第27卷第1期 数 学 研 究 与 评 论 Vol.27, No.1
2007年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2007

Article ID: 1000-341X(2007)01-0075-06 **Document code**: A

Note on Convergence Theorems of Iterative Sequences for Asymptotically Non-Expansive Mapping in a Uniformly Convex Banach Space

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Abstract: In this paper, we approximate fixed point of asymptotically nonexpansive mapping T on a closed, convex subset C of a uniformly convex Banach space. Our argument removes the boundedness assumption on C, generalizing theorems of Liu and Xue.

Key words: asymptotically nonexpansive mapping; modified Ishikawa iterative sequence; fixed points.
MSC(2000): 47H09; 47H10
CLC number: O177.91; O174.41

1. Introduction and preliminaries

Let E be a real normed linear space, E^* its dual, and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined for each $x \in E$ by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

It is well known that if E is smooth then J is single-valued.

Definition 1.1 *Let $T : D(T) \subset E \to E$ be a mapping. T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall x, y \in D(T), n = 1, 2, 3, \cdots.$$

It is well known that if T is nonexpansive, then T is asymptotically nonexpansive with a constant sequence $\{1\}$.

Definition 1.2 *Let C be a nonempty convex subset of E, $T : C \to C$ be a mapping and $x_1 \in C$ be a given point. If sequences $\{x_n\}, \{y_n\} \subset C$ are defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, n \ge 1, \end{aligned} \tag{1.1}$$

Received date: 2005-09-13; **Accepted date**: 2007-07-02
Foundation item: the National Natural Science Foundation of China(10561007); the Natural Science Foundation of Jiangxi Province(0411036).

then $\{x_n\}$ *is called the modified Ishikawa iterative sequence of* T, *where* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, *with* $0 < \delta \leq \alpha_n, \beta_n \leq 1-\delta < 1$.

In (1.1) if $\beta_n = 0$ *for all* $n \geq 0$, *then* $y_n = x_n$. *The sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \geq 1 \tag{1.2}$$

is called the modified Mann iterative sequence of T, *where* $\{\alpha_n\} \subset [0,1]$, *with* $0 < \delta \leq \alpha_n \leq 1-\delta < 1$.

The concept of asymptotically nonexpansive mapping was first introduced and studied by Goebel and Kirk[1] in 1972. They proved that if D is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive selfmapping T defined on D has a fixed point.

In 2000, Liu and Xue[2] proved the convergence of iterative sequence in a uniformly convex Banach space for asymptotically nonexpansive mappings. They got the following main theorem.

Theorem LX *Let* T *be a completely continuously asymptotically nonexpansive mapping with sequence* $\{k_n\}$ *in a bounded closed convex subset* C *of a uniformly convex Banach space and* $k_n \geq 1, \sum_{n=1}^{\infty}(k_n-1) < +\infty$, $x_1 \in C$, $\{x_n\}$ *defined by (1.1), where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy* $0 < \alpha \leq \alpha_n \leq \frac{1}{2}, 0 \leq \beta_n \leq \frac{1}{2}$ *and* $\lim\limits_{n\to\infty} \beta_n = 0$, *then the iterative sequence* $\{x_n\}$ *converges to a fixed point of* T.

In this paper, our results generalize Theorem LX, in that we remove the assumption that C is bounded and we approximate fixed point of asymptotically nonexpansive mapping T on a closed, convex subset C of a uniformly convex Banach space.

We shall need the following results.

Lemma 1.1[3,4] *Let* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$ *and* $\sum_{n=1}^{\infty} c_n < +\infty$, *then* $\lim_{n\to\infty} a_n$ *exists. In particular, if* $c_n \equiv 0$, *then* $\lim_{n\to\infty} a_n$ *also exists.*

Lemma 1.2[5] *Let* $\{\rho_n\}_{n=1}^{\infty}$ *and* $\{\sigma_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$\rho_{n+1} \leq \rho_n + \sigma_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \sigma_n < +\infty$, *then* $\lim_{n\to\infty} \rho_n$ *exists. In particular, if* $\{\rho_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} \rho_n = 0$.

Lemma 1.3[6] *Suppose that* E *is a uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ *for all positive intergers* n. *Also suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences of* E *such that* $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ *and* $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ *hold for some* $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

2. Main results

Lemma 2.1 *Let E be a real normed linear space, C be a nonempty convex subset of E, and $T : C \to C$ be an asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1, +\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Let $x_1 \in C$, $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, then for any given $q \in F(T)$, $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof For any given $q \in F(T)$, using iterates (1.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)\| \\ &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n k_n \|y_n - q\|. \end{aligned} \tag{2.1}$$

Otherwise,

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)\| \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n k_n \|x_n - q\| \\ &= [1 + (k_n - 1)\beta_n]\|x_n - q\| \\ &\leq k_n \|x_n - q\|. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n k_n^2 \|x_n - q\| \\ &= [1 + \alpha_n(k_n^2 - 1)]\|x_n - q\|. \end{aligned} \tag{2.3}$$

Set $a_n = \|x_n - q\|, b_n = \alpha_n(k_n^2 - 1) = \alpha_n(k_n + 1)(k_n - 1)$. Then Inequality (2.3) is equal to

$$a_{n+1} \leq (1 + b_n)a_n.$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, $\{k_n\}$ is a bounded sequence and $\alpha_n \in [0, 1]$, so $\sum_{n=1}^{\infty} b_n < +\infty$. By Lemma 1.1, we know $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n - q\|$ exists.

Lemma 2.2 *Let E be a uniformly convex Banach space, C be a nonempty convex subset of E, and $T : C \to C$ be an asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1, +\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$. Let $x_1 \in C$, $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof Since $F(T) \neq \emptyset$, let $q \in F(T)$. By Lemma 2.1, we know $\lim_{n\to\infty} \|x_n - q\|$ exists. Let $\lim_{n\to\infty} \|x_n - q\| = c, c \geq 0$.

Step 1. We prove $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$.

From (1.1), we have

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - T^n q)\| \\ &\leq [1 + (k_n - 1)\beta_n]\|x_n - q\| \\ &\leq k_n \|x_n - q\|. \end{aligned}$$

Taking lim sup on both sides of the above inequality, we have

$$\limsup_{n\to\infty}\|y_n-q\|\le c. \tag{2.4}$$

Next, consider $\|T^ny_n-q\|\le k_n\|y_n-q\|$. Taking lim sup on both sides of the above inequality and then using (2.4), we get that

$$\limsup_{n\to\infty}\|T^ny_n-q\|\le c.$$

Furthermore, $\lim_{n\to\infty}\|x_{n+1}-q\|=c$ means that

$$\lim_{n\to\infty}\|(1-\alpha_n)(x_n-q)+\alpha_n(T^ny_n-q)\|=c.$$

Hence applying Lemma 1.3, we obtain that

$$\lim_{n\to\infty}\|x_n-T^ny_n\|=0.$$

Next,

$$\|x_n-q\|\le\|x_n-T^ny_n\|+\|T^ny_n-q\|\le\|x_n-T^ny_n\|+k_n\|y_n-q\|$$

gives that

$$c\le\liminf_{n\to\infty}\|y_n-q\|\le\limsup_{n\to\infty}\|y_n-q\|\le c.$$

That is $\lim_{n\to\infty}\|y_n-q\|=c$. Now $\lim_{n\to\infty}\|y_n-q\|=c$ can be expressed as

$$\lim_{n\to\infty}\|(1-\beta_n)(x_n-q)+\beta_n(T^nx_n-q)\|=c.$$

Observe that $\|T^nx_n-q\|\le k_n\|x_n-q\|$. Taking lim sup on both the sides in the above inequality, we have $\limsup_{n\to\infty}\|T^nx_n-q\|\le c$. So again by Lemma 1.3, we have $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$.

Step 2. We prove $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

For convenience, let $\rho_n=\|x_n-T^nx_n\|$.

Now consider

$$\begin{aligned}\|x_n-x_{n+1}\|&=\|\alpha_n(x_n-T^ny_n)\|\\&\le\alpha_n\|x_n-T^nx_n\|+\alpha_n\|T^nx_n-T^ny_n\|\\&\le\alpha_n\|x_n-T^nx_n\|+\alpha_nk_n\|x_n-y_n\|\\&\le\alpha_n\|x_n-T^nx_n\|+\alpha_n\beta_nk_n\|x_n-T^nx_n\|\\&=\alpha_n\rho_n+\alpha_n\beta_nk_n\rho_n.\end{aligned}$$

That is

$$\|x_n-x_{n+1}\|\le\alpha_n\rho_n+\alpha_n\beta_nk_n\rho_n.$$

Next consider

$$\begin{aligned}\|x_{n+1}-Tx_{n+1}\| &\le \|x_{n+1}-T^{n+1}x_{n+1}\|+\|T^{n+1}x_{n+1}-Tx_{n+1}\|\\ &\le \rho_{n+1}+k_1\|T^n x_{n+1}-x_{n+1}\|\\ &\le \rho_{n+1}+k_1\|T^n x_{n+1}-T^n x_n\|+k_1\|T^n x_n-x_n\|+k_1\|x_n-x_{n+1}\|\\ &\le \rho_{n+1}+k_1k_n\|x_{n+1}-x_n\|+k_1\|T^n x_n-x_n\|+k_1\|x_n-x_{n+1}\|\\ &= \rho_{n+1}+k_1(k_n+1)\|x_n-x_{n+1}\|+k_1\rho_n\\ &\le \rho_{n+1}+k_1(k_n+1)\alpha_n\rho_n+k_1k_n(k_n+1)\alpha_n\beta_n\rho_n+k_1\rho_n.\end{aligned}$$

From Step 1, we have $\lim_{n\to\infty}\rho_n=0$, $k_n\to 1(n\to\infty)$ and $0\le\alpha_n,\beta_n\le 1$, so

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Theorem 2.1 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset of E, and $T:C\to C$ be a completely continuously asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1,+\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<+\infty$. Let $x_1\in C$, $\{x_n\}$ be the modified Ishikawa iterative sequence defined by* (1.1)*. If $F(T)\ne\emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof Since $F(T)\ne\emptyset$, by Lemma 2.2, we know

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{2.5}$$

Since T is completely continuous, by Lemma 2.1, we have $\{x_n\}$ is bounded and C is a closed subset, so there must exist $\{Tx_{n_k}\}_{k=1}^{+\infty}\subset\{Tx_n\}_{n=1}^{+\infty}$. Set

$$\lim_{k\to+\infty}Tx_{n_k}=q. \tag{2.6}$$

It follows from (2.5) and (2.6), we get

$$\lim_{k\to+\infty}x_{n_k}=q. \tag{2.7}$$

Since T is completely continuous, T is obviously continuous. It follows from (2.6) and (2.7) that so $\|q-Tq\|=0$. Thus q is a fixed point of T.

From (1.1), we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|(1-\alpha_n)x_n+\alpha_nT^ny_n-q\|=\|(1-\alpha_n)(x_n-q)+\alpha_n(T^ny_n-q)\|\\ &\le (1-\alpha_n)\|x_n-q\|+\alpha_nk_n\|y_n-q\|.\end{aligned} \tag{2.8}$$

Next, consider

$$\begin{aligned}\|y_n-q\| &= \|(1-\beta_n)x_n+\beta_nT^nx_n-q\|=\|(1-\beta_n)(x_n-q)+\beta_n(T^nx_n-q)\|\\ &\le (1-\beta_n)\|x_n-q\|+\beta_nk_n\|x_n-q\|=[1+(k_n-1)\beta_n]\|x_n-q\|.\end{aligned} \tag{2.9}$$

Using (2.9) in (2.8), we obtain

$$\begin{aligned}\|x_{n+1}-q\| &\le (1-\alpha_n)\|x_n-q\|+\alpha_n k_n[1+(k_n-1)\beta_n]\|x_n-q\|\\ &= \|x_n-q\|+(k_n-1)(1+k_n\beta_n)\alpha_n\|x_n-q\|. \end{aligned}\tag{2.10}$$

By Lemma 2.1, for all $n\ge 0$, $\|x_n-q\|$ is bounded, $k_n\to 1(n\to\infty)$ and $0\le \alpha_n,\beta_n\le 1$, so there exists $M>0$ such that

$$(1+k_n\beta_n)\alpha_n\|x_n-q\|\le M. \tag{2.11}$$

Submitting it into (2.10), we have

$$\|x_{n+1}-q\|\le \|x_n-q\|+M(k_n-1).$$

Since $\sum_{n=1}^{\infty}(k_n-1)<+\infty$, by Lemma 1.2, we also know $\lim_{n\to\infty}\|x_n-q\|$ exists. Again by (2.7) and Lemma 1.2, we know $\{x_n\}$ converges strongly to q.

Corollary 2.1 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset of E, and $T:C\to C$ be a completely continuously asymptotically nonexpansive mapping with a real sequence $\{k_n\}$ in $[1,+\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<+\infty$. Let $x_1\in C$, $\{x_n\}$ be the modified Mann iterative sequence defined by (1.2). If $F(T)\neq\emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof Taking $\beta_n=0$ in Theorem 2.1, we know Corollary 2.1 is true.

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关于一致凸的 Banach 空间上的渐近非扩张映象的迭代序列的收敛性定理的注记

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摘要: 本文研究了在一致凸 Banach 空间中定义在闭凸集 C 上渐近非扩张映象 T 不动点的迭代问题，我们的讨论去掉了在刘和薛 [2] 中 C 是有界的假设.

关键词: 渐近非扩张映射；修改的 Ishikawa 迭代序列；不动点.