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Implicit Iteration Process with Errors for Common Fixed Points of a Finite Family of Strictly Pseudocontractive Maps

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Abstract: Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$, where $F(T_i)=\{x\in K: T_ix=x\}$, $\{\alpha_n\}\subset[0,1]$ be a real sequence, and $\{u_n\}\subset K$ be a sequence satisfying the conditions:

(i) $0<a\leq\alpha_n\leq 1$;
(ii) $\sum_{n=1}^\infty(1-\alpha_n)=+\infty$;
(iii) $\sum_{n=1}^\infty\|u_n\|<+\infty$.

Let $x_0\in K$ and $\{x_n\}_{n=1}^\infty$ be defined by

$$x_n=\alpha_nx_{n-1}+(1-\alpha_n)T_nx_n+u_{n-1},\quad n\geq 1,$$

where $T_n=T_{n \bmod N}$, then

(i) $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$;
(ii) $\lim_{n\to\infty}d(x_n,F)$ exists, where $d(x_n,F)=\inf_{p\in F}\|x_n-p\|$;
(iii) $\liminf_{n\to\infty}\|x_n-T_nx_n\|=0$.

Another result is that if $\{\alpha_n\}_{n=1}^\infty\subset[1-2^{-n},1]$, then $\{x_n\}$ is convergent. This paper generalizes and improves the results of Osilike in 2004. The ideas and proof lines used in this paper are different from those of Osilike in 2004.

Key words: strictly pseudocontractive mappings; implicit iteration process with error; common fixed points; convergence theorems.
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1. Introduction

Let E be a real Banach space and J denote the normalized duality mapping from E into 2^{E^*} given by $J(x)=\{f\in E^*:\langle x,f\rangle=\|x\|^2=\|f\|^2\}$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. A mapping T with domain $D(T)$ and range $R(T)$ in E is called strictly pseudocontractive in the terminology of Browder and Petryshyn [1] if there exists $\lambda>0$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leq\|x-y\|^2-\lambda\|x-y-(Tx-Ty)\|^2, \tag{1}$$

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for all $x, y \in D(T)$ and some $j(x-y) \in J(x-y)$. Without loss of generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then (1) can be written in the form

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \lambda \|(I-T)x-(I-T)y\|^2. \tag{2}$$

The class of strictly pseudocontractive mappings has been studied by various authors[1−7,10].

Let K be a nonempty convex subset of Banach space E, and $T_i, i=1,2,3,\cdots,N$, be a finite family of nonexpansive self-maps of K. In [9], Xu and Ori introduced the following implicit iteration process: For $x_0 \in K$ and $\{\alpha_n\} \subset (0,1)$, the sequence $\{x_n\}$ is generated by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, n \geq 1, \tag{3}$$

where $T_n = T_{n \bmod N}$.

In [10], Osilike considered the scheme (3) for finite family of strictly pseudocontractive self-maps $T_i, i=1,2,3,\cdots,N$, of K and proved some convergence theorems for finite family of strictly pseudocontractive mappings which extended the results of Xu and Ori[9].

In this paper, we will continue to investigate the problems of approximation of common fixed points of a finite family of strictly pseudocontractive mappings by implicit iteration process with errors. We generalize and improve the results of Osilike[10]. The ideas and proof lines used in this paper are different from those of Osilike[10].

If K is a nonempty convex subset of Banach space E and $T: K \to K$ is a strictly pseudocontractive mapping, then for every $u, v \in K$ and $\alpha \in (0,1]$, the operator $S_\alpha : K \to K$ defined by

$$S_\alpha x = \alpha u + (1-\alpha)Tx + v$$

satisfies

$$\langle S_\alpha x - S_\alpha y, j(x-y)\rangle = (1-\alpha)\langle Tx-Ty, j(x-y)\rangle \leq (1-\alpha)\|x-y\|^2,$$

for all $x, y \in K$, thus S_α is a strongly pseudocontractive mapping. Since S_α is also Lipschitz, it follows from [1,10] that S_α has a unique fixed point $x_\alpha \in K$. Thus there exists a unique $x_\alpha \in K$ such that $x_\alpha = \alpha u + (1-\alpha)Tx_\alpha + v$. This implies that the following implicit iteration scheme with errors

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n + u_n \tag{4}$$

can be employed for the approximation of common fixed points of a finite family of strictly pseudocontractive mappings, where $\{u_n\} \subset K$ is a sequence, and (3) is special form of (4) when $u_n = 0$.

Lemma OAA[8] *Let $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be three sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < +\infty$ and $\sum_{n=1}^\infty b_n < +\infty$,then $\lim_{n\to\infty} a_n$ exists.

2. Main results

Theorem 1 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$, where $F(T_i)=\{x\in K: T_ix=x\}$, and let $\{\alpha_n\}\subset[0,1]$ be a real sequence and $\{u_n\}\subset K$ be a sequence satisfying the conditions:*

(i) $0<a\le\alpha_n\le 1$;

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)=+\infty$;

(iii) $\sum_{n=1}^{\infty}\|u_n\|<+\infty$.

Let $x_0\in K$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (4). Then

(i) $\lim_{n\to\infty}\|x_n-p\|$ *exists for all* $p\in F$;

(ii) $\lim_{n\to\infty}d(x_n,F)$ *exists, where* $d(x_n,F)=\inf_{p\in F}\|x_n-p\|$.

(iii) $\liminf_{n\to\infty}\|x_n-T_nx_n\|=0$.

Proof For any $p\in F$, we have that

$$\begin{aligned}\|x_n-p\|^2&=\langle x_n-p,j(x_n-p)\rangle\\&=\alpha_n\langle x_{n-1}-p,j(x_n-p)\rangle+(1-\alpha_n)\langle T_nx_n-p,j(x_n-p)\rangle+\langle u_n,j(x_n-p)\rangle\\&\le\alpha_n\|x_{n-1}-p\|\|x_n-p\|+(1-\alpha_n)\|x_n-p\|^2+\\&\quad\|u_{n-1}\|\|x_n-p\|-(1-\alpha_n)\lambda\|x_n-T_nx_n\|^2\end{aligned}$$

$$\|x_n-p\|\le\|x_{n-1}-p\|+\frac{1}{\alpha_n}\|u_n\|-\frac{(1-\alpha_n)\lambda}{\alpha_n\|x_n-p\|}\|x_n-T_nx_n\|^2 \tag{5}$$

$$\|x_n-p\|\le\|x_{n-1}-p\|+\frac{1}{\alpha_n}\|u_{n-1}\|.$$

From condition (i), we have

$$\|x_n-p\|\le\|x_{n-1}-p\|+\frac{1}{a}\|u_{n-1}\|. \tag{6}$$

Since $\sum_{n=1}^{\infty}\|u_n\|<+\infty$, by Lemma OAA, we obtain that the limit $\lim_{n\to\infty}\|x_n-p\|$ exists. The proof of conclusion (i) is completed.

It follows from Inequality (6) that

$$0\le d(x_n,F)\le d(x_{n-1},F)+\frac{1}{\alpha}\|u_{n-1}\|,$$

by Lemma OOA, we obtain conclusion (ii).

It follows from conclusion (i) that, $\{x_n\}$ is bounded, then there exists a constant $M>0$, such that for any $n\ge 1$, we have $\|x_n-p\|\le M$. Therefore, it follows from Inequality (5) and condition (i) that

$$\|x_n-p\|\le\|x_{n-1}-p\|+\frac{1}{a}\|u_n\|-\frac{1}{M}(1-\alpha_n)\lambda\|x_n-T_nx_n\|^2$$

$$\frac{\lambda}{M}\sum_{j=1}^{n}(1-\alpha_j)\|x_j-T_jx_j\|^2\le\|x_0-p\|-\|x_n-p\|+\frac{1}{a}\sum_{j=1}^{n}\|u_j\|$$

$$\frac{\lambda}{M}\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-T_nx_n\|^2 \le \|x_0-p\|+\frac{1}{a}\sum_{n=1}^{\infty}\|u_n\|.$$

From condition (iii), we know that

$$\frac{\lambda}{M}\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-T_nx_n\|^2 < +\infty. \tag{7}$$

By condition (ii), we know

$$\liminf_{n\to\infty}\|x_n-T_nx_n\|=0.$$

This completes the proof of Theorem 1.

In Theorem 1, let $\{u_n\}=\{0\}$ and the condition (i) be substituted by the condition $\sum_{n=1}^{+\infty}(1-\alpha_n)^2<+\infty$, then the result of Theorem 1 is the theorem of Osilike-1[10].

Theorem 2 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F=\cap_{i=1}^N F(T_i)\neq\emptyset$, where $F(T_i)=\{x\in K: T_ix=x\}$, $\{\alpha_n\}\subset[0,1]$ be a real sequence, and $\{u_n\}\subset K$ be a sequence satisfying the conditions:*

(i) $0<a\le\alpha_n\le\beta<1$;

(ii) $\sum_{n=1}^{\infty}\|u_n\|<+\infty$.

Let $x_0\in K$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (4). Then

(i) $\lim_{n\to\infty}\|x_n-p\|$ *exists for all* $p\in F$;

(ii) $\lim_{n\to\infty}d(x_n,F)$ *exists, where* $d(x_n,F)=\inf_{p\in F}\|x_n-p\|$;

(iii) $\lim_{n\to\infty}\|x_n-T_nx_n\|=0$.

Proof It follows from Condition (i) and Inequality (7) that

$$\frac{\lambda}{M}\sum_{n=1}^{\infty}(1-\beta)\|x_n-T_nx_n\|^2 \le \frac{\lambda}{M}\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-T_nx_n\|^2 < +\infty. \tag{8}$$

Thus from Inequality (8) we have that $\lim_{n\to\infty}\|x_n-T_nx_n\|=0$. The proofs of conclusions (i) and (ii) are the same as Theorem 1. This completes the proof of Theorem 2.

Theorem 3 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F=\cap_{i=1}^N F(T_i)\neq\emptyset$, where $F(T_i)=\{x\in K: T_ix=x\}$, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0<\alpha<\alpha_n<1$;

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)=+\infty$;

(iii) $\sum_{n=1}^{+\infty}\|u_n\|<+\infty$.

Let $x_0\in K$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (4). Then $\{x_n\}$ converges strongly to a common fixed point $p\in F$ if and only if $\liminf_{n\to\infty}d(x_n,F)=0$.

Proof Suppose that $\{x_n\}$ converges strongly to a common fixed point $p\in F$. In view of the fact that $0\le d(x_n,F)\le\|x_n-p\|$, we see that

$$\liminf_{n\to\infty}d(x_n,F)=0.$$

Conversely, assume that $\liminf_{n\to\infty} d(x_n,F)=0$, by Theorem 1, then we have also $\lim_{n\to\infty} d(x_n,F)=0$. Hence there must exist $p_n\in F$ such that $\lim_{n\to\infty}\|x_n-p_n\|=0$. It follows from Inequality (6) that

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \|x_{n+m}-p_n\|+\|x_n-p_n\|\\ &\le \|x_{n+m-1}-p_n\|+\frac{1}{\alpha}\|u_{n+m-2}\|+\|x_n-p_n\|\\ &\le \frac{1}{\alpha}\sum_{i=n}^{n+m-2}\|u_i\|+2\|x_n-p_n\|\to 0,\quad \text{as}\quad n\to\infty.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Suppose $\lim_{n\to\infty} x_n=q$, then it follows from $\lim_{n\to\infty}\|x_n-p_n\|=0$ that $\lim_{n\to\infty} p_n=q$. Since strictly pseudoconrtactive mappings are Lipschitz mappings, we have

$$\begin{aligned}\|q-T_lq\| &\le \|q-p_n\|+\|p_n-T_lq\|\le\|T_lp_n-T_lq\|\\ &\le \|q-p_n\|+L\|p_n-q\|\to 0,\quad \text{as}\quad n\to\infty,\end{aligned}$$

for all $l=1,2,3\cdots,N$, that is $q\in F$. This completes the proof of Theorem 3.

Theorem 4 *Let T_n, K, $\{\alpha_n\}$ and $\{x_n\}$ be as in Theorem 2. If $\{x_n\}$ converges strongly to a point $q\in K$, then q must be a common fixed point of $\{T_n\}_{n=1}^N$.*

Proof If $\{x_n\}$ converges strongly to a point $q\in K$, then

$$\|T_nx_n-T_nq\|\le L\|x_n-q\|\to 0,\quad \text{as}\quad n\to\infty.$$

Thus it follows from $\lim_{n\to\infty}\|x_n-T_nx_n\|=0$ that $\|q-T_nq\|\to 0$, $(n\to\infty)$. That is, for any $l=1,2,3,\cdots,N$, we have $q=T_lq$. This completes the proof of Theorem 4.

Lemma *Let $a_1,a_2,a_3,\cdots,a_n$ be real numbers, then*

$$(\sum_{i=1}^n a_i)^2\le\sum_{i=1}^{n-1}2^ia_i^2+2^{n-1}a_n^2.$$

Proof If $n=2$, then $(a_1+a_2)^2\le 2a_1^2+2a_2^2$. If $n=3$, then $(a_1+a_1+a_3)^2\le 2a_1^2+2(a_2+a_3)^2\le 2a_1^2+2(2a_2^2+2a_3^2)\le 2a_1^2+2^2a_2^2+2^2a_3^2$, which leads to

$$(\sum_{i=1}^n a_i)^2\le\sum_{i=1}^{n-1}2^ia_i^2+2^{n-1}a_n^2,\quad \forall n\ge 2.$$

The proof is done.

Theorem 5 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K such that $F=\cap_{i=1}^N F(T_i)\ne\emptyset$, where $F(T_i)=\{x\in K: T_ix=x\}$, and let $\{\alpha_n\}_{n=1}^\infty\subset[1-2^{-n},1]$ be a real sequence. Let $x_0\in K$ and let $\{x_n\}_{n=1}^\infty$ be defined by*

$$x_n=\alpha_nx_{n-1}+\beta_nT_nx_n+\gamma_nu_n,\ \alpha_n+\beta_n+\gamma_n=1. \tag{9}$$

Here $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$ *are real sequences and* $\{u_n\} \in K$ *is bounded. Then* $\{x_n\}$ *is convergent.*

Note It is easy to prove that the implicit iteration processes (4) and (9) are equivalent.

Proof It is now well known that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{10}$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$. Let $p \in F$, it follows from Inequality (10) that

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n(x_{n-1}-p) + \beta_n(T_nx_n - p) + \gamma_n(u_n - p)\|^2 \\ &\le [\|\alpha_n(x_{n-1}-p) + \beta_n(T_nx_n - p)\| + \|\gamma_n(u_n-p)\|]^2 \\ &\le {\alpha_n}^2\|x_{n-1}-p\|^2 + 2\beta_n\langle T_nx_n - p, j(x-y)\rangle + \\ &\quad 2\gamma_n\|u_n - p\|\|\alpha_n(x_{n-1}-p) + \beta_n(T_nx_n-p)\| + {\gamma_n}^2\|u_n-p\|^2. \end{aligned} \tag{11}$$

Since $T_i : K \to K, I = 1,2,3,\cdots,N$ is strictly pseudocontractive, we have

$$\langle T_ix - T_iy, j(x-y)\rangle \le \|x-y\|^2 - \lambda_i\|x - T_ix - (y - T_iy)\|^2 (\lambda_i \in (0,1)).$$

Let $\lambda = \min_{1\le i\le N}\{\lambda_i\}$, then

$$\langle T_ix - T_iy, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|x - T_ix - (y - T_iy)\|^2 (\lambda \in (0,1)).$$

Thus, it follows from (11) that

$$\begin{aligned}\|x_n - p\|^2 \le& {\alpha_n}^2\|x_{n-1}-p\|^2 + 2\beta_n\|x_n - p\|^2 - 2\lambda\beta_n\|x_n - T_nx_n\|^2 + \\ & 2\gamma_n\|u_n-p\|\|\alpha_n(x_{n-1}-p) + \beta_n(T_nx_n - p)\| + {\gamma_n}^2\|u_n - p\|^2. \end{aligned} \tag{12}$$

Now we prove that $\{x_n\}$ is bounded. For $x_0 = y_0 \in K$, $\{y_n\}$ is defined by

$$y_n = \alpha_ny_{n-1} + (\beta_n + \gamma_n)T_ny_n.$$

It follows from [10] that $\{y_n\}$ is bounded. Since $\{y_n\}$ and $\{u_n\}$ are bounded, we have

$$\begin{aligned}\|y_n - x_n\| &\le \alpha_n\|y_{n-1} - x_{n-1}\| + \beta_n\|T_ny_n - T_nx_n\| + \gamma_n\|T_ny_n - u_n\| \\ &\le \alpha_n\|y_{n-1} - x_{n-1}\| + L\beta_n\|y_n - x_n\| + \gamma_nM,\end{aligned}$$

which leads to

$$\begin{aligned}\|y_n - x_n\| &\le \frac{\alpha_n}{1 - L\beta_n}\|y_{n-1} - x_{n-1}\| + \frac{\gamma_nM}{1 - L\beta_n} \\ &\le [1+\sigma_n]\|y_{n-1} - x_{n-1}\| + \gamma_nM_1.\end{aligned}$$

Using the assumptions of theorem and Lemma OAA, we know that $\lim_{n\to\infty}\|y_n - x_n\|$ exists. Since $\{y_n\}$ is bounded, it follows that $\{x_n\}$ is bounded. Therefore, it follows from (12) that

$$\|x_n - p\|^2 \le {\alpha_n}^2\|x_{n-1}-p\|^2 + 2\beta_n\|x_n - p\|^2 - 2\lambda\beta_n\|x_n - T_nx_n\|^2 + 2\gamma_nM_2 + {\gamma_n}^2M_2, \tag{13}$$

where M_2 is a constant. Since $\lim_{n\to\infty}\alpha_n=1, \lim_{n\to\infty}\beta_n=0$, there exists a positive integer N such that $\beta_n<(1-\lambda)/2, \forall n\geq N$. Thus $1-2\beta_n\geq\lambda, \forall n\geq N$. Hence it follows from (13) that for all $n\geq N$

$$\begin{aligned}\|x_n-p\|^2 &\leq \frac{\alpha_n^2}{1-2\beta_n}\|x_{n-1}-p\|^2-\frac{2\lambda\beta_n}{1-2\beta_n}\|x_n-T_nx_n\|^2+\frac{2\gamma_nM_2+{\gamma_n}^2M_2}{1-2\beta_n}\\ &=[1+\frac{{\alpha_n}^2+2\beta_n-1}{1-2\beta_n}]\|x_{n-1}-p\|^2-\frac{2\lambda\beta_n}{1-2\beta_n}[\alpha_n\|x_{n-1}-T_nx_n\|+\gamma_n\|u_n-T_nx_n\|]^2+\\ &\quad\frac{2\gamma_nM_2+{\gamma_n}^2M_2}{\lambda}\\ &\leq[1+\frac{2\beta_n}{\lambda}]\|x_{n-1}-p\|^2-\lambda\beta_n\|x_{n-1}-T_nx_n\|^2+\frac{2\gamma_nM_2+{\gamma_n}^2M_2}{\lambda}. \end{aligned}\tag{14}$$

Since $\alpha_n\in[1-2^{-n},1]$ and $\alpha_n+\beta_n+\gamma_n=1$, we have $\sum_{i=1}^\infty\frac{2\beta_n}{\lambda}<\infty$ and $\sum_{i=1}^\infty\frac{2\gamma_nM_2+{\gamma_n}^2M_2}{\lambda}<\infty$. It follows from Lemma OAA that $\lim_{n\to\infty}\|x_n-p\|$ exists. Therefore, $\{\|x_n-p\|\}$ is bounded. Thus, there exists a positive integer number R such that $\|x_n-p\|^2\leq R, \forall n\geq 1$. From (14) we have

$$\lambda\sum_{i=N+1}^n\beta_i\|x_{i-1}-T_ix_i\|^2\leq\|x_N-p\|^2+R\sum_{i=N+1}^n\sigma_i+\sum_{i=N+1}^n\frac{2\gamma_iM_2+{\gamma_i}^2M_2}{\lambda}.$$

Hence

$$\sum_{n=1}^\infty\beta_n\|x_{n-1}-T_nx_n\|^2<\infty.\tag{15}$$

From implicit iteration process (9), we obtain that

$$\|x_n-x_{n-1}\|\leq\beta_n\|T_nx_n-x_{n-1}\|+\gamma_n\|u_n-x_{n-1}\|.$$

$$\|x_n-x_{n-1}\|^2\leq{\beta_n}^2\|T_nx_n-x_{n-1}\|^2+2\beta_n\gamma_n\|T_nx_n-x_{n-1}\|\|u_n-x_{n-1}\|+{\gamma_n}^2\|u_n-x_{n-1}\|^2.\tag{16}$$

Since $\{x_n\}$ is bounded, it follows from (16) that

$$\|x_n-x_{n-1}\|^2\leq{\beta_n}^2\|T_nx_n-x_{n-1}\|^2+\gamma_nM_3+{\gamma_n}^2M_3.\tag{17}$$

Since

$$\|x_{n+m}-x_{n-1}\|\leq\sum_{i=n-1}^{n+m-1}\|x_{i+1}-x_i\|,$$

it follows from Lemma OAA that

$$\|x_{n+m}-x_{n-1}\|^2\leq\sum_{i=n-1}^{n+m-2}2^i\|x_{i+1}-x_i\|^2+2^{n+m-1}\|x_{n+m}-x_{n+m-1}\|^2.\tag{18}$$

Combining (17) and (18), we obtain that

$$\begin{aligned}\|x_{n+m}-x_{n-1}\|^2 &\leq\sum_{i=n-1}^{n+m-2}2^i{\beta_{i+1}}^2\|x_{i+1}-x_i\|^2+\\ &\sum_{i=n-1}^{n+m-2}(\gamma_{i+1}M_3+{\gamma_{i+1}}^2M_3)+2^{n+m-1}{\beta_{n+m}}^2\|T_{n+m}x_{n+m}-x_{n+m-1}\|^2+\\ &\gamma_{n+m}M_3+{\gamma_{n+m}}^2M_3.\end{aligned}\tag{19}$$

Since $\beta_n \le 2^{-n}$, from (19) we obtain

$$\|x_{n+m} - x_{n-1}\|^2 \le \sum_{i=n}^{n+m-1} \beta_i \|x_{i-1} - T_i x_i\|^2 + \sum_{i=n-1}^{n+m-2} (\gamma_{i+1} M_3 + {\gamma_{i+1}}^2 M_3) + 2^{n+m-1} {\beta_{n+m}}^2 \|T_{n+m} x_{n+m} - x_{n+m-1}\|^2 + \gamma_{n+m} M_3 + {\gamma_{n+m}}^2 M_3. \tag{20}$$

It follows from (15) and $\lim_{n\to\infty} \beta_n = 0, \lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} (\gamma_n M_3 + {\gamma_n}^2 M_3) < \infty$ that

$$\lim_{n\to\infty} \|x_{n+m} - x_{n-1}\| = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence, and $\{x_n\}$ converges strongly to a point $p \in E$. This completes the proof of Theorem 5.

Theorem 6 *Let the assumptions of Theorem 5 hold and $\{x_n\}$ be defined by (9). Then $\{x_n\}$ converges strongly to a common fixed point $p \in F$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.*

Proof Suppose that $\{x_n\}$ converges strongly to a common fixed point $p \in F$. In view of fact that $0 \le d(x_n, F) \le \|x_n - p\|$, we see that $\lim_{n\to\infty} d(x_n, F) = 0$.

Conversely, assume that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.1, we have $x_n \to p$. Hence $d(p, F) = 0$. It is easy to prove that the set of fixed points of strictly pseudocontractive mappings is closed, so F is closed and $p \in F$, that is, $\{x_n\}$ converges strongly to a common fixed point $p \in F$. This completes the proof of Theorem 6.

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具误差隐格式迭代逼近严格伪压缩映像族公共不动点

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摘要: 设 K 是实 Banach 空间 E 中非空闭凸集， $\{T_i\}_i=1^N$ 是 N 个具公共不动点集 F 的严格伪压缩映像， $\{\alpha_n\}\subset[0,1]$ 是实数列， $\{u_n\}\subset K$ 是序列，且满足下面条件
(i) $0<\alpha\le\alpha_n\le 1$;
(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)=+\infty$;
(iii) $\sum_{n=1}^{\infty}\|u_n\|<+\infty$.
设 $x_0\in K$, $\{x_n\}$ 由下式定义

$$x_n=\alpha_n x_{n-1}+(1-\alpha_n)T_n x_n+u_{n-1},\ \ n\ge 1,$$

其中 $T_n=T_n\mathrm{mod}N$, 则有下面结论
(i) $\lim_{n\to\infty}\|x_n-p\|$ 存在，对所有 $p\in F$;
(ii) $\lim_{n\to\infty}d(x_n,F)$ 存在，当 $d(x_n,F)=\inf_{p\in F}\|x_n-p\|$;
(iii) $\liminf_{n\to\infty}\|x_n-T_nx_n\|=0$.
文中另一个结果是, 如果 $\{x_n\}\subset[1-2^{-n},1]$, 则 $\{x_n\}$ 收敛. 文中结果改进与扩展了 Osilike(2004) 最近的结果，证明方法也不同.

关键词: 严格伪压缩映像；具误差隐格式迭代；公共不动点；收敛定理.