0\subset T$. $(m',n')\bar{\in}T$, if $P_{H_X(T)}X(m',n')=P_{H_X(T_0)}X(m',n')$, then we say that $H_X(T)$ has the Morkov property for $H_X(T_0)$ at $X(m',n')$. In this paper, three types are posed and discussed:

1). $T=\{(m,n),-\infty<m<\infty,n\le 0,(m,n)\ne(0,0)\}$; $T_0=\{(m,-1),-\infty<m<\infty$, or $(m,0),-\infty<m<\infty,m\ne 0\}$, $X(m',n')=X(0,0)$.

2). $T=\{(m,n),-\infty<m,n<\infty,n\ne 0$, or $(m,0),-\infty<m\le -1\}$;
$T_0=\{(m,n),-\infty<m<\infty,n=\pm 1$, or $(m,0),-\infty<m\le -1\}$, $X(m',n')=X(0,0)$.

3). $T=\{(m,n),-\infty<m,n<\infty,(m,n)\ne(0,0)\}$;
$T_0=\{(m,n),-\infty<m<\infty,n=\pm 1$, or $(m,0),-\infty<m\le -1\}$, $X(m',n')=X(0,0)$.

Key words: linear prediction; predictor; predictor error; Markov property.