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Chromatic Choosability of a Class of Complete Multipartite Graphs

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Abstract: A graph G is called to be chromatic choosable if its choice number is equal to its chromatic number. In 2002, Ohba conjectured that every graph G with $2\chi(G)+1$ or fewer vertices is chromatic choosable. It is easy to see that Ohba's conjecture is true if and only if it is true for complete multipartite graphs. But at present only for some special cases of complete multipartite graphs, Ohba's conjecture have been verified. In this paper we show that graphs $K_{6,3,2*(k-6),1*4}$ $(k \geq 6)$ is chromatic choosable and hence Ohba's conjecture is true for the graphs $K_{6,3,2*(k-6),1*4}$ and all complete k-partite subgraphs of them.

Key words: list coloring; complete multipartite graph; chromatic choosable graph; Ohba's conjecture.
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1. Introduction

For a graph $G=(V,E)$ and each vertex $u \in V(G)$, let $L(u)$ denote a set (or a list) of colors available for u. $L=\{L(u)|u \in V(G)\}$ is said to be a list assignment of G. If $|L(u)|=k$ for all $u \in V(G)$, L is called a k-list assignment of G. A list-coloring (or L-coloring for short) from a given list assignment is a proper coloring c such that $c(u)$ is chosen from $L(u)$. We call a graph G L-colorable if G admits an L-coloring. A graph G is called k-choosable if G is L-colorable for every k-list assignment L. The choice number $Ch(G)$ of a graph G is the smallest k such that G is k-choosable. The concept of list coloring was introduced independently by V.G. Vizing[7], P. Erdös, A.L. Rubin, and H. Taylar[2]. For a recent survey, we refer the interested reader to the Ref. [8].

Clearly, $Ch(G) \geq \chi(G)$ holds for every graph G, where $\chi(G)$ denotes the chromatic number of G. On the other hand, Erdös et al[2] showed that bipartite graphs can have arbitrarily large choice number. Therefore, it is significant to investigate the condition or give some graph classes, in which each graph satisfies $Ch(G)=\chi(G)$. For convenience, a graph G is called chromatic choosable, if $Ch(G)=\chi(G)$[5]. About the chromatic choosable graph, a glamorous conjecture was given by K. Ohba. K. Ohba showed that

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Conjecture 1.1[5] *If $|V(G)| \le 2\chi(G)+1$, then $Ch(G)=\chi(G)$.*

Because every χ-chromatic graph is a subgraph of complete χ-partite graph, Ohba's conjecture is true if and only if it is true for complete χ-partite graph. Namely, Conjecture 1.1 is equivalent to Conjecture 1.2 in the following.

Conjecture 1.2 *If G is a complete k-partite graph with $|V(G)| \le 2k+1$, then $Ch(G)=\chi(G)=k$.*

We use the notation K_{r*s} for a complete s-partite graph in which each part is of size r. Notations such as $K_{r*s,t}$, etc. are used similarly.

For Conjecture 1.2, at present we know that only some special cases have been verified and all of them were obtained from the results of choice number of some complete multipartite graphs in the following.

Theorem 1.1[1] *If $k \ge s+1$ and $m \le 2s+1$, $Ch(K_{m,2*(k-s-1),1*s})=k$.*

In Theorem 1.1, let $m=s+3$, we know that $Ch(K_{s+3,2*(k-s-1),1*s})=k$ $(k \ge s+1)$. Namely Conjecture 1.2 is true for complete multipartite graphs with precisely one partite set of size greater than 2 as a general situation.

Theorem 1.2[6] $Ch(K_{3*r,1*t})=\max(r+t,\lceil\frac{4r+2t-1}{3}\rceil)$.

In Theorem 1.2, let $r=t+1$ and $k=r+t$, we know that $Ch(K_{3*(t+1),1*t})=k$. Namely Conjecture 1.2 is true for complete multipartite graphs $K_{3*(t+1),1*t}$ and all complete k-partite subgraphs of them, where $k=r+t$.

Theorem 1.3[3] *If $k \ge 3$, $Ch(K_{3*2,2*(k-2)})=k$.*

Theorem 1.3 showed that if $k \ge 3$, $Ch(K_{3*2,2*(k-3),1})=k$. Namely Conjecture 1.2 is true for complete multipartite graphs $K_{3*2,2*(k-3),1}$ and all complete k-partite subgraphs of them.

Another partial result is that the authors have showed that $Ch(K_{4,3,2*(k-4),1*2})=k$ $(k \ge 4)$ and $Ch(K_{5,3,2*(k-5),1*3})=k$ $(k \ge 5)$ in another paper. Namely Conjecture 1.2 is true for graphs $K_{4,3,2*(k-4),1*2}$ and $K_{5,3,2*(k-5),1*3}$, and all complete k-partite subgraphs of them.

In this paper, we will show that Conjecture 1.2 is true for graphs $K_{6,3,2*(k-6),1*4}$ and all complete k-partite subgraphs of them, where $k \ge 6$. In Section 2, we will introduce some propositions as a preparation to prove our main result. In Section 3, using these propositions, we will show that $Ch(K_{6,3,2*(k-6),1*4})=k$.

2. Some propositions

For a graph $G=(V,E)$ and a subset $X \subset V$, let $G[X]$ denote the subgraph of G induced by X. For a list assignment L of G, let $L|_X$ denote L restricted to X, and $L(X)$ denote the union $\bigcup_{u\in X} L(u)$. If A is a set of colors, let $L\backslash A$ denote the list assignment obtained from L by removing the colors in A from each $L(u)$ with $u \in V(G)$. When A consists of a single color a, we write $L-a$ instead of $L\backslash\{a\}$.

We say that G with L satisfies Hall's condition in G, if $|L(X)| \ge |X|$ for every subset

$X \subset V(G)$. It is clear that if G with L satisfies Hall's condition, then by Hall's marriage theorem, there exists an L-coloring for G, in which all vertices receive distinct colors.

The following proposition is proved in [4], which is the foundation of our proof. Here the statement is slightly different.

Proposition 2.1[4] *Let L be a list assignment for a graph $G=(V,E)$. Then G is L-colorable if $G[X]$ is $L|_X$-colorable for a maximal non-empty subset $X \subset V(G)$ such that $|L(X)| < |X|$.*

Let $G = K_{m_1,m_2,2*r,1*s}$ be any complete k-partite graph with $|V(G)| = 2k+1$, where $m_1 \geq m_2 \geq 3$, $r \geq 0$, $s \geq 1$, $2+r+s=k$ and $m_1+m_2+2r+s=2k+1$. Write the k parts of G with $V_1=\{x_1,x_2,\ldots,x_{m_1}\}$, $V_2=\{y_1,y_2,\ldots,y_{m_2}\}$, $U_i=\{u_i,v_i\}$ for $i=1,2,\ldots,r$, and $W_i=\{w_i\}$ for $i=1,2,\ldots,s$. Suppose that L is a k-list assignment of G such that G is not L-colorable. Under the above assumption, we have the following propositions.

Proposition 2.2 $\bigcap_{x_i\in V_1} L(x_i)=\Phi$, $\bigcap_{y_i\in V_2} L(y_i)=\Phi$.

Proof Suppose that there exists a color $a \in \bigcap_{x_i\in V_1} L(x_i)$. Then, assign a to all vertices x_i for $i=1,2,\ldots,m_1$. Note that $G'=G-V_1=K_{m_2,2*r,1*s}$ with $m_2=2k+1-m_1-2r-s \leq 2(2+r+s)+1-m_1-2r-s=s+5-m_1 \leq s+2 \leq 2s+1$, and $L'=L-a$ with $|L'(u)| \geq k-1$ for all $u \in V(G')$. By Theorem 1.1, G' is $(k-1)$-choosable. Hence we can obtain an L-coloring of G, a contradiction. Similarly, $\bigcap_{y_i\in V_2} L(y_i)=\Phi$.

Proposition 2.3 *If $r=0$ or $L(u_i)\cap L(v_i)=\Phi$ for $i=1,2,\ldots,r$, $r\neq 0$, then there exist $x_{i_1},x_{i_2}\in V_1$ and $y_{i_1},y_{i_2}\in V_2$ such that $L(x_{i_1})\cap L(x_{i_2})\neq\Phi$ and $L(y_{i_1})\cap L(y_{i_2})\neq\Phi$.*

Proof Without loss of generality, suppose that $L(x_1),L(x_2),\ldots,L(x_{m_1})$ are pairwise disjoint, then there must exist two vertices $y_{i_1},y_{i_2}\in V_2$ such that $L(y_{i_1})\cap L(y_{i_2})\neq\Phi$. Otherwise, it is obvious that G with L satisfies Hall's condition. This is a contradiction to that G is not L-colorable. Let A be a largest subset of V_2 such that $\bigcap_{y\in A} L(y)\neq\Phi$. By Proposition 2.2, we know that $2\leq|A|\leq m_2-1$. Choose a color $a\in\bigcap_{y\in A}L(y)$, and let $G'=G-A$, $L'=L-a$. Then $|L'(x_i)\cup L'(x_j)|\geq 2k-1$ for every $i,j=1,2,\ldots,m_1, i\neq j$; $|L'(y_i)|=k$ for every $y_i\in V_2\backslash A$; and $|L'(u_i)\cup L'(v_i)|\geq 2k-1$ for $i=1,2,\ldots,r$. Since G is not L-colorable, G' is not L'–colorable. In particular, G' with L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)|<|X|$. Clearly, $|X\cap V_1|\leq 1$ and $|X\cap U_i|\leq 1$ for $i=1,2,\ldots,r$. Otherwise, $2k-1\leq|L'(X)|<|X|\leq|V(G')|\leq 2k-1$, a contradiction. Hence $|X\backslash V_2|\leq k-1$. Note that $|L'(u)|\geq k-1$ for every $u\in X\backslash V_2$ and $|L'(u)|=k$ for every $u\in X\cap V_2$. It is obvious that $G'[X]$ is $L'|_X$-colorable. By Proposition 2.1, G' is L'-colorable. This is a contradiction.

The following three propositions are all obvious (we omit the proof of them), but all of them are very useful in the proof of our main result.

Proposition 2.4 *Let $G=K_{3,1}$ with two parts $U=\{u_1,u_2,u_3\}$ and $V=\{v\}$, and L be a list assignment on the vertices of G with $|L(u_1)|\geq 1$, $|L(u_2)|\geq 2, |L(u_3)|\geq 2$ and $|L(v)|\geq 2$. Then G is L-colorable.*

Proposition 2.5 *Let $G = K_{3,1}$ with two parts $U = \{u_1, u_2, u_3\}$ and $V = \{v\}$, and L be a list assignment on the vertices of G with $|L(u_i)| \geq 2$ for $i = 1, 2, 3$ and $|L(v)| = 1$. Then G is L-colorable.*

Proposition 2.6 *Let $G = K_{3,1}$ with two parts $U = \{u_1, u_2, u_3\}$ and $V = \{v\}$, and L be a list assignment on the vertices of G with $|L(u_1)| = |L(u_2)| = |L(u_3)| = 1$ and $|L(v)| \geq 4$. Then G is L-colorable.*

3. $Ch(K_{6,3,2*(k-6),1*4}) = k$

In order to prove $Ch(K_{6,3,2*(k-6),1*4}) = k$ by induction, we show $Ch(K_{6,3,1*4}) = 6$ first.

Theorem 3.1 $Ch(K_{6,3,1*4}) = 6$.

Proof For $G = K_{6,3,1*4}$, write its 6 parts with $V_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V_2 = \{y_1, y_2, y_3\}$, $W_i = \{w_i\}$ for $i = 1, 2, 3, 4$. By contradiction, assume that L is a list assignment with $|L(v)| = 6$ for each $v \in V(G)$ such that G is not L-colorable.

Let A be the largest subset of V_1 such that $\bigcap_{x\in A} L(x) \neq \Phi$. Then we know that $2 \leq |A| \leq 5$ by Propositions 2.2 and 2.3. Choose a color $c_1 \in \bigcap_{x\in A} L(x)$ to color all the vertices in A. Let $G' = G - A$, $L' = L - c_1$. As G is not L–colorable, G' is not L'-colorable. In particular, G' with L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. Clearly, $(X \cap V_1) \subset V_1 \backslash A$ and $|X \cap V_1| \leq 4$. In the following, we will prove that $G'[X]$ is $L'|_X$-colorable. Then G' is L'-colorable by Proposition 2.1. Thus we obtain a contradiction.

By the maximality of A, we have $|L'(x)| = 6$ for every $x \in V_1 \backslash A$. And from Proposition 2.2 we know that $|L'(y_i)| \geq 5$ for $i = 1, 2, 3$, and in $\{|L'(y_1)|, |L'(y_2)|, |L'(y_3)|\}$ there exists at least one being of 6. Without loss of generality, let $|L'(y_1)| \geq 5, |L'(y_2)| \geq 5$ and $|L'(y_3)| = 6$. We also know $|L'(w_i)| \geq 5$ for $i = 1, 2, 3, 4$.

Case 1 $|X \cap V_2| \leq 1$.

In this case, $|X \backslash V_1| \leq 5$. As $|L'(v)| \geq 5$ for every $v \in X \backslash V_1$, and $|L'(x)| = 6$ for every $x \in X \cap V_1$, it is obvious that $G'[X]$ is $L'|_X$-colorable.

Case 2 $|X \cap V_2| = 2$.

Let $\{y_p, y_q\} \subset X$, $\{p, q\} \subset \{1, 2, 3\}$, then $y_t \notin X$, where $t = \{1, 2, 3\} \backslash \{p, q\}$. Clearly, $X \subset \{y_p, y_q, w_1, w_2, w_3, w_4\} \cup (V_1 \backslash A)$, then $|X| \leq 10$.

If $L'(y_p) \cap L'(y_q) = \Phi$, we have $10 \leq |L(X)| < |X| \leq 10$. This is a contradiction.

If $L'(y_p) \cap L'(y_q) \neq \Phi$, choose a color $b \in L'(y_p) \cap L'(y_q)$. Note that $|L'(x)| = 6$ for every $x \in V_1 \backslash A$, $|L'(y_i)| \geq 5$ for $i = 1, 2, 3$, and $|L'(w_i)| \geq 5$ for $i = 1, 2, 3, 4$, we can color both verties y_p and y_q with color b, and then color w_1, w_2, w_3, w_4, and every vertex $x \in V_1 \backslash A$ in that order. Thus, $G'[X]$ is $L'|_X$-colorable.

Case 3 $|X \cap V_2| = 3$.

In this case, $\{y_1, y_2, y_3\} \subset X$, and we have Claim 3.1 as follows.

Claim 3.1 $|X \cap V_1| \leq 3$.

Otherwise, since $2 \leq |A| \leq 5$, then $|X \cap V_1| = 4$ and $|A| = 2$. Without loss of generality, let $A = \{x_5, x_6\}, X \cap V_1 = \{x_1, x_2, x_3, x_4\}$. Clearly, in this case, $|X| \leq 11$. And $|X| \leq 11$ implies that $|L'(x_i) \cap L'(x_j)| \geq 2$ for $i, j = 1, 2, 3, 4, i \neq j$; and $|L'(x_1) \cup L'(x_2) \cup L'(x_3) \cup L'(x_4)| \leq 10$. Otherwise, $11 \leq |L(X)| < |X| \leq 11$, a contradiction. On the other hand, we have $L'(x_p) \cap L'(x_q) \cap L'(x_r) = \Phi$ by the maximality of A and $|A| = 2$, where p, q and r are pairwise different and $\{p, q, r\} \subset \{1, 2, 3, 4\}$. So we have $|L'(x_1) \cup L'(x_2) \cup L'(x_3) \cup L'(x_4)| \geq 12$. This contradicts to $|L'(x_1) \cup L'(x_2) \cup L'(x_3) \cup L'(x_4)| \leq 10$.

Since $|X \cap V_1| \leq 3$, according to the size of $X \cap V_1$, we need to consider some subcases.

Subcase 3.1 $|X \cap V_1| \leq 1$ (no matter $|A| = 2, 3, 4$ or 5).

$|X \cap V_1| \leq 1$ implies that $|X| \leq 8$. If $L'(y_1) \cap L'(y_2) = \Phi$, then $10 \leq |L'(X)| < |X| \leq |V(G')| \leq 8$. This is a contradiction. If $L'(y_1) \cap L'(y_2) \neq \Phi$, choose a color $b \in L'(y_1) \cap L'(y_2)$. Note that $b \notin L'(y_3)$, we can color both y_1 and y_2 with color b, and then color w_1, w_2, w_3, w_4, vertex $x \in X \cap V_1$(if there exists a vertex x in $X \cap V_1$), and y_3 in that order. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.2 $|X \cap V_1| = 2$ (no matter $|A| = 2, 3$ or 4).

$|X \cap V_1| = 2$ implies that $|X| \leq 2+3+4 = 9$. Without loss of generality, say $X \cap V_1 = \{x_1, x_2\}$. It is obvious that $|L'(y_1) \cap L'(y_2)| \geq 2$ and $|L'(x_1) \cap L'(x_2)| \geq 4$. Otherwise, $9 \leq |L'(X)| < |X| \leq 9$, a contradiction.

Subcase 3.2.1 $L'(w_1), L'(w_2), L'(w_3)$ and $L'(w_4)$ are not the same color lists.

Choose a color $c_2 \in L'(y_1) \cap L'(y_2)$ and a color $c_3 \in L'(x_1) \cap L'(x_2)$ such that $c_3 \neq c_2$. Assign c_2 to both y_1 and y_2, and c_3 to both x_1 and x_2. Since $L'(w_1), L'(w_2), L'(w_3)$ and $L'(w_4)$ are not the same color lists, we know that $|L'(w_i) \backslash \{c_2, c_3\}| \geq 3$ for $i = 1, 2, 3, 4$, and at the same time $L'(w_1) \backslash \{c_2, c_3\}$, $L'(w_2) \backslash \{c_2, c_3\}$, $L'(w_3) \backslash \{c_2, c_3\}$ and $L'(w_4) \backslash \{c_2, c_3\}$ are not the same color lists. Hence, for $i = 1, 2, 3, 4$, we can choose a color d_i from $L'(w_i) \backslash \{c_2, c_3\}$ to color w_i. Note that $c_2 \notin L'(y_3)$, we can choose a color b from $L'(y_3) \backslash \{c_3, d_1, d_2, d_3, d_4\}$ to color y_3 afterwards. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.2.2 $L'(w_1) = L'(w_2) = L'(w_3) = L'(w_4)$.

Write $\{e_1, e_2, e_3, e_4, e_5\} \subset L'(w_i)$ for $i = 1, 2, 3, 4$. Clearly, $L'(y_1) \cap L'(y_2) \subset \{e_1, e_2, e_3, e_4, e_5\}$. Otherwise, it is easy to see that $G'[X]$ is $L'|_X$-colorable similarly to Subcase 3.2.1.

As $|L'(x_i)| = 6$, for $i = 1, 2$, there exists $a_i \in L'(x_i) \backslash \{e_1, e_2, e_3, e_4, e_5\}$, for $i = 1, 2$, respectively. Since $|L'(y_1) \cap L'(y_2)| \geq 2$ and $L'(y_1) \cap L'(y_2) \subset \{e_1, e_2, e_3, e_4, e_5\}$, we have $L'(y_3) \backslash \{e_1, e_2, e_3, e_4, e_5\} \geq 3$ by Proposition 2.2. Write $\{b_{34}, b_{35}, b_{36}\} \subset L'(y_3) \backslash \{e_1, e_2, e_3, e_4, e_5\}$. Use colors in $\{e_1, e_2, e_3, e_4, e_5\}$ to color y_1, y_2, w_1, w_2, w_3 and w_4, and then color x_1, x_2, y_3 with a_1, a_2 and a color in $\{b_{34}, b_{35}, b_{36}\} \backslash \{a_1, a_2\}$, respectively. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.3 $|X \cap V_1| = 3$ and $|A| = 2$.

Without loss of generality, let $A = \{x_5, x_6\}, X \cap V_1 = \{x_1, x_2, x_3\}$. Clearly, in this subcase, $|X| \le 10$. And $|X| \le 10$ implies that $|L'(x_i) \cap L'(x_j)| \ge 3$ for $i, j = 1, 2, 3, i \ne j$; $|L'(y_1) \cap L'(y_2)| \ge 1$, $|L'(y_1) \cap L'(y_3)| \ge 2$, $|L'(y_2) \cap L'(y_3)| \ge 2$, and $|L'(x_1) \cup L'(x_2) \cup L'(x_3)| \le 9$. Otherwise, $10 \le |L(X)| < |X| \le 10$, a contradiction. As $|A| = 2$, furthermore, we have Claim 3.2 as follows.

Claim 3.2 $|L'(x_i) \cap L'(x_j)| = 3$ for $i, j = 1, 2, 3, i \ne j$.

Otherwise, suppose that there exist $p, q \in \{1, 2, 3\}$, such that $|L'(x_p) \cap L'(x_q)| \ge 4$. Without loss of generality, say $|L'(x_1) \cap L'(x_2)| \ge 4$. Note that $|A| = 2$, by the maximality of A, we have $|L'(x_1) \cap L'(x_2) \cap L'(x_3)| = \Phi$. Therefore, $|L'(x_1) \cap L'(x_3)| \le 2$ by $|L'(x_1)| = 6$. This contradicts to $|L'(x_i) \cap L'(x_j)| \ge 3$ for $i, j = 1, 2, 3, i \ne j$.

As $|L'(y_1) \cap L'(y_2)| \ge 1$, choose a color c_2 to color both y_1 and y_2, choose color $d_i \in L'(w_i) - c_2$ to color w_i for $i = 1, 2, 3, 4$, respectively. Now we consider the induced subgraph $G'' = G'[x_1, x_2, x_3, y_3] = K_{3,1}$ and its list assignment $L'' = L' \backslash \{c_2, d_1, d_2, d_3, d_4\}$. Since $c_2 \notin L'(y_3)$ by Proposition 2.2, we have that $|L''(y_3)| \ge 2$. Since $|L'(x_i) \cap L'(x_j)| = 3$ for $i, j = 1, 2, 3, i \ne j$, by Claim 3.2, we have that $|L''(x_i)| \ge 1$ for $i, j = 1, 2, 3$, and in $\{|L''(x_1)|, |L''(x_2)|, |L''(x_3)|\}$ there exist at least two being of 3 or more. Without loss of generality, say $|L''(x_1)| \ge 1, |L''(x_2)| \ge 3$, and $|L''(x_3)| \ge 3$. Therefore, G'' is L''-colorable by Proposition 2.4. And hence $G'[X]$ is $L'|_X$-colorable.

Subcase 3.4 $|X \cap V_1| = 3$ and $|A| = 3$.

Without loss of generality, let $A = \{x_4, x_5, x_6\}, X \cap V_1 = \{x_1, x_2, x_3\}$. In this subcase, similarly to Subcase 3.3, we also have that $|X| \le 10$. And $|X| \le 10$ implies that $|L'(x_i) \cap L'(x_j)| \ge 3$ for $i, j = 1, 2, 3, i \ne j$, $|L'(y_1) \cap L'(y_2)| \ge 1$, $|L'(y_1) \cap L'(y_3)| \ge 2$, $|L'(y_2) \cap L'(y_3)| \ge 2$, and $|L'(x_1) \cup L'(x_2) \cup L'(x_3)| \le 9$.

If $|L'(x_i) \cap L'(x_j)| \le 4$ for any $i, j = 1, 2, 3$ and $i \ne j$, it is easy to see that $G'[X]$ is $L'|_X$-colorable similarly to Subcase 3.3.

If there exist $p, q \in \{1, 2, 3\}$, such that $|L'(x_p) \cap L'(x_q)| \ge 5$, it is clear that $|L'(x_1) \cap L'(x_2) \cap L'(x_3)| \ge 2$. Otherwise, we have that $|L'(x_1) \cup L'(x_2) \cup L'(x_3)| \ge 10$. This contradicts to $|L'(x_1) \cup L'(x_2) \cup L'(x_3)| \le 9$.

Subcase 3.4.1 $L'(w_1), L'(w_2), L'(w_3)$ and $L'(w_4)$ are not the same color lists.

As $|L'(y_1) \cap L'(y_2)| \ge 1$, and $|L'(x_1) \cap L'(x_2) \cap L'(x_3)| \ge 2$, we can choose a color $c_2 \in L'(y_1) \cap L'(y_2)$ and a color $c_3 \in L'(x_1) \cap L'(x_2) \cap L'(x_3)$ such that $c_3 \ne c_2$. Assign c_2 to both y_1 and y_2, and c_3 to all vertices in $\{x_1, x_2, x_3\}$. Since $L'(w_1), L'(w_2), L'(w_3)$ and $L'(w_4)$ are not the same color lists, we know that $|L'(w_i) \backslash \{c_2, c_3\}| \ge 3$ for $i = 1, 2, 3, 4$, and $L'(w_1) \backslash \{c_2, c_3\}$, $L'(w_2) \backslash \{c_2, c_3\}$, $L'(w_3) \backslash \{c_2, c_3\}$ and $L'(w_4) \backslash \{c_2, c_3\}$ are not the same color lists. Hence, for $i = 1, 2, 3, 4$, we can choose a color d_i from $L'(w_i) \backslash \{c_2, c_3\}$ to color w_i. Note that $c_2 \notin L'(y_3)$, we can choose a color b from $L'(y_3) \backslash \{c_3, d_1, d_2, d_3, d_4\}$ to color y_3 afterwards. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.4.2 $L'(w_1) = L'(w_2) = L'(w_3) = L'(w_4)$.

Without loss of generality, write $\{e_1, e_2, e_3, e_4, e_5\} \subset L'(w_i)$ for $i = 1, 2, 3, 4$.

Claim 3.3 $L'(y_1) \cap L'(y_2) \subset \{e_1, e_2, e_3, e_4, e_5\}$ and $L'(x_1) \cap L'(x_2) \cap L'(x_3) \subset \{e_1, e_2, e_3, e_4, e_5\}$.

Otherwise, it is easy to see that $G'[X]$ is $L'|_X$–colorable similarly to subcase 3.4.1.

Since that $|L'(x_1) \cap L'(x_2) \cap L'(x_3)| \geq 2$ and $L'(x_1) \cap L'(x_2) \cap L'(x_3) \subset \{e_1, e_2, e_3, e_4, e_5\}$, without loss of generality, let $\{e_1, e_2\} \subset L'(x_1) \cap L'(x_2) \cap L'(x_3)$.

Subcase 3.4.2.1 $c_1 \notin L(y_1)$ and $c_1 \notin L(y_2)$.

In this subcase, we have that $|L'(y_1)| = |L'(y_2)| = |L'(y_3)| = 6$, namely $|L'(y_i)\backslash\{e_1, e_2, e_3, e_4, e_5\}| \geq 1$. It is obvious that we can use colors in $\{e_1, e_2, e_3, e_4, e_5\}$ to color the vertices in $\{x_1, x_2, x_3, w_1, w_2, w_3, w_4\}$, and colors $b_i \in L'(y_i)\backslash\{e_1, e_2, e_3, e_4, e_5\}$ for $i = 1, 2, 3$, to color vertices y_i for $i = 1, 2, 3$, afterwards. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.4.2.2 $c_1 \in L(y_1)$ and $c_1 \in L(y_2)$.

In this subcase, we can show that G itself is L-colorable. This is a contradiction to the hypothesis that G is not L-colorable.

In fact, let $c(y_1) = c(y_2) = c_1$, $c(x_1) = c(x_2) = c(x_3) = e_1$, $c(w_1) = e_2$, $c(w_2) = e_3$, $c(w_3) = e_4$, $c(w_4) = e_5$. And then we consider the induced subgraph $G'' = G[x_4, x_5, x_6, y_3] = K_{3,1}$ and its list assignment $L''' = L\backslash\{c_1, e_1, e_2, e_3, e_4, e_5\}$. Since $c_1 \notin L'(y_3)$ by Proposition 2.2, we have that $|L'''(y_3)| \geq 1$. Since $\{e_1, e_2\} \subset L'(x_1) \cap L'(x_2) \cap L'(x_3)$, and $\{e_1, e_2\} \cap L(x_i) = \Phi$ for $i = 4, 5, 6$, by the maximality of A and $|A| = 3$, we have that $|L'''(x_i)| \geq 2$ for $i = 4, 5, 6$. Therefore, G'' is L'''-colorable by Proposition 2.5. And hence G is L-colorable.

Subcase 3.4.2.3 $c_1 \in L(y_1)$ but $c_1 \notin L(y_2)$, or $c_1 \in L(y_2)$ but $c_1 \notin L(y_1)$.

Without loss of generality, say $c_1 \in L(y_1)$ but $c_1 \notin L(y_2)$. Namely, $|L'(y_1)| = 5$ and $|L'(y_2)| = |L'(y_3)| = 6$.

Claim 3.4 $L'(y_1) = \{e_1, e_2, e_3, e_4, e_5\}$.

Otherwise, it is obvious that $|L'(y_i)\backslash\{e_1, e_2, e_3, e_4, e_5\}| \geq 1$. And we can use colors in $\{e_1, e_2, e_3, e_4, e_5\}$ to color the vertices in $\{x_1, x_2, x_3, w_1, w_2, w_3, w_4\}$, and colors $b_i \in L'(y_i)\backslash\{e_1, e_2, e_3, e_4, e_5\}$ for $i = 1, 2, 3$, to color vertices y_i for $i = 1, 2, 3$, afterwards. Thus, $G'[X]$ is $L'|_X$-colorable.

Subcase 3.4.2.3.1 $1 \leq |L'(y_1) \cap L'(y_2)| \leq 2$.

As $L'(y_1) = \{e_1, e_2, e_3, e_4, e_5\}$ and $|L'(y_1) \cap L'(y_2)| \leq 2$, we have that $|L'(y_2)\backslash\{e_1, e_2, e_3, e_4, e_5\}| \geq 4$. On the other hand, since $|L'(y_1) \cap L'(y_3)| \geq 2$ and $L'(y_1) = \{e_1, e_2, e_3, e_4, e_5\}$, we have $L'(y_1) \cap L'(y_3) \subset \{e_1, e_2, e_3, e_4, e_5\}$. Therefore, we can use colors in $\{e_1, e_2, e_3, e_4, e_5\}$ to color the vertices in $\{y_1, y_3, w_1, w_2, w_3, w_4\}$. Now we consider the induced subgraph $G'' = G'[x_1, x_2, x_3, y_2] = K_{3,1}$ and its list assignment $L'' = L'\backslash\{e_1, e_2, e_3, e_4, e_5\}$. Clearly, $|L''(y_2)| \geq 4$. Since $|L'(x_i)| = 6$ for $i = 1, 2, 3$, we have that $|L''(x_i)| \geq 1$ for $i = 1, 2, 3$. Therefore, G'' is L''-colorable by Proposition 2.6. And hence $G'[X]$ is $L'|_X$-colorable.

Subcase 3.4.2.3.2 $|L'(y_1) \cap L'(y_2)| \geq 3$.

From Claim 3.3, $L'(y_1) \cap L'(y_2) \subset \{e_1, e_2, e_3, e_4, e_5\}$, so in this subcase we have $|L'(y_3) \backslash\{e_1, e_2, e_3, e_4, e_5\}| \geq 4$ by Proposition 2.2. Firstly, we use colors in $\{e_1, e_2, e_3, e_4, e_5\}$ to color the vertices in $\{y_1, y_2, w_1, w_2, w_3, w_4\}$. Secondly, we consider the induced subgraph $G'' = G'[x_1, x_2, x_3, y_3] = K_{3,1}$ and its list assignment $L'' = L' \backslash \{e_1, e_2, e_3, e_4, e_5\}$. Clearly, $|L''(y_3)| \geq 4$. Since $|L'(x_i)| = 6$ for $i = 1, 2, 3$, we have that $|L''(x_i)| \geq 1$ for $i = 1, 2, 3$. Therefore, G'' is L''-colorable by Proposition 2.6. And hence $G'[X]$ is $L'|_X$-colorable.

Combine all discussions above, we have shown that Theorem 3.1 holds.

Theorem 3.2 $Ch(K_{6,3,2*(k-6),1*4}) = k$, where $k \geq 6$.

Proof For $G = K_{6,3,2*(k-6),1*4}$, write its k parts with $V_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V_2 = \{y_1, y_2, y_3\}$, $U_i = \{u_i, v_i\}$ for $i = 1, 2, \ldots, k-6$, $W_i = \{w_i\}$ for $i = 1, 2, 3, 4$. We will use induction on k. If $k = 6$, by Theorem 3.1, we are done. Suppose that $k \geq 7$, and Theorem 3.2 is true for smaller value of k. By contradiction, assume that $Ch(K_{6,3,2*(k-6),1*4}) \neq k$, and L is a list assignment of G such that G is not L-colorable.

Claim 3.5 $L(u_i) \cap L(v_i) = \Phi$ for every $1 \leq i \leq k-6$.

Otherwise, suppose that there exists a color $a \in L(u_i) \cap L(v_i)$. Then assign a to both u_i and v_i, and apply induction to $G - U_i$ and $L - a$. Thus we can obtain an L-coloring of G, a contradiction.

Let A be the largest subset of V_1 such that $\bigcap_{x \in A} L(x) \neq \Phi$. Then we know that $2 \leq |A| \leq 5$ by Propositions 2.2 and 2.3. Choose a color $c_1 \in \bigcap_{x \in A} L(x)$ to color the vertices in A. Let $G' = G - A$, $L' = L - c_1$. As G is not L-colorable, G' is not L'-colorable. In particular, G' with L' does not satisfy Hall's condition. Let X be a maximal subset of $V(G')$ such that $|L'(X)| < |X|$. In the following, we will prove that $G'[X]$ is $L'|_X$-colorable. Then G' is L'-colorable by Proposition 2.1. Thus we obtain a contradiction.

By the maximality of A, we have $|L'(x)| = k$ for every $x \in V_1 \backslash A$. And by Proposition 2.2 we know that $|L'(y_i)| \geq k-1$ for $i = 1, 2, 3$, and in $\{|L'(y_1)|, |L'(y_2)|, |L'(y_3)|\}$ there exists at least one being of k. Without loss of generality, let $|L'(y_1)| \geq k-1, |L'(y_2)| \geq k-1$ and $|L'(y_3)| = k$. And we know $|L'(u_i)| \geq k-1$, $|L'(v_i)| \geq k-1$ and $|L'(u_i) \cup L'(v_i)| \geq 2k-1$, for $i = 1, 2, \ldots, k-6$, by Claim 3.5. We also know $|L'(w_i)| \geq k-1$ for $i = 1, 2, 3, 4$.

Claim 3.6 $|X \cap U_i| \leq 1$ for every $i = 1, 2, \ldots, k-6$.

Otherwise, by Claim 3.5, $2k-1 \leq |L'(X)| < |X| \leq |V(G')| \leq 2k-1$, a contradiction.

By Claim 3.6, for every $z_i \in \bigcup_{1 \leq i \leq k-6}(X \cap U_i)$, choose a color b_i from $L'(z_i)$ and assign it to z_i such that $b_1, b_2, \ldots, b_t$ are pairwise different, where $t = |\bigcup_{1 \leq i \leq k-6}(X \cap U_i)|$, $0 \leq t \leq k-6$. Let $G'' = G' - (\bigcup_{1 \leq i \leq k-6} U_i)$, $X' = X \backslash (\bigcup_{1 \leq i \leq k-6}(X \cap U_i))$, $L'' = L' - \{b_1, b_2, \ldots, b_t\}$. We only need to prove $G''[X']$ is $L''|_{X'}$-colorable. Note that in $V(G'')$, $|L''(x)| \geq k-t \geq 6$ for every $x \in V_1 \backslash A$, $|L''(y_1)| \geq k-1-t \geq 5$, $|L''(y_2)| \geq k-1-t \geq 5$, $|L''(y_3)| \geq k-t \geq 6$, and $|L''(w_i)| \geq k-1-t \geq 5$ for $i = 1, 2, 3, 4$.

Finally, in the proof of Theorem 3.1, replace $G'[X]$ by $G''[X']$ and L' by L'', we can show that $G''[X]$ is $L''|_{X'}$-colorable similarly.

Remark Though we have only proved that $Ch(K_{s+2,3,2*(k-s-2),1*s}) = k$ in this paper for $s = 4$, and in another paper for $s = 2, 3$ (from Theorem 1.3 we know that the above equality holds for $s = 1$), we believe that Ohba's conjecture can be proved for complete multipartite graphs $K_{s+2,3,2*(k-s-2),1*s}$ $(s \geq 1)$ and $K_{m_1,m_2,2*(k-s-2),1*s}$ $(m_1 + m_2 + 2r + s \leq 2k + 1)$ by using the same method of this paper.

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一类完全多部图的色可选择性

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摘要: 如果一个图 G 的选择数等于它的色数，则称该图 G 是色可选择的．在 2002 年， Ohba 给出如下猜想：每一个顶点个数小于等于 $2\chi(G)+1$ 的图 G 是色可选择的．容易发现 Ohba 猜想成立的条件是当且仅当它对完全多部图成立，但是目前只是就某些特殊的完全多部图的图类证明了 Ohba 猜想的正确性．在本文我们证明图 $K_{6,3,2*(k-6),1*4}(k \geq 6)$ 是色可选择的，从而对图 $K_{6,3,2*(k-6),1*4}(k \geq 6)$ 和它们的所有完全 $k-$ 部子图证明了 Ohba 猜想成立.

关键词: 列表染色；完全多部图；色可选择图； Ohba 猜想.