第27卷第1期 数 学 研 究 与 评 论 Vol.27, No.1
2007年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2007

Article ID: 1000-341X(2007)01-0035-06 **Document code**: A

H-Difference, H-Differentiability and S-Differentiability for Fuzzy Valued Functions

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Abstract: In this paper, we introduce the concept of monotonicity of interval functions and give the characterization of fuzzy valued functions which satisfies the H-difference. Furthermore, relations among H-difference, H-differentiability and S-differentiability are discussed.

Key words: fuzzy number; fuzzy valued functions; H-difference.
MSC(2000): 26A39; 26A42; 26A45
CLC number: O159.2

1. Introduction

The differential and integral calculus for the fuzzy-valued functions, shortly fuzzy calculus, have been developed in the recent papers of O.Kaleva[4], M.L.Puri, D.A.Ralescu[5], Gong Zengtai and Wu Congxin[2]. In [4], in order to show the existence of the solution of fuzzy differential equations, Kaleva discussed the properties of differentiability of fuzzy-valued mappings by the concept of H-differentiability. However, the discussion of H-differentiability is very difficult because the function considered must satisfy H-difference. H-difference was first presented by Puri and Ralescu[5] in 1983. For H-differentiability of fuzzy-valued functions, we have pointed out that there exists a fuzzy-valued function which is Kaleva integrable on $[0,1]$, but its primitive is not differentiable almost everywhere[1]. Another definition of fuzzy-valued functions was given by Seikkala in 1987[6]. We call it S-differentiability. In this paper, first we have to recall some basic results of fuzzy numbers and definition of H-difference of fuzzy-valued functions. Next, we introduce the definition of the monotonicity of interval functions and use it to characterize H-difference. In addition, relations among H-difference, H-differentiability, and Seikkala differentiability are discussed.

2. Notations and preliminaries

Let $F(R)$ be the class of all fuzzy subsets on R. For $\tilde{A} \in F(R)$, let $\tilde{A}$ satisfy the following conditions:

(1) $\tilde{A}$ is normal, i.e., there exists $x_0 \in R$, such that $A(x_0) = 1$;

Received date: 2005-04-01; **Accepted date**: 2005-07-12
Foundation item: the Natural Scientific Fund of Gansu Province (3ZS041-A25-004); the National Natural Science Stree Fund of China (40235053)

(2) $\tilde{A}$ is a convex fuzzy set, i.e., $A(tx+(1-t)y) \geq \min(A(x), A(y))$, for any $x, y \in R$, $t \in [0,1]$;

(3) $A(x)$ is upper semi-continuous;

(4) $[A]_0 = \overline{\{x \in R : A(x) > 0\}}$ is compact.

Then we say $\tilde{A}$ is a fuzzy number. Let E^1 denote the set of all fuzzy numbers[3−5].

For $\tilde{A}, \tilde{B} \in E^1$, $k \in R$, we define $\tilde{A} + \tilde{B} = \tilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda, \lambda \in [0,1]$, iff $A_\lambda^+ + B_\lambda^+ = C_\lambda^+, A_\lambda^- + B_\lambda^- = C_\lambda^-$, for any $\lambda \in [0,1]$. $[kA]_\lambda = kA_\lambda, \lambda \in [0,1]$, where $A_\lambda = \{x | A(x) \geq \lambda\}$. We easily prove that A_λ is a close interval, and write $[A_\lambda^-, A_\lambda^+]$[3−5].

Define $D(\tilde{A}, \tilde{B}) = \sup\limits_{\lambda \in [0,1]} \max(|A_\lambda^- - B_\lambda^-|, |A_\lambda^+ - B_\lambda^+|)$.

Definition 2.1[4,5] *Let $\tilde{f} : [a,b] \to E^1$. We say $\tilde{f}$ satisfies H-difference on $[a,b]$, if for any $x_1, x_2 \in [a,b]$ satisfying $x_1 < x_2$, there exists $\tilde{A} \in E^1$ such that $\tilde{f}(x_2) = \tilde{f}(x_1) + \tilde{A}$, denoted by $\tilde{f}(x_2) - \tilde{f}(x_1) = \tilde{A}$.*

3. Characterize of H-difference

Lemma 3.1[3] *If $\tilde{A} \in E^1$, then*

(1) A_λ^- is nondecreasing function on $[0,1]$,

(2) A_λ^+ is nonincreasing function on $[a,b]$,

(3) A_λ^-, A_λ^+ are bounded and left continuous on $(0,1]$, and right continuous at $\lambda = 0$, and

(4) $A_1^- \leq A_1^+$.

Conversely, if $a(\lambda), b(\lambda)$ satisfy (1)–(4), then there exists a unique $\tilde{A} \in E^1$ such that $A_\lambda = [a(\lambda), b(\lambda)]$ for any $\lambda \in [0,1]$.

In order to study the characterization of H-difference condition, we will give the concept of interval function and its monotonicity.

Definition 3.2 *Let $f : [a,b] \times [c,d] \to R$ be two variable function. $F(I)$ is called the interval function induced by f, if*

$$F(I) = f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1),$$

for nondegenerate interval $I = [x_1, x_2] \times [y_1, y_2]$, where $[x_1, x_2] \subset [a,b], [y_1, y_2] \subset [c,d]$. In particular, for degenerate interval $I_y = [x_1, x_2] \times [y, y] (x_1 < x_2)$, $F(I_y) = f(x_2, y) - f(x_1, y)$. For degenerate interval $I^x = [x, x] \times [y_1, y_2] (y_1 < y_2)$, $F(I^x) = f(x, y_2) - f(x, y_1)$. For degenerate interval $I_y^x = [x, x] \times [y, y]$, $F(I_y^x) = f(x, y)$.

Definition 3.3 *Let $F(I)$ be the interval function induced by f. $F(I)$ is said to be nondecreasing (nonincreasing), if $F(I) \geq 0$ ($F(I) \leq 0$), for any $I \subset [a,b] \times [0,1]$.*

Theorem 3.4 *Let $\tilde{f} : [a,b] \to E^1$, and $[\tilde{f}(x)]_\lambda = [f_\lambda^-(x), f_\lambda^+(x)]$. Then $\tilde{f}(x)$ satisfies $H-$difference if and only if:*

(1) $F^+(I_1) \geq F^-(I_1)$,

(2) $F^-(I)$ is nondecreasing,

(3) $F^+(I)$ is nonincreasing.

Here $I_1 = [x_1, x_2] \times [1, 1], ([x_1, x_2] \subset [a, b]$ and $x_1 < x_2)$, for nondegenerate interval $I \subset [a, b] \times [0, 1]$, and $F^-(I)$ and $F^+(I)$ are the interval functions induced by f^- and f^+, respectively.

Proof If $\tilde{f}(x)$ satisfies H-difference on $[a, b]$, then for any $x_1, x_2 \in [a, b]$ satisfying $x_1 < x_2$, there exists $\tilde{A} \in E^1$ such that $\tilde{f}(x_2) = \tilde{f}(x_1) + \tilde{A}$. This gives that

(1) $F^+(I_1) = f_1^+(x_2) - f_1^+(x_1) = A_1^+, F^-(I_1) = f_1^-(x_2) - f_1^-(x_1) = A_1^-$. Since $\tilde{A}$ is a fuzzy number, by Lemma 3.1, we have $A_1^- \le A_1^+$, i.e., $F^-(I_1) \le F^+(I_1)$.

(2) By Lemma 3.1, we have that $A_\lambda^- = f_\lambda^-(x_2) - f_\lambda^-(x_1)$ is nondecreasing. For any $0 \le \lambda_1 \le \lambda_2 \le 1$, we have $f_{\lambda_2}^-(x_2) - f_{\lambda_2}^-(x_1) \ge f_{\lambda_1}^-(x_2) - f_{\lambda_1}^-(x_1)$, then $F^-(I) = f_{\lambda_2}^-(x_2) - f_{\lambda_2}^-(x_1) - f_{\lambda_1}^-(x_2) + f_{\lambda_1}^-(x_1) \ge 0$. Here $I = [x_1, x_2] \times [\lambda_1, \lambda_2]$.

Hence $F^-(I)$ is nondecreasing.

(3) By Lemma 3.1, we have that $A_\lambda^+ = f_\lambda^+(x_2) - f_\lambda^+(x_1)$ is nonincreasing. For any $0 \le \lambda_1 \le \lambda_2 \le 1$, we have $f_{\lambda_2}^+(x_2) - f_{\lambda_2}^+(x_1) \le f_{\lambda_1}^+(x_2) - f_{\lambda_1}^+(x_1)$, that is $F^+(I) = f_{\lambda_2}^+(x_2) - f_{\lambda_2}^+(x_1) - f_{\lambda_1}^+(x_2) + f_{\lambda_1}^+(x_1) \le 0$. Here $I = [x_1, x_2] \times [\lambda_1, \lambda_2]$.

Hence $F^+(I)$ is nonincreasing.

Conversely, for any $x_1, x_2 \in [a, b]$ satisfying $x_1 < x_2$ and each $\lambda \in [0, 1]$, let $[a(\lambda), b(\lambda)] = [f_\lambda^-(x_2) - f_\lambda^-(x_1), f_\lambda^+(x_2) - f_\lambda^+(x_1)]$. We can show that $a(\lambda)$ and $b(\lambda)$ $(\lambda \in [0, 1])$ satisfy the conditions of Lemma 3.1.

(1) For any $\lambda_1, \lambda_2 \in [0, 1]$ satisfying $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} a(\lambda_2) - a(\lambda_1) &= (f_{\lambda_2}^-(x_2) - f_{\lambda_2}^-(x_1)) - (f_{\lambda_1}^-(x_2) - f_{\lambda_1}^-(x_1)) \\ &= f_{\lambda_2}^-(x_2) - f_{\lambda_2}^-(x_1) - f_{\lambda_1}^-(x_2) + f_{\lambda_1}^-(x_1) = F^-(I) \ge 0. \end{aligned}$$

Here $I = [x_1, x_2] \times [\lambda_1, \lambda_2]$. Hence $a(\lambda_2) \ge a(\lambda_1)$, i.e., $a(\lambda)$ is a nondecreasing function on $[0, 1]$.

(2) For any $\lambda_1, \lambda_2 \in [0, 1]$ satisfying $\lambda_1 < \lambda_2$, we have

$$\begin{aligned} b(\lambda_2) - b(\lambda_1) &= (f_{\lambda_2}^+(x_2) - f_{\lambda_2}^+(x_1)) - (f_{\lambda_1}^+(x_2) - f_{\lambda_1}^+(x_1)) \\ &= f_{\lambda_2}^+(x_2) - f_{\lambda_2}^+(x_1) - f_{\lambda_1}^+(x_2) + f_{\lambda_1}^+(x_1) = F^+(I) \le 0. \end{aligned}$$

Here $I = [x_1, x_2] \times [\lambda_1, \lambda_2]$. i.e. $b(\lambda_2) \le b(\lambda_1)$. Hence $b(\lambda)$ is nonincreasing on $[0, 1]$.

(3) Obviously, $a(\lambda), b(\lambda)$ are bounded and left continuous on $(0, 1]$, and right continuous at $\lambda = 0$.

(4) $b(1) = f_1^+(x_2) - f_1^+(x_1) = F^+(I_1), a(1) = f_1^-(x_2) - f_1^-(x_1) = F^-(I_1)$. Hence, $a(1) \le b(1)$.

By Lemma 3.1, $a(\lambda), b(\lambda)$ determine a fuzzy number $\tilde{B} \in E^1$ such that $\tilde{f}(x_2) - \tilde{f}(x_1) = \tilde{B}$. This is, there exists $\tilde{B} \in E^1$ such that $\tilde{f}(x_2) = \tilde{f}(x_1) + \tilde{B}$.

The proof is complete.

Proposition 3.5 *Let $\tilde{f}, \tilde{g} : [a, b] \to E^1$. For any $x \in [a, b]$, the H-difference $\tilde{f}(x) - \tilde{g}(x)$ exists if and only if:*

(1) $(F^- - G^-)(I_1^x) \le (F^+ - G^+)(I_1^x)$,

(2) $(F^- - G^-)(I^x)$ is nondecreasing,

(3) $(F^+ - G^+)(I^x)$ is nonincreasing.

Here $(F-G)(I)$ denote interval function $F(I)-G(I)$. $F^-(I), F^+(I), G^-(I)$ and $G^+(I)$ are the interval functions induced by f^-, f^+, g^- and g^+, respectively.

Proof If $\tilde{f}(x) = \tilde{g}(x) + \tilde{A}(x)$, for any $x \in [a,b]$, i.e., $\tilde{f}(x) - \tilde{g}(x) = \tilde{A}(x) \in E^1$, then

(1) $(F^- - G^-)(I_1^x) = f_1^-(x) - g_1^-(x) = A_1^-(x) \le A_1^+(x) = f_1^+(x) - g_1^+(x) = (F^+ - G^+)(I_1^x)$,

(2) $A_\lambda^-(x) = f_\lambda^-(x) - g_\lambda^-(x)$ is nondecreasing on $[0,1]$. For any $\lambda_1, \lambda_2 \in [0,1]$ satisfying $\lambda_1 < \lambda_2$, we have

$$(F^- - G^-)(I^x) = f_{\lambda_2}^-(x) - g_{\lambda_2}^-(x) - (f_{\lambda_1}^-(x) - g_{\lambda_1}^-(x)) = A_{\lambda_2}^-(x) - A_{\lambda_1}^-(x) \ge 0,$$

i.e., $(F^- - G^-)(I^x)$ is nondecreasing.

(3) The proof is similar to (2).

Conversely, for any $x \in [a,b]$, we can prove that $f_\lambda^-(x) - g_\lambda^-(x), f_\lambda^+(x) - g_\lambda^+(x)(\lambda \in [0,1])$ satisfy the condition of Lemma 3.1.

(i) For any $\lambda_1, \lambda_2 \in [0,1]$ satisfying $\lambda_1 < \lambda_2$, we have

$$(f_{\lambda_2}^-(x) - g_{\lambda_2}^-(x)) - (f_{\lambda_1}^-(x) - g_{\lambda_1}^-(x)) = (F^- - G^-)(I^x) \ge 0,$$

i.e., $f_{\lambda_2}^-(x) - g_{\lambda_2}^-(x) \ge (f_{\lambda_1}^-(x) - g_{\lambda_1}^-(x))$. Hence, $f_\lambda^-(x) - g_\lambda^-(x)$ is nondecreasing on $[0,1]$.

(ii) Similarly, $f_\lambda^+(x) - g_\lambda^+(x)$ is nonincreasing on $[0,1]$.

(iii) Obviously, $f_\lambda^-(x) - g_\lambda^-(x), f_\lambda^+(x) - g_\lambda^+(x)$ are bounded and left continuous on $(0,1]$, and right continuous at $\lambda = 0$.

(iv) $f_1^-(x) - g_1^-(x) = (F^- - G^-)(I_1^x) \le (F^+ - G^+)(I_1^x) = f_1^+(x) - g_1^+(x)$.

By Lemma 3.1, $[f_\lambda^-(x) - g_\lambda^-(x), f_\lambda^+(x) - g_\lambda^+(x)]$ $(\lambda \in [0,1])$ determines a fuzzy $\tilde{A}(x) \in E^1$, such that $\tilde{f}(x) - \tilde{g}(x) = \tilde{A}(x)$. i.e., $\tilde{f}(x) = \tilde{g}(x) + \tilde{A}(x)$.

Hence, H-difference $\tilde{f}(x) - \tilde{g}(x)$ exists, for all $x \in [a,b]$.

4. Seikkala differentiability, H-differentiability and H-difference

Remark 4.1 Let $\tilde{f} : [a,b] \to E^1$. For each $x \in [a,b]$ there exists a $\beta(x) > 0$ such that the H-differences $\tilde{f}(x+h) - \tilde{f}(x)$ and $\tilde{f}(x) - \tilde{f}(x-h)$ exist for all $0 \le h < \beta(x)$. Then $\tilde{f}$ satisfies the H-difference on $[a,b]$.

Proof By the Heine-Borel covering theorem we easyly prove.

Definition 4.2[6] *Let $\tilde{f} : [a,b] \to E^1$. $\tilde{f}$ is said to be Seikkala differentiable at $x \in [a,b]$ if there exists $f'(x) \in E^1$ such that*

$$[\tilde{f}'(x)]_\lambda = [(f_\lambda^-(x))', (f_\lambda^+(x))'],$$

for any $\lambda \in [0,1]$. In this case, $\tilde{f}$ is called S-differentiable at $x \in [a,b]$.

Theorem 4.3 *Let $\tilde{f} : [a,b] \to E^1$. If $\tilde{f}$ is S-differentiable on $[a,b]$, then $\tilde{f}$ satisfies the H-difference on $[a,b]$.*

Proof For any $x \in [a,b]$, since $\tilde{f}(x)$ is S-differentiable at x, there exists $\tilde{f}'(x) \in E^1$ such that $[\tilde{f}'(x)]_\lambda = [(f_\lambda^-(x))', (f_\lambda^+(x))']$.

(1) By Lemma 3.1, $(f_\lambda^-(x))'$ is nondecreasing on $\lambda \in [0,1]$. For any $\lambda_1, \lambda_2 \in [0,1]$ satisfying $\lambda_1 < \lambda_2$, we have $(f_{\lambda_1}^-(x))' \le (f_{\lambda_2}^-(x))'$, i.e., $(f_{\lambda_2}^-(x) - f_{\lambda_1}^-(x))' \ge 0$. Hence $f_{\lambda_2}^-(x) - f_{\lambda_1}^-(x)$ is nondecreasing on $[a,b]$. For any $x_1, x_2 \in [a,b]$ satisfying $x_1 < x_2$, we have $f_{\lambda_2}^-(x_2) - f_{\lambda_1}^-(x_2) \ge f_{\lambda_2}^-(x_1) - f_{\lambda_1}^-(x_1)$, i.e., $f_{\lambda_2}^-(x_2) - f_{\lambda_1}^-(x_2) - f_{\lambda_2}^-(x_1) + f_{\lambda_1}^-(x_1) \ge 0$. This implies that $F^-(I)$ is nondecreasing, where $I = [x_1, x_2] \times [\lambda_1, \lambda_2] \subset [a,b] \times [0,1]$ is arbitrary nondegenerate interval.

(2) $(f_\lambda^+(x))'$ is nondecreasing on $\lambda \in [0,1]$. For any $\lambda_1, \lambda_2 \in [0,1]$ satisfying $\lambda_1 < \lambda_2$, we have $(f_{\lambda_2}^+(x))' \le (f_{\lambda_1}^+(x))'$, i.e., $(f_{\lambda_2}^-(x) - f_{\lambda_1}^-(x))' \le 0$. So $f_{\lambda_2}^+(x) - f_{\lambda_1}^+(x)$ is nondecreasing on $[a,b]$. For any $x_1, x_2 \in [a,b]$ satisfying $x_1 < x_2$, we have $f_{\lambda_2}^+(x_2) - f_{\lambda_1}^+(x_2) \ge f_{\lambda_2}^+(x_1) - f_{\lambda_1}^+(x_1)$, i.e., $f_{\lambda_2}^+(x_2) - f_{\lambda_1}^+(x_2) - f_{\lambda_2}^+(x_1) + f_{\lambda_1}^+(x_1) \le 0$. This implies that $F^+(I)$ is nonincreasing, where $I = [x_1, x_2] \times [\lambda_1, \lambda_2] \subset [a,b] \times [0,1]$ is arbitrary nondegenerate interval.

(3) By Lemma 3.1, $(f_1^-(x))' \le (f_1^+(x))'$, for any $x \in [a,b]$. There exists a $\beta(x) > 0$ such that

$$f_1^-(x_2) - f_1^-(x_1) \le f_1^+(x_2) - f_1^+(x_1),$$

for any $0 < h < \beta(x), x_1, x_2 \in [x-h, x+h]$ satisfying $x_1 < x_2$, i.e., $F^-(I_1) \le F^+(I_1)$.

By using Theorem 3.4 and Remark 4.1, $\tilde{f}$ satisfies the H-difference on $[a,b]$.

Definition 4.4[4,5] *Let $\tilde{f} : [a,b] \to E^1$ satisfy H-difference. $\tilde{f}$ is said to be H-differentiable at $x_0 \in [a,b]$ if there exists $f'(x) \in E^1$ such that the limits*

$$\lim_{h \to 0^+} \frac{\tilde{f}(x_0+h) - \tilde{f}(x_0)}{h} \quad \textit{and} \quad \lim_{h \to 0^+} \frac{\tilde{f}(x_0) - \tilde{f}(x_0-h)}{h}$$

exist and equal $f'(x)$.

Here the limit is taken in the metric space (E^1, D). At the end points of $[a,b]$ we consider only the one-sided derivatives.

Lemma 4.5[7] *Let $\tilde{f} : [a,b] \to E^1$. Then $\tilde{f}$ satisfies H-difference and H-differentiable on $[a,b]$ if and only if $f_\lambda^-(x)$ and $f_\lambda^+(x)$ ($\lambda \in [0,1]$) are differentiable, and*

$$G_h^-(\lambda) = \frac{f_\lambda^-(x+h) - f_\lambda^-(x)}{h}, \quad G_h^+(r) = \frac{f_\lambda^+(x+h) - f_\lambda^+(x)}{h}$$

converge to $(f_\lambda^-(x))'$, $(f_\lambda^+(x))'$ uniformly, respectively, and $(f_\lambda^-(x))'$, $(f_\lambda^+(x))'$ determine a fuzzy number, for any $x \in [a,b]$.

Example 4.7 shows that the following Proposition 4.6 hold.

Proposition 4.6 *Let $\tilde{f} : [a,b] \to E^1$, and $\tilde{f}$ be S-differentiable on $[a,b]$. Although $\tilde{f}$ satisfies the H-difference on $[a,b]$, it does not imply that $\tilde{f}$ is H-differentiable on $[a,b]$.*

Example 4.7 Define

$$\widetilde{G}(x)(s) = \begin{cases} 1, & s = 0, \\ x - \frac{s}{2}, & 0 \le s \le 2x, \\ 0, & \text{otherwise}, \end{cases}$$

the λ-level set is

$$[G(x)]_\lambda = \begin{cases} [0,0], & x < \lambda \le 1, \\ [0, 2(x-\lambda)], & 0 \le \lambda \le x. \end{cases}$$

Obviously, $\widetilde{G}(x)$ is S-differentiable on $[0,1]$, and

$$\widetilde{G}'(x) = \widetilde{f}(x),$$

where

$$\widetilde{f}(x)(s) = \begin{cases} 1, & s = 0, \\ x, & 0 \le s \le 2, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$[f(x)]_\lambda = \begin{cases} [0,0], & x < \lambda \le 1, \\ [0,2], & x < \lambda \le 1. \end{cases}$$

Furthermore,

$$\widetilde{G}(x) = \widetilde{G}(0) + \int_0^x \widetilde{f}(t)\mathrm{d}t$$

satisfies H-difference. However, $\widetilde{G}(x)$ is not H-differentiable. In fact, for any $x \in [0,1]$ and $h > 0$, we have

$$\begin{aligned} D(\frac{\tilde{G}(x+h) - \tilde{G}(x)}{h}, \widetilde{f}(x)) &\ge \frac{1}{h} \sup_{\lambda \in (x, x+h]} |[G(x+h) - G(x)]^+_\lambda - h[f(x)]^+_\lambda| \\ &= \frac{1}{h} \sup_{\lambda \in (x, x+h]} (2(x+h-\lambda)) = 2. \end{aligned}$$

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模糊数值函数的 H- 差, H- 可导性和 S- 可导性

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摘要: 本文提出了区间值函数单调的概念, 并利用所定义的区间值函数刻划了模糊数值函数的 H- 差, H- 可导性和 S- 可导性及其相互关系.

关键词: 模糊数; 模糊数值函数; H- 差.