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Self-Converse Mendelsohn Designs with Odd Prime Block Size

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Abstract: A Mendelsohn design MD(v,k,λ) is a pair $(X,\mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of k-tuples from X such that each ordered pair from X is contained in exactly λ k-tuples of $\mathcal{B}$. An MD(v,k,λ) is called self-converse and denoted by SCMD$(v,k,\lambda)=(X,\mathcal{B},f)$, if there exists an isomorphic mapping f from $(X,\mathcal{B})$ to $(X,\mathcal{B}^{-1})$. In this paper, using difference method, we give a constructive proof for the existence of SCMD$(4mp,p,1)$, where p is an odd prime and m is a positive integer.

Key words: self-converse Mendelsohn design; difference cycle; SDC; UDC; CDC.
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1. Introduction

Let X be a v-set, and k be an integer, $3\le k\le v$. A cyclic k-tuple from X is a collection of k ordered pairs $(x_0,x_1),(x_1,x_2),\cdots,(x_{k-2},x_{k-1})$ and (x_{k-1},x_0), where $x_0,x_1,\cdots,x_{k-1}$ are distinct elements of X, denoted by $\langle x_0,x_1,\cdots,x_{k-1}\rangle$ or its any cyclic shift $\langle x_i,x_{i+1},\cdots,x_{k-1},x_0,x_1,\cdots,x_{i-1}\rangle$. A (v,k,λ)-Mendelsohn design (or MD(v,k,λ)) is a pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a collection of cyclic k-tuples (called blocks) from X, such that each ordered pair of distinct elements of X belongs to exactly λ blocks of $\mathcal{B}$.

For an MD$(v,k,\lambda)=(X,\mathcal{B})$, we define

$$\mathcal{B}^{-1}=\{B^{-1}=\langle x_{k-1},x_{k-2},\cdots,x_1,x_0\rangle;B=\langle x_0,x_1,\cdots,x_{k-1}\rangle\in\mathcal{B}\}.$$

Obviously, $(X,\mathcal{B}^{-1})$ is also an MD(v,k,λ), which is called converse of $(X,\mathcal{B})$. If there exists an isomorphic mapping f from $(X,\mathcal{B})$ to $(X,\mathcal{B}^{-1})$, then the MD(v,k,λ) is called self-converse and denoted by SCMD$(v,k,\lambda)=(X,\mathcal{B},f)$. Especially, it is denoted by k-SCMD(v) when $\lambda=1$.

It is well known that the necessary condition for the existence of an MD(v,k,λ) is $\lambda v(v-1)\equiv 0(\mathrm{mod}\ k)$. C.J.Colbourn and A.Rosa posed an open problem in their survey[1]: For what orders do self-converse MTS (i.e SCMD$(v,3,1)$) exist? The existence of SCMD$(v,3,\lambda)$, SCMD$(v,4,1)$ and SCMD$(v,5,1)$; SCMD$(v,2p^m,1)$; SCMD$(mk+1,k,1)$ and SCMD$(2k,k,1)$ had been completely settled in [3],[4],[5] and [6], respectively, where p is an odd prime, $m>0$ and k (>5) is odd. In this paper, by using difference method, the following result is obtained.

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Theorem 1.1 *There exists a p-SCMD$(4mp)$, for any odd prime p and positive integer m.*

2. Definition and main ideas

Let $X = Z_v$, where Z_v is the ring of integers modulo v, $\mathcal{S} = \{(x_1, x_2)|x_1, x_2 \in X, x_1 \neq x_2\}$ and $\mathcal{C}$ is the set of all cyclic k-tuple of X. For a permutation f on X, $B = \langle x_0, x_1, \cdots, x_{k-1}\rangle \in \mathcal{C}$ and $P = (y_1, y_2) \in \mathcal{S}$, we define

$$f(B) = \langle f(x_0), f(x_1), \cdots, f(x_{k-1})\rangle,$$

$$Rf(B) = (f(B))^{-1} = \langle f(x_{k-1}), \cdots, f(x_1), f(x_0)\rangle,$$

$$f(P) = (f(y_1), f(y_2)); Rf(P) = (f(P))^{-1} = (f(y_2), f(y_1)).$$

The finite permutation group on $\mathcal{S}$ and $\mathcal{C}$ generated by Rf gives an orbit-partition of $\mathcal{S}$ and $\mathcal{C}$ respectively. The orbit of $\mathcal{S}$ is called a pair orbit and that of $\mathcal{C}$ is called a block orbit. The pair-orbit containing pair P is denoted by $O(P)$. Similarly, the block-orbit containing block B is denoted by $O(B)$. Obviously, if $(X, \mathcal{B})$ is an MD(v, k, λ) and Rf$(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$, then $(X, \mathcal{B}, f)$ is an SCMD(v, k, λ). So to construct an SCMD$(v, k, 1) = (X, \mathcal{B})$, we need only to choose some block-orbits of $\mathcal{C}$, such that each $P \in \mathcal{S}$ is contained in exactly one block of a unique block-orbit. In the following, using difference method, we choose such a family of block orbits.

Let $d \in Z_v^* = Z_v \backslash \{0\}$. We call d a difference on Z_v. For each difference d on Z_v, the set of pairs $\{(a, a+d) \mid a \in Z_v\}$ corresponding to d is called a *pair set with* d, denoted by $PS(d)$. It is easy to see there are totally $v-1$ differences on Z_v and all the PSs form a partition of $\mathcal{S}$. Let $d_1, d_2, \cdots, d_m$ be differences on Z_v. The ordered sequence $D = (d_1, d_2, \cdots, d_m)$ is called a difference tuple on Z_v with size m, denoted simply by m-$DT(D)$. The corresponding number tuple $(i, i+d_1, i+d_1+d_2, \cdots, i+\sum_{j=1}^{m} d_j), i \in Z_v$, is denoted by $\bar{D}_i$. Furthermore, define

$$\begin{aligned}
-D =&(-d_1, -d_2, \cdots, -d_m),\\
A(D) =&(d_1, -d_2, \cdots, (-1)^{m-1}d_m),\\
D^{-1} =&(d_m, d_{m-1}, \cdots, d_1),\\
[a, a+mk]_k =&(a, a+k, \cdots, a+mk), a \in Z_v,\\
abs(D) =&\{|d_1|, |d_2|, \cdots, |d_m|\},\\
abs(D_i) =&\{|i|, |i+d_1|, |i+d_1+d_2|, \cdots, |i+\sum_{j=1}^{m} d_j|\},
\end{aligned}$$

where k is a positive integer and the subscript k may be omitted when $k=1$.

If the terms in $\bar{D}_0$ are distinct, then D is called a difference path, denoted by m-DP(D). Obviously, if D is a DP then both $-D$ and D^{-1} are DPs. If the tail of $\bar{D}_0$ is 0 and all the other terms in $\bar{D}_0$ are distinct then D is called a difference cycle, denoted by m-DC(D). Similarly, if D is a DC then D^{-1} is a DC too. There is a set of k-block orbits $\{O(\bar{D}_i); i \in Z_v\}$ corresponding to each k-DC(D). However, there usually exist repeated pairs in these block orbits. So to construct $\mathcal{B}$, we need only to choose some particular k-DCs such that:

(i) If d (or $-d$) is contained in a DC(D) then each pair contained in PS($\pm d$) repeats just once in the block orbits corresponding to DC(D).

(ii) Every d or $-d$ on Z_v^* appears exactly in a unique DC (we call these DCs form a partition of $[1,\frac{v}{2}]$).

If we take the set of blocks contained in the orbits corresponding to those chosen k-DCs as $\mathcal{B}$ then $(X,\mathcal{B},f)$ is an SCMD$(v,k,1)$.

In this paper, we discuss the case $X=Z_v$, $k=p$ and $f=(0,1,\cdots,v-1)$, where $v=4sp$, s is a positive integer, and p is an odd prime. It is easy to see that if $d\in {Z_v}^*\backslash\{\frac{v}{2}\}$, the two pair-orbits (called complementary pair-orbits): $\{(2i,2i+d),(2i+d+1,2i+1);0\le i\le\frac{v}{2}-1\}$ and $\{(2i+1,2i+d+1),(2i+d,2i);0\le i\le\frac{v}{2}-1\}$ exactly cover PS($\pm d$). But for the difference $\frac{v}{2}$, the only pair-orbit: $\{(i,i+\frac{v}{2});0\le i\le v-1\}$ exactly cover $PS(\frac{v}{2})$.

Let D be a DC on Z_v. The following kinds of DCs are used to give $\mathcal{B}$.

(1) SDC(D), where D satisfies: 1) D contains exactly one $\frac{v}{2}$. 2) For any two pairs in $\bar{D}_0$, their pair-orbits are distinct. 3) For any pair $P=(x,x+d)$ in $\bar{D}_0,\ d\ne\frac{v}{2}$, there exists a pair P' in $\bar{D}_0$ such that $O(P)$ and $O(P')$ are complementary.

(2) CDC(D), where D satisfies: $\frac{v}{2}\notin D$ and all the elements in abs(D) are distinct.

(3) UDC(D), where D is formed by the same difference repeating p times.

Obviously, from the above definitions and discussion, we have the following lemma.

Lemma 2.1 *Under the group generated by Rf, there hold the following facts.*

(1) There is only one block-orbit with length v corresponding to each SDC(D).

(2) There are two block-orbits with length v corresponding to each CDC(D).

(3) There are two orbits with length $4s$ corresponding to each UDC(D).

It is easy to see that $\{\bar{D}_{2i},\ \bar{D}_{2i+1}^{-1};\ 0\le i\le\frac{v}{2}-1\}$ and $\{\bar{D}_{2i+1},\ \bar{D}_{2i}^{-1};\ 0\le i\le\frac{v}{2}-1\}$ are two complementary block-orbits corresponding to CDC(D), and that $\{\bar{D}_{2i},\ \bar{D}_{2i+1}^{-1};\ 0\le i\le 2s-1\}$ and $\{\bar{D}_{2i+1},\ \bar{D}_{2i}^{-1};\ 0\le i\le 2s-1\}$ are two complementary block-orbits corresponding to an UDC(D). But $\{\bar{D}_{2i},\ \bar{D}_{2i+1}^{-1};\ 0\le i\le\frac{v}{2}-1\}$ is the only orbit corresponding to an SDC(D). Further, if d (or $-d$) is contained in a DC(D) given above, then each pair contained in PS($\pm d$) repeats just once in the blocks contained in the block-orbits corresponding to the DC(D).

3. Sub-structures

In this section we will give some specific p-SCD, p-CDC and p-UDC constructions on $Z_v, v=4sp$.

Remark In this paper, if $d\in Z_v$ then d is limited to $[1-\frac{v}{2},\frac{v}{2}]$.

First, we have Lemma 3.1 and Corollary 3.1 from [4].

Lemma 3.1 *For* DT(D) $=(d_1,d_2,\cdots,d_n)$ *and* $0<d_1<\cdots<d_n\le\frac{v}{2},\pm A(D)$ *are both* DPs.

Corollary 3.1 *Let $0<d<m,a>0$ and $a+km\le\frac{v}{2}$. For* DT(D) $=[a,a+km]_k$ *or* $[a,a+km]_k\backslash\{a+kd\},\pm A(D)$ *are both* DPs.

Lemma 3.2 *Let $M=(d_1,\cdots,d_m)$ be a* DP *on Z_v. If $|\sum_{i=1}^m d_i|=\frac{v}{4}(\mathrm{mod} v), \frac{v}{2}\notin \mathrm{abs}(\bar{M}_o)$ and the terms in $\mathrm{abs}(\bar{M}_o)$ are distinct, then $D=(M,\frac{v}{2},M^{-1})$ is a $(2m+1)$-DC. Furthermore, $(M,\frac{v}{2},M^{-1})$ is an* SDC *when the differences in M are all odd and all the terms in $abs(M)$ are distinct.*

Proof Let $\bar{M}_0=(0,x_1,\cdots,x_m), 0<|x_i|<\frac{v}{2}$. Since $|x_m|=\frac{v}{4}$, it is easy to see that

$$\bar{D}_0=(0,x_1,\cdots,x_m,-x_m,\cdots,-x_1,0).$$

Because the terms in $abs(\bar{M}_0)$ are distinct and $\frac{v}{2}\notin abs(\bar{M}_0)$, we know D is a DC. Furthermore, if the differences in M are all odd, then $O(P)\cup O(P')=\mathrm{PS}(\pm d)$, where $P,P'\in\bar{D}_0\cap\mathrm{PS}(d)$ and $d\in M$. Since the terms in $\mathrm{abs}(M)$ are distinct, so $O(P)\cap O(P')=\phi$, where $P\neq P'\in\bar{D}_0$. D is an SDC by the definition. □

Corollary 3.2a *Let $s,\ t\ (>2)$ be positive integers, $p=4t+1, l=8s, v=lp, n=\left\lfloor\frac{2t-1}{l}\right\rfloor$,*

$$(c,P)=\begin{cases}(l-1,[2l-1,nl-1]_l,[nl+l+1,2tl-3l+1]_l,2tl-2l+4+(-1)^n); & n>0\\ (l+1,[2l+1,2tl-3l+1]_l,2tl-2l+1); & n=0,s>1\\ (l+3,[2l+1,2tl-4l+1]_l,2tl-3l+3,2tl-2l+1); & n=0,s=1\end{cases}$$

*$M=(c,A(P),\frac{8lt-l-4}{4},\frac{v}{2}-1)$, then $D_0=(M,\frac{v}{2},M^{-1})$ is a p-*SDC.

Proof First, let $M=(d_1,\ d_2,\cdots,d_m)$. It is easy to see that $m=2t$, $|\sum_{d\in M}d|\equiv\frac{v}{4}(\mathrm{mod} v)$ and $d_i, i=1,\cdots,2t$, are all odd. If $n>0$, let $\bar{M}_0=(0,x_1,\cdots,x_{2t})$. We have $x_1=l-1, x_2=3l-2, x_3=-1, x_{2t-2}=lt+l+2, x_{2t}=\frac{v}{4}=lt+2s$. Furthermore, the odd elements $(x_{2i-1}, i=1,\cdots,t)$ in $\bar{M}_0$ are monotonic decreasing. The even elements in $\bar{M}_0$ are monotonic increasing except for x_{2t}. So the elements in $abs(\bar{M}_0)$ are distinct. Similarly, we have the other cases. Therefore, $(M,\frac{v}{2},M^{-1})$ is a p-SDC by Lemma 3.2. □

Corollary 3.2b *Let $s,\ t\ (>0)$ be non-negative integers, $p=4t+3$, $l=8s+4$, $v=lp$, $n=\left\lfloor\frac{2t}{l}\right\rfloor$, and*

$$M=A([l-1,nl-1]_l,[nl+l+1,2tl-2l+1]_l,2lt-\frac{l}{2}-(-1)^n,2lt+l-1,2lt+\frac{5l}{4}).$$

*Then $D_0=(M,\frac{v}{2},M^{-1})$ is a p-*SDC.

Proof Let $M=(d_1,d_2,\cdots,d_m)$. It is easy to see that $m=2t+1$, $|\sum_{d\in M}d|\equiv\frac{v}{4}(\mathrm{mod} v)$ and $d_i, i=1,\cdots,2t+1$, are all odd. Furthermore $\mathrm{abs}(M)$ is monotonically increasing and $-\frac{v}{2}<d_{2i}<0, 0<d_{2i-1}<\frac{v}{2}$. From the above structure of M, we know: Let $\bar{M}_0=(0,x_1,\cdots,x_{2t+1})$ then $-\frac{v}{2}<x_{2t}<\cdots<x_4<x_2<0$ and $0<x_1<x_3<\cdots<2t+1<\frac{v}{2}$, where x_{2i} are all even and x_{2i-1} are all odd. So all the elements in $\mathrm{abs}(\bar{M}_0)$ are distinct and $(M,\frac{v}{2},M^{-1})$ is a p-SDC by Lemma 3.2. □

Lemma 3.3 *Let $M=(d_1,d_2,\cdots,d_m)$ be a DP on Z_v. If M satisfies:*

(I) The terms in $\mathrm{abs}(M)$ are distinct; $d_2,\cdots,d_m$ are all odd and $d_1\ (\neq\ \frac{v}{2})$ is even; $|\sum_{i=1}^m d_i|\equiv v/4(\mathrm{mod}\ v)$;

(II) $|d_2|, v/2 \notin \mathrm{abs}(\bar{M}_0)$ *and the terms in* $\mathrm{abs}(\bar{M}_0)$ *are distinct, then* $D = (M, v/2, (M\backslash\{d_1, d_2\})^{-1}, d_1, d_2)$ *is an* SDC.

Proof Since d_1 is even and d_2 is odd, we have the proof similar to the that of Lemma 3.2. □

Corollary 3.3a *Let* s, t (> 2) *be nonnegative integers,* $p = 4t+1, l = 8s+4, v = lp, n = \left\lfloor \frac{2t-1}{l} \right\rfloor$, $d_1 = 2lt - \frac{l}{4} - 1$, $d' = 2tl - 2l + (-1)^n + 4$ *and*

$$M = \begin{cases} (d_1, l-1, A([2l-1, nl-1]_l, [nl+l+1, 2tl-3l+1]_l, d'), \frac{v}{2}-1), & n > 0, \\ (d_1, l+1, A([2l+1, 2tl-2l+1]_l), \frac{v}{2}-1), & n = 0, \end{cases}$$

then $D_0 = (M, \frac{v}{2}, (M\backslash\{d_1, d_2\})^{-1}, d_1, d_2)$ *is a* p-SDC*, where* d_2 *is the second term of* M*.*

Proof Let $M = (d_1, d_2, \cdots, d_m)$. It is easy to see that $m = 2t$, $|\sum_{d \in M} d| \equiv \frac{v}{4} (\mathrm{mod} v)$, $d_1 (\neq \frac{v}{2})$ is even, d_i, $i = 2, \cdots, 2t$, are all odd and the terms in $\mathrm{abs}(M)$ are distinct. If $n > 0$, let $\bar{M}_0 = (0, x_1, \cdots, x_{2t})$, then we have: $x_1 = d_1, x_2 = -2tl - \frac{l}{4} - 2$ ($x_2 = 8t+1$ when $s = 0$), $x_{2t-1} = 1 - \frac{v}{4}, x_{2t} = \frac{v}{4}$ and x_{2i} are all odd, x_{2i-1} are all even for $i = 1, \cdots, t$. By Lemma 3.1, $\{x_{2i}\}_{(i>1)}$ is monotonically decreasing by l (where $x_{2i} > 0, x_4 = 2tl - \frac{l}{4} - 2$) and $\{x_{2i+1}\}_{(i \geq 1)}$ is monotonically increasing by l (where $x_{2i+1} < 0, x_3 = -2tl + \frac{7l}{4} - 3$). So $|d_2|, \frac{v}{2} \notin abs(\bar{M}_0)$ and the terms in $\mathrm{abs}(\bar{M}_0)$ are distinct. Similarly we have the case of $n = 0$. Thus D_0 is a p-SDC by Lemma 3.3. □

Corollary 3.3b *Let* s, t *be positive integers,* $p = 4t+3, l = 8s, v = lp, n = \left\lfloor \frac{2t}{l} \right\rfloor, d_1 = \frac{8lt+5l}{4}$, *and*

$$M = (d_1, A([l-1, nl-1]_l, [nl+l+1, 2tl-2l+1]_l, 2tl - \frac{l}{2} - (-1)^n, 2tl+l-1)).$$

Then $D_0 = (M, \frac{v}{2}, (M\backslash\{d_1, d_2\})^{-1}, d_1, d_2)$ *is a* p-SDC, where

$$d_2 = \begin{cases} l-1, & n > 0, \\ l+1, & n = 0, t > 1, \\ 12s-1, & n = 0, t = 1 \end{cases}$$

is the second term of M.

Proof The proof is similar to that of Corollary 3.3a. □

Lemma 3.4 *Let* s, t *be positive integers,* $p = 2t+1$, $v = 4sp$ *and* $i \in [1, 2t]$, *then the* $p - DT(D) = (4si, 4si, \cdots, 4si)$ *is an* UDC (*denoted simply by* $U(i)$).

Proof Since p is an odd prime and $i \in [1, p-1]$, $(im, p) = 1$, where $m \in [1, p-1]$. So D is an UDC by the definition. □

Lemma 3.5 *Let* s, t (> 2), x, i *be positive integers and* $t + 2i - 1 < x - i$, $k = 2t+1$, $v = 4sk$, $M = (d_1, \cdots, d_{2t-2})$,

$$(c_1, c_2, c_3) = \begin{cases} (x+i, t-2i-1, i-x) & \text{Case 1 (where } 2i+1 < t) \\ (x-i, -x-i, t+2i-1) & \text{Case 2} \\ (x-i+1, -x-i, t+2i-2) & \text{Case 3} \\ (x+i, t-2i, i-x-1) & \text{Case 4 (where } 2i < t) \end{cases}.$$

$D=(M,c_1,c_2,c_3)$ satisfies:

(1) $\mathrm{abs}(M)\subset[1,\frac{v}{2}-1]\backslash\{|c_1|,|c_2|,|c_3|\}$ *and* $\mathrm{abs}(M)$ *is monotonically increasing.*

(2) $d_{2j-1}+d_{2j}=-1, j=1,\cdots,t-1; |d_{2t-2}|<x\pm i<\frac{v}{2}$.

Then D is $k-CDCs$ for the above four cases of (c_1,c_2,c_3).

Proof By the structure of D and conditions (1),(2), it is easy to see that $|\sum_{d\in D}d|\equiv 0(\mathrm{mod}\ v)$ and the positive elements in $\bar{D}_0$ are monotonically increasing, the negative elements in $\bar{D}_0$ are monotonically decreasing. So D are CDCs by the definition of CDC. □

Lemma 3.6 *Let s, t (>2), x be positive integers, $i=\pm1, x+i>2t-2i-1$ and $k=4t+1, v=4sk$, $M=(d_1,\cdots,d_{4t-2})$.*

$$D=(d_1,\ d_2,\cdots,d_{4t-4},\ x+i,\ d_{4t-2},\ d_{4t-3},\ 2t-2i-1,i-x)$$

and satisfies:

(1) $\mathrm{abs}(M)$ *is monotonically increasing and* $d_{2j-1}+d_{2j}=-1, i=1,\cdots,2t-1;$

(2) $2t-2i-1\notin\mathrm{abs}(M)$ *and* $|d_{4t-4}|+1<x+i<|d_{4t-3}|<|d_{4t-2}|<\frac{v}{2}$.

Then D is a CDC.

Proof By the structure of D and conditions (1), (2), we have $|\sum_{d\in D}d|\equiv 0(\mathrm{mod}\ v)$ and the negative elements in $\bar{D}_0$ are monotonically decreasing, the positive elements in $\bar{D}_0$ are

$$d_1,d_3-1,\cdots,d_{4t-5}-2t+3,x-2t+i+2,x-2t+i+1,x-i.$$

It is easy to see the above sequence is monotonically increasing except for the term $x-2t+i+1$. Since $|d_{4t-4}|+1<x+i$, so $x-2t+i+1>d_{4t-5}-2t+3$ and the positive elements in $\bar{D}_0$ are distinct. Thus D is a CDC by the definition of CDC. □

4. The Proof of Theorem 1.1

In this section, we will give a constructive proof for p-SCMD$(4mp)=(X,\mathcal{B},f)$, where p is an odd prime, the point set X is $Z_v, v=4mp$, and the mapping $f=(0,1,\cdots,v-1)$. Also, the block set $\mathcal{B}$ consists of a unique SDC, $\frac{p-1}{2}$ UDCs and n CDCs, where $\frac{v(v-1)}{p}=2nv+2\cdot\frac{p-1}{2}\cdot\frac{v}{p}+v$ by Lemma 2.1. So the number n of CDCs should be $\frac{v-2p}{2p}=2m-1$. Furthermore, in order to verify the correction of the given construction, we need only to show:

(i) Each given DC(D) is an SDC(D) or CDC(D) or UDC(D) by the conclusion in Section 3.

(ii) For the chosen DC$(D)s$, all the abs$(D)s$ form a partition of $[1,\frac{v}{2}]$.

Theorem 4.1 *Let s, $t(>2)$ be positive integers and $4t+1$ be a prime. Then there exists a* $(4t+1)$-SCMD$(8s(4t+1))$.

Construction Let $v=8s(4t+1), X=Z_v$, $n=\left\lfloor\frac{2t-1}{8s}\right\rfloor$, $d=16st+2s-2$, $d'=16st-2s-1$.

Part 1. SDC(D_0) given in Corollary 3.2a.

Part 2. UDC(D) $= U(i), i = 1, \cdots, 2t$, given by Lemma 3.4.

Part 3. CDC(D_i)$, i = 1, \cdots, 4s-1$, where D_i is given as follows:

$$D_i = \begin{cases} (d_{(4t-2)(i-1)+1}, \cdots, d_{(4t-2)i}, R_i), & 1 \le i \le 2s-1, \\ (d_{(4t-2)(i-1)+1}, \cdots, d_{(4t-2)i}, T_{i-2s+2}), & 2s \le i \le 4s-2, \\ (d_{(4t-2)(i-1)+1}, \cdots, d_{(4t-2)i-2}, a_1, d_{(4t-2)i}, d_{(4t-2)i-1}, a_2, a_3), & i = 4s-1. \end{cases}$$

Here (Case 1. $2t-1-8sn > 4s$; Case 2. $2t-1-8sn < 4s$)

$$R_1 = \begin{cases} (16st-16s-1, 2t-4s, 20s-16st) & \text{Case } 1 \\ (16st-20s, 16s-16st+1, 2t+4s-2) & \text{Case } 2 \end{cases},$$

$$R_i = \begin{cases} (d+i+1, 2t-2i, -(d-i+2)) & \text{Case } 1 \\ (d-i+2, -(d+i+1), 2t+2i-2) & \text{Case } 2 \end{cases}, i = 2, \cdots, 2s-1,$$

$$T_i = \begin{cases} (d'+i, 2t-2i-1, i-d') & \text{Case } 1 \\ (d'-i, -(d'+i), 2t+2i-1) & \text{Case } 2 \end{cases}, i = 2, \cdots, 2s,$$

$$(a_1, a_2, a_3) = T_1 = \begin{cases} (d'+1, 2t-3, 1-d') & \text{Case } 1 \\ (d'-1, 2t+1, -(d'+1)) & \text{Case } 2 \end{cases},$$

$$(d_1, \cdots, d_{(4t-2)(4s-1)}) = A([1, \frac{v}{2}] \backslash \{abs(D_0) \cup (\bigcup_{i=1}^{2t} 8is) \cup (\bigcup_{i=1}^{2s-1} abs(R_i)) \cup (\bigcup_{i=1}^{2s} abs(T_i))\}).$$

Proof Obviously, the DTs in Part 3 are all CDCs by Lemmas 3.5 and 3.6. The differences in the three Parts form a partition of $[1, \frac{v}{2}]$ by the structure of CDCs in Part 3. In addition, the number of blocks $v + 2t \times 2 \times 8s + 2v \times (4s-1) = \frac{v(v-1)}{4t+1}$, as expected. □

Theorem 4.2 *Let s, t (> 0) be nonnegative integers and $4t+3$ be a prime, then there exists a $(4t+3)$-SCMD$((8s+4)(4t+3))$.*

Construction Let $v = (8s+4)(4t+3)$, $X = Z_v$, $d = \frac{v}{2} - 6s - 5$, $d' = \frac{v}{2} - 2s - 1$, $n = \left\lfloor \frac{2t}{8s+4} \right\rfloor$.

Part 1. SDC(D_0) given in Corollary 3.2b.

Part 2. UDC(D) $= U(i),\ i = 1, \cdots, 2t+1$, given by Lemma 3.4.

Part 3. CDC(D_i)$,\ i = 1, \cdots, 4s+1$, given as follows:

$$D_i = \begin{cases} (P_i, \quad R_i) & i = 1, \cdots, 2s+1 \\ (P_i, \quad T_{i-2s-1}) & i = 2s+2, \cdots, 4s+1 \end{cases}.$$

Here (Case 1. $2t-(8s+4)n > 4s+2$; Case 2. $2t-(8s+4)n < 4s+2$)

$$R_i = \begin{cases} (d+i, \quad 2t-2i+1, \quad -(d-i+1)) & \text{Case } 1 \\ (d-i+1, \quad -(d+i), \quad 2t+2i-1) & \text{Case } 2 \end{cases}, i = 1, \cdots, 2s+1.$$

$$T_i = \begin{cases} (d'+i, \quad 2t-2i, \quad -(d'-i)) & \text{Case } 1 \\ (d'-i, \quad -(d'+i), \quad 2t+2i) & \text{Case } 2 \end{cases}, i = 1, \cdots, 2s$$

$$P_i = (d_{4(i-1)t+1}, \cdots, d_{4it}), \quad 1 \le i \le 4s+1$$

$$(d_1, \cdots, d_{4t(4s+1)}) = A([1, \quad \frac{v}{2}] \backslash \{abs(D_0) \cup (\bigcup_{i=1}^{2t+1} 8si+4i) \cup (\bigcup_{i=1}^{2s+1} abs(R_i)) \cup (\bigcup_{i=1}^{2s} abs(T_i))\}).$$

Proof The proof is similar to that of Theorem 4.1. □

Theorem 4.3 *Let s, t (>2) be nonnegative integers and $4t+1$ be a prime. Then there exists a $(4t+1)$-SCMD$((8s+4)(4t+1))$.*

Construction Let $v=(8s+4)(4t+1)$, $X=Z_v$, $n=\left\lfloor\frac{2t-1}{8s+4}\right\rfloor$, $d=16st+8t+2s-1$, $d'=16st+8t-2s-2$.

Part 1. SDC(D_0) given in Corollary 3.3a.

Part 2. UDC(D) $=U(i)$, $i=1,\cdots,2t$, given by Lemma 3.4.

Part 3. CDC(D_i), $i=1,\cdots,4s+1$, where D_i is given as follows:

$$D_i=\begin{cases}(d_{(4t-2)(i-1)+1},\cdots,d_{(4t-2)i},\ \ R_i), & 1\le i\le 2s,\\ (d_{(4t-2)(i-1)+1},\cdots,d_{(4t-2)i},\ \ T_{i-2s+2}), & 2s+1\le i\le 4s,\\ (d_{(4t-2)(i-1)+1},\cdots,d_{(4t-2)i-2},\ \ a_1,\ \ d_{(4t-2)i},\ \ d_{(4t-2)i-1},\ \ a_2,\ \ a_3), & i=4s+1.\end{cases}$$

Remark When $s=0$ the only CDC : $D_1=(d_1,\cdots,d_{4t-2},\ \ a_1,\ \ a_2,\ \ a_3)$.

Here (Case 1. $2t-3-(8s+4)n>4s$; Case 2. $2t-3-(8s+4)n<4s$)

$$R_1=\begin{cases}(d-10s-4,\ \ 2t-4s-2,\ \ -(d-14s-5)) & \text{Case 1}\\ (d-14s-5,\ \ -(d-10s-4),\ \ 2t+4s) & \text{Case 2}\end{cases}$$

$$R_i=\begin{cases}(d+i+1,\ \ 2t-2i,\ \ -(d-i+2)) & \text{Case 1}\\ (d-i+2,\ \ -(d+i+1),\ \ 2t+2i-2) & \text{Case 2}\end{cases},\quad i=2,\cdots,2s.$$

$$T_i=\begin{cases}(d'+i,\ \ 2t-2i-1,\ \ -(d'-i)) & \text{Case 1}\\ (d'-i,\ \ -(d'+i),\ \ 2t+2i-1) & \text{Case 2}\end{cases},\quad i=2,\cdots,2s+1.$$

$$(a_1,\ \ a_2,\ \ a_3)=T_1=\begin{cases}(d'+1,\ \ 2t-3,\ \ -(d'-1)) & \text{Case 1}\\ (d'-1,\ \ 2t+1,\ \ -(d'+1)) & \text{Case 2}\end{cases}$$

$$(d_1,\cdots,d_{(4t-2)(4s+1)})=A([1,\ \ \frac{v}{2}]\backslash\{abs(D_0)\cup(\bigcup_{i=1}^{2t}8is+4i)\cup(\bigcup_{i=1}^{2s}abs(R_i))\cup(\bigcup_{i=1}^{2s+1}abs(T_i))\}).$$

Proof The proof is similar to that of Theorem 4.1. □

Theorem 4.4 *Let s, t (>1) be positive integers and $4t+3$ be a prime. Then there exists a $(4t+3)$-SCMD$(8s(4t+3))$.*

Construction Let $v=8s(4t+3)$, $X=Z_v$, $n=\lfloor\frac{2t}{8s}\rfloor$, $d=16st+6s-2$, $d'=16st+10s$.

Part 1. SDC(D_0) given in Corollary 3.3b.

Part 2. UDC(D) $=U(i)$, $i=1,\cdots,2t+1$, given by Lemma 3.4.

Part 3. CDC(D_i), $i=1,\cdots,4s-1$, where D_i is given as follows:

$$D_i=\begin{cases}(d_{4t(i-1)+1},\cdots,d_{4ti},\ \ R_i), & 1\le i\le 2s,\\ (d_{4t(i-1)+1},\cdots,d_{4ti},\ \ T_{i-2s}), & 2s+1\le i\le 4s-1.\end{cases}$$

Here (Case 1. $2t-8sn>4s$; Case 2. $2t-8sn<4s$)

$$R_i=\begin{cases}(d+i,\ \ 2t-2i+1,\ \ -(d-i+1)) & \text{Case 1}\\ (d-i+1,\ \ -(d+i),\ \ 2t+2i-1) & \text{Case 2}\end{cases},\quad i=1,\cdots,2s.$$

$$T_i = \begin{cases} (d'+i, \quad 2t-2i, \quad -(d'-i)) & \text{Case } 1 \\ (d'-i, \quad -(d'+i), \quad 2t+2i) & \text{Case } 2 \end{cases}, \quad i = 1, \cdots, 2s-1.$$

$$(d_1, \cdots, d_{4t(4s-1)}) = A([1, \ \frac{v}{2}] \backslash \{abs(D_0) \cup (\bigcup_{i=1}^{2t+1} 8is) \cup (\bigcup_{i=1}^{2s} abs(R_i)) \cup (\bigcup_{i=1}^{2s-1} abs(T_i))\}).$$

Proof The proof is similar to that of Theorem 4.1. □

The Proof of Theorem 1.1 According to the range of p and m, there are the following cases

(1) $p = 3, 5$, see [3], [4]

(2) $p = 4t+1, \ \ m = 2s$, see Theorem 4.1;

$p = 4t+1$, $m = 2s+1$, see Theorem 4.3;

$p = 4t+3$, $m = 2s$, see Theorem 4.4;

$p = 4t+3$, $m = 2s+1$, see Theorem 4.2. □

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区组长为奇素数的自反 MD 设计

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摘要: 本文利用差方法对自反 MD 设计 SCMD$(4mp, p, 1)$ 的存在性给出了构造性证明，这里 p 为奇素数， m 为正整数.

关键词: 自反 MD 设计；差圈； SDC; UDC; CDC.