第27卷第2期 数 学 研 究 与 评 论 Vol.27, No.2
2007年5月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION May, 2007

Article ID: 1000-341X(2007)02-0289-09 **Document code**: A

Orthogonal Multiple Vector-Valued Wavelet Packets

CHEN Qing-jiang[1,2], *CHENG Zheng-xing*[1], *LI Xue-zhi*[3]

(1. School of Science, Xi'an University of Arch. & Tech., Shaanxi 710055, China;
2. School of Science, Xi'an Jiaotong University, Shaanxi 710049, China;
3. Department of Mathematics, Xinyang Normal University, Henan 464000, China)
(E-mail: chenxj6684@mail.xjtu.edu.cn)

Abstract: The multiple vector-valued wavelet packets are defined and investigated. A procedure for constructing the multiple vector-valued wavelet packets is presented. The properties of multiple vector-valued wavelet packets are discussed by using integral transformation and operator theory. Finally, new orthogonal bases of $L^2(R, C^{s\times s})$ is constructed from the orthogonal multiple vector-valued wavelet packets.

Key words: vector-valued multiresolution analysis; multiple vector-valued scaling functions; multiple vector-valued wavelet packets; refinement equation.
MSC(2000): 42C40
CLC number: O174.2

1. Introduction

In order to improve the localization of the frequency field of wavelet bases, Coifman and Meyer firstly introduced the notion of orthogonal univariate wavelet packets. They have been applied to signal processing[1], image compression[2], solving integral equations[3] and so on. Yang[4] constructed a-scale orthogonal multiwavelet packets which are more flexible in applications. Xia and Suter[5] introduced the notion of multiple vector-valued wavelets and investigated the construction of multiple vector-valued wavelets. Multiple vector-valued wavelets are a class of generalized multiwavelets[6]. However, multiwavelets and multiple vector-valued wavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms[7] but not necessary for discrete multiple vector-valued wavelet transforms. Video image and medical CT formation are nevertheless multiple vector-valued signals. Thus, it is necessary to extend the notion of orthogonal wavelet packets to the case of orthogonal multiple vector-valued wavelets. Inspired by [4] and [5], we shall give the definition of the orthogonal multiple vector-valued wavelet packets and investigate their properties.

Throughout this paper, let R and C be sets of all real and complex numbers, respectively. Let Z stand for all integers and $Z_+ = \{z: z \geq 0, z \in Z\}$. Assume s is a constant and $2 \leq s \in Z$. Write by $\ell^2(Z)^{s\times s} = \{\mathbf{Q}: Z \longrightarrow C^{s\times s}, \|\mathbf{Q}\|_2 = (\sum_{\imath,\jmath=1}^{s}\sum_{k\in Z}|q_{\imath,\jmath}(k)|^2)^{\frac{1}{2}} < +\infty\}$. By $\mathbf{I}_s$ and $\mathbf{O}$, we denote the $s\times s$ identity matrix and zero matrix, respectively. The space $L^2(R, C^{s\times s})$ is defined as the set of all multiple vector-valued function $H(t)$, i.e.,

Received date: 2005-10-17; **Accepted date**: 2006-07-03
Foundation item: the National Natural Science Foundation of China (10371105).

$$L^2(R,C^{s\times s}) := \left\{ H(t) := \begin{pmatrix} h_{11}(t) & h_{12}(t) & \cdots & h_{1\,s}(t) \\ h_{21}(t) & h_{22}(t) & \cdots & h_{2\,s}(t) \\ \cdots & \cdots & \cdots & \cdots \\ h_{s1}(t) & h_{s2}(t) & \cdots & h_{s\,s}(t) \end{pmatrix} : \begin{array}{c} t \in R,\ h_{k\,l}(t) \in L^2(R), \\ k,\ l = 1,2,\ldots,s \end{array} \right\}.$$

For any $H \in L^2(R,C^{s\times s})$, $\|H\|$ denotes the norm of H, i.e., $\|H\| := (\sum_{k,l=1}^{s} \int_R |h_{k,\,l}(t)|^2 \mathrm{d}t)^{1/2}$. Its integration $\int_R H(t)\mathrm{d}t$ is defined as $\int_R H(t)\mathrm{d}t := (\int_R h_{k,l}(t)\,\mathrm{d}t\,)_{k,l=1}^{s}$, and the Fourier transform of $H(t)$ is defined as $\widehat{H}(\omega) := \int_R H(t)\exp\{-i\omega t\}\,\mathrm{d}t$, $\omega \in R$. For two multiple vector-valued functions $H,\ G \in L^2(R,C^{s\times s})$, their symbol inner product is defined to be $\langle\, H, G\,\rangle := \int_R H(t)G(t)^*\mathrm{d}t$. Here and afterwards, * means the transpose and the complex conjugate.

Definition 1 *A multiple vector-valued function $H(t) \in L^2(R,C^{s\times s})$ is said to be orthogonal, if its integer translations satisfy*

$$\langle\, H(\cdot\,), H(\cdot - k)\,\rangle = \delta_{0,\,k}\mathbf{I}_s, \quad k \in Z, \tag{1}$$

where $\delta_{0,\,k}$ is the Kronecker symbol, i.e., $\delta_{0,\,k} = 1$ when $k = 0$ and $\delta_{0,\,k} = 0$ when $k \neq 0$.

Definition 2 *We say a sequence of multiple vector-valued functions $\{\,H_k(t)\}_{k\in Z} \subset \mathbf{U} \subset L^2(R,C^{s\times s})$ is an orthogonal basis of $\mathbf{U}$ if it satisfies formula (1) and for any $\Gamma(t) \in \mathbf{U}$, there exists a unique sequence $\{P_k\}_{k\in Z}$, of which each element is an $s\times s$ matrix such that*

$$\Gamma(t) = \sum_{k\in Z} P_k H_k(t),\ \ t \in R.$$

This paper is organized as follows: In Section 2, we briefly recall the concept of vector-valued multiresolution analysis. In Section 3, we give our main result, the definition of the multiple vector-valued wavelet packets and their properties. In the final section, new orthogonal bases of $L^2(R,C^{s\times s})$ will be constructed.

2. Vector-valued multiresolution analysis

We begin with the generic setting of a vector-valued multiresolution analysis of $L^2(R,C^{s\times s})$.

Let $\Upsilon\,(t) \in L^2(R,C^{s\times s})$ satisfy the following refinement equation:

$$\Upsilon\,(t) = 4\cdot \sum_{k\in Z} A_k \Upsilon(4\,t - k), \tag{2}$$

where $\{A_k\}_{k\in Z} \in \ell^2(Z)^{s\times s}$ is a finitely supported sequence of $s\times s$ matrices.

Without loss of generality, we assume $\widehat{\Upsilon}(\omega)$ is continuous at the origin and $\widehat{\Upsilon}(0) = \mathbf{I}_s$.

Define a closed subspace $\mathbf{V}_j \subset L^2(R,C^{s\times s})$ by

$$\mathbf{V}_j = \mathbf{clos}_{L^2(R,C^{s\times s})}(\mathrm{Span}\{\,\Upsilon(4^j\ \cdot - k) : k \in Z\,\}),\ \ j \in Z. \tag{3}$$

If the closed subspace sequence $\{\,\mathbf{V}_j\}_{j\in Z}$ defined in (3) satisfies the following conditions:

(i). $\cdots \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \cdots$;

(ii). $\bigcap_{j\in Z} \mathbf{V}_j = \{\mathbf{O}\}$; $\bigcup_{j\in Z} \mathbf{V}_j$ is dense in $L^2(R,C^{s\times s})$;

(iii). $H(\cdot)\in\mathbf{V}_j\Longleftrightarrow H(4\cdot)\in\mathbf{V}_{j+1},\ \forall j\in Z$;

(iv). The families $\{\Upsilon(t-k),k\in Z\}$ forms an orthogonal basis for $\mathbf{V}_0$;

then we say $\Upsilon(t)$ in (2) generates a vector-valued multiresolution analysis $\{\{\mathbf{V}\}_{j\in Z},\,\Upsilon(t)\}$ and $\Upsilon(t)$ is called a multiple vector-valued scaling function.

Let $\mathbf{W}_j$, $j\in Z$, be the orthocomplementary space of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$ and there exist three multiple vector-valued function $\Gamma_\nu(t)\in L^2(R,C^{s\times s}),\nu=1,2,3$, such that their translations and dilations form a *Riesz* basis of $\mathbf{W}_j$, i.e.,

$$\mathbf{W}_j=\mathbf{clos}_{L^2(R,C^{s\times s})}\langle\Gamma_\nu(4^j\ \cdot-k):\,k\in Z,\ \nu=1,2,3\rangle,\ j\in Z. \tag{4}$$

Since $\Gamma_\imath(t)\in\mathbf{W}_0\subset\mathbf{V}_1$, $\imath=1,2,3$, there exist three finitely supported sequences of $s\times s$ matrices $\{B_k^{(\imath)}\}_{k\in Z}\in\ell^2(Z)^{s\times s}$ such that $\Gamma_\imath(t)=4\cdot\sum_{k\in Z}B_k^{(\imath)}\Upsilon_\imath(4t-k)$, $\imath=1,2,3$.

If $\Upsilon(t)\in L^2(R,C^{s\times s})$ is an orthogonal multiple vector-valued scaling function, then by Definition 1, we have

$$\langle\,\Upsilon(\cdot),\Upsilon(\cdot-n)\,\rangle=\delta_{0,\,n}\mathbf{I}_s,\quad n\in Z. \tag{5}$$

We say that $\Gamma_\imath(t)\in L^2(R,C^{s\times s}),\imath=1,2,3$, are orthogonal multiple vector-valued wavelet functions associated with the orthogonal multiple vector-valued scaling function $\Upsilon(t)$ if

$$\langle\,\Upsilon(\cdot),\,\Gamma_\imath(\cdot-n)\,\rangle=\mathbf{O},\qquad \imath=1,2,3,\qquad n\in Z, \tag{6}$$

and $\{\,\Gamma_\imath(t-k),\,k\in Z,\,\imath=1,2,3\}$ is an orthogonal basis of $\mathbf{W}_0$. Then, we have

$$\langle\,\Gamma_\imath(\cdot),\,\Gamma_\jmath(\cdot-n)\,\rangle=\delta_{0,\,n}\mathbf{I}_s,\ \imath,\,\jmath\in\{1,2,3\},\ n\in Z. \tag{7}$$

Lemma 1 *Let $H(t)\in L^2(R,C^{s\times s})$. Then $H(t)$ is an orthogonal multiple vector-valued function if and only if*

$$\sum_{l\in Z}\widehat{H}(\omega+2l\pi)\widehat{H}(\omega+2l\pi)^*=\mathbf{I}_s. \tag{8}$$

Proof If $H(t)$ is an orthonormal function, then we get from (1) that

$$\begin{aligned}\delta_{0,\,k}\mathbf{I}_s=\langle\,H(\cdot)\,,\,H(\cdot-k)\,\rangle&=\frac{1}{2\pi}\int_R\widehat{H}(\omega)\widehat{H}(\omega)^*\cdot\exp\{ik\omega\}\mathrm{d}\omega\\&=\frac{1}{2\pi}\int_0^{2\pi}\sum_{l\in Z}\widehat{H}(\omega+2l\pi)\widehat{H}(\omega+2l\pi)^*\cdot\exp\{ik\omega\}\mathrm{d}\omega\end{aligned}$$

which implies (8). The converse is obvious. □

By Lemma 1, Formulas (5)–(7) and Fourier transformation, we can obtain the following Lemma 2.

Lemma 2[5] *Let $\Upsilon(t)\in L^2(R,C^{s\times s})$ be an orthogonal multiple vector-valued scaling function. Assume $\Gamma_\imath(t)\in L^2(R,C^{s\times s})$, $\imath=1,2,3$, are orthogonal multiple vector-valued wavelet functions associated with $\Upsilon(t)$. Then*

$$\sum_{\sigma=0}^{3}\mathcal{A}\,[\,\omega+(\sigma\pi)/2]\,\mathcal{A}[\,\omega+(\sigma\pi)/2]^*=\mathbf{I}_s,\quad \omega\in R. \tag{9}$$

$$\sum_{\sigma=0}^{3}\mathcal{A}\,[\,\omega+(\sigma\pi)/2]\,\mathcal{B}^{(\imath)}[\,\omega+(\sigma\pi)/2]^{*}=\mathbf{O},\ \imath=1,2,3,\ \ \omega\in R. \tag{10}$$

$$\sum_{\sigma=0}^{3}\mathcal{B}^{(\imath)}[\,\omega+(\sigma\pi)/2]\,\mathcal{B}^{(\jmath)}[\,\omega+(\sigma\pi)/2]^{*}=\delta_{\imath,\jmath}\mathbf{I}_s,\imath,\ \jmath=1,2,3. \tag{11}$$

We now present multiple vector-valued Meyer wavelets as a special family of multiple vector-valued wavelets. For details on scalar-valued Meyer wavelets, see [8]. Let

$$\widehat{\Upsilon}(\omega)=\begin{cases}\mathbf{I}_s, & |\omega|<\frac{2\pi}{3},\\ \cos[\,\frac{\pi}{2}f(\frac{3}{2\pi}|\omega|-1)\,]\Lambda(\omega), & \frac{2\pi}{3}\le|\omega|\le\frac{4\pi}{3},\\ 0, & \text{otherwise},\end{cases} \tag{12}$$

where $\Lambda(\omega)$ is paraunitary and $\Lambda(\frac{2\pi}{3})=\Lambda(-\frac{2\pi}{3})=\mathbf{I}_s$, and $f(t)$ is a scalar-valued smooth function such that

$$f(t)=\begin{cases}1, & t\ge 1,\\ 0, & t\le 0,\end{cases}\quad\text{and}\ \ f(t)+f(1-t)=1,\quad\text{for}\ \ t\in(0,\,1).$$

Then, after computation, for $\omega\in R$, we obtain that $\sum_{k\in Z}\widehat{\Upsilon}(\omega+2k\pi)\widehat{\Upsilon}(\omega+2k\pi)^{*}=\mathbf{I}_s$.

By Lemma 1, $\Upsilon(t)$ is orthogonal. This implies that the multiple vector-valued functions $\Upsilon(t)$, defined in (12), is a multiple vector-valued scaling function. Similar to the scalar-valued Meyer wavelets[8,p138] the corresponding lowpass filter $\mathcal{A}(\omega)$ is

$$\mathcal{A}(\omega)=\sum_{k\in Z}\widehat{\Upsilon}(2(\omega+2k\pi)),\quad \omega\in R.$$

By using paraunitary vector filter theory[5], we can obtain three filter functions $\mathcal{B}^{(1)}(\omega)$, $\mathcal{B}^{(2)}(\omega)$, $\mathcal{B}^{(3)}(\omega)$ satisfying (10) and (11). Let $\widehat{\Gamma}_\imath(\omega)=\mathcal{B}^{(\imath)}(\omega/4)\widehat{\Upsilon}(\omega/4)$, $\imath=1,2,3$. Then, $\Gamma_\imath(t)$, $\imath=1,2,3$, are multiple vector-valued Meyer wavelets[5].

3. The properties of multiple vector-valued wavelet packets

Xia and Suter[5] introduced the multiple vector-valued wavelets and studied the construction of the multiple vector-valued wavelets. In this section, we shall define the multiple vector-valued wavelet packets and discuss their properties. First, we set

$$\mathbf{\Psi}_0(t)=\Upsilon(t),\quad \mathbf{\Psi}_\imath(t)=\Gamma_\imath(t);\quad P_k^{(0)}=A_k,\quad P_k^{(\imath)}=B_k^{(\imath)},\quad \imath=1,2,3,\quad k\in Z.$$

Definition 3 *The family of multiple vector-valued functions* $\{\,\mathbf{\Psi}_{4n+\lambda}(t),\ n=0,1,2,\ldots,\lambda=0,1,2,3\,\}$ *is called a multiple vector-valued wavelet packets with respect to the orthogonal multiple vector-valued scaling function* $\Upsilon(t)$ *where*

$$\mathbf{\Psi}_{4n+\lambda}(t)=4\cdot\sum_{k\in Z}P_k^{(\lambda)}\,\mathbf{\Psi}_n(4t-k),\ \ \lambda=0,1,2,3. \tag{13}$$

By implementing the Fourier transform for the both sides of (13), we have

$$\widehat{\mathbf{\Psi}}_{4n+\lambda}(\omega) = \mathcal{P}^{(\lambda)}(\omega/4)\, \widehat{\mathbf{\Psi}}_{n}(\omega/4),\ \lambda = 0,1,2,3, \tag{14}$$

where

$$\mathcal{P}^{(\lambda)}(\omega) = \sum_{k\in Z} P_k^{(\lambda)} \cdot \exp\{-ik\omega\},\quad \omega \in R. \tag{15}$$

Then $\mathcal{P}^{(0)}(\omega) = \mathcal{A}(\omega),\ \mathcal{P}^{(\imath)}(\omega) = \mathcal{B}^{(\imath)}(\omega),\ \imath = 1,2,3$. Formulas (9)–(11) can jointly be written as

$$\sum_{\sigma=0}^{3} \mathcal{P}^{(\imath)}(\,\omega + \frac{\sigma\pi}{2})\,\mathcal{P}^{(\jmath)}(\,\omega + \frac{\sigma\pi}{2})^* = \delta_{\imath,\jmath}\mathbf{I}_s,\quad \imath,\,\jmath \in \{\,0,1,2,3\},\ \ \omega \in R. \tag{16}$$

It is evident that Formula (16) is equivalent to

$$\sum_{\sigma\in Z} P^{(\imath)}_{\sigma+4k}(P^{(\jmath)}_{\sigma+4l})^* = \frac{1}{4}\delta_{\imath,\jmath}\delta_{k,\,l}\mathbf{I}_s,\quad \imath,\,\jmath \in \{\,0,1,2,3\},\quad k,\,l \in Z. \tag{17}$$

In the following, we will discuss properties of the multiple vector-valued wavelet packets.

Theorem 1 *If* $\{\,\mathbf{\Psi}_n(t),\ n \in Z_+\}$ *are multiple vector-valued wavelet packets with respect to the orthogonal multiple vector-valued scaling function* $\Upsilon(t)$, *then for every* $n \in Z_+$, *we have*

$$\langle\, \mathbf{\Psi}_n(\cdot - j\,)\,,\, \mathbf{\Psi}_n(\cdot - k)\,\rangle =\, \delta_{j,\,k}\,\mathbf{I}_s,\quad j,\,k \in Z. \tag{18}$$

Proof (i) Formula (18) follows from (5) for the case of $n = 0$. (ii) Assume that Formula (18) holds, for the case of $0 \le n < 4^{\mathcal{L}}$, where $\mathcal{L}$ is a positive integer. Then, when $4^{\mathcal{L}} \le n < 4^{\mathcal{L}+1}$, we have $4^{\mathcal{L}-1} \le [\frac{n}{4}] < 4^{\mathcal{L}}$ where $[\rho] = \max\{\nu \in Z,\, \nu \le \rho\}$. Order $n = 4[\frac{n}{4}] + \lambda,\ \lambda = 0,1,2,3$. By induction assumption and Lemma 1, we obtain

$$\langle\, \mathbf{\Psi}_{[\frac{n}{4}]}(\cdot - j),\, \mathbf{\Psi}_{[\frac{n}{4}]}(\cdot - k)\,\rangle =\, \delta_{j,\,k}\mathbf{I}_s \Longleftrightarrow \sum_{l\in Z} \widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega + 2l\pi)\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega + 2l\pi)^* = \mathbf{I}_s. \tag{19}$$

According to Lemma 1 and Formulas (14), (16), (19), we have

$$\begin{aligned}
&\langle\, \mathbf{\Psi}_n(\cdot - j\,)\,,\, \mathbf{\Psi}_n(\cdot - k)\,\rangle \\
&= \frac{1}{2\pi}\int_R \widehat{\mathbf{\Psi}}_n(\omega)\widehat{\mathbf{\Psi}}_n(\omega)^* \cdot \exp\{-i(j-k)\omega\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_{l\in Z}\int_{8l\pi}^{8(l+1)\pi} P^{(\lambda)}(\frac{\omega}{4})\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\frac{\omega}{4})\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\frac{\omega}{4})^* P^{(\lambda)}(\frac{\omega}{4})^* \cdot e^{-i(j-k)\omega}\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^{2\pi} P^{(\lambda)}(\omega)\{\sum_{l\in Z}\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega + 2l\pi)\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega + 2l\pi)^*\}P^{(\lambda)}(\omega)^* \cdot e^{-4i(j-k)\omega}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sum_{\sigma=0}^{3}\mathcal{P}^{(\lambda)}(\,\omega + \frac{\sigma\pi}{2})\,\mathcal{P}^{(\lambda)}(\,\omega + \frac{\sigma\pi}{2})^* \cdot e^{-4i(j-k)\omega}\mathrm{d}\omega \\
&= \delta_{j,\,k}\mathbf{I}_s.
\end{aligned}$$

Theorem 2 *If $\{\mathbf{\Psi}_n(t),\ n\in Z_+\}$ are multiple vector-valued wavelet packets with respect to the orthogonal multiple vector-valued scaling function $\Upsilon(t)$, then for every $n\in Z_+$, we have*

$$\langle\,\mathbf{\Psi}_{4n+\lambda}(\cdot)\,,\ \mathbf{\Psi}_{4n+\mu}(\cdot-k)\,\rangle=\ \delta_{\lambda,\,\mu}\,\delta_{0,\,k}\mathbf{I}_s,\quad \lambda,\ \mu=\{\,0,1,2,3\},\ \ k\in Z. \tag{20}$$

Proof By Formulas (14) and (16), we get that

$$\begin{aligned}\langle\,\mathbf{\Psi}_{4n+\lambda}(\cdot)\,,\ \mathbf{\Psi}_{4n+\mu}(\cdot-k)\,\rangle&=\frac{1}{2\pi}\int_R P^{(\lambda)}(\frac{\omega}{4})\widehat{\mathbf{\Psi}}_n(\frac{\omega}{4})\,\widehat{\mathbf{\Psi}}_n(\frac{\omega}{4})^*P^{(\mu)}(\frac{\omega}{4})^*\cdot e^{ik\omega}\mathrm{d}\omega\\&=\frac{1}{2\pi}\int_0^{8\pi}P^{(\lambda)}(\frac{\omega}{4})\{\sum_{l\in Z}\widehat{\mathbf{\Psi}}_n(\frac{\omega}{4}+2l\pi)\,\widehat{\mathbf{\Psi}}_n(\frac{\omega}{4}+2l\pi)^*\}P^{(\mu)}(\frac{\omega}{4})^*\,e^{ik\omega}\,\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{2\pi}P^{(\lambda)}(\omega)P^{(\mu)}(\omega)^*\cdot\exp\{4ik\omega\}\,\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sum_{\sigma=0}^{3}\mathcal{P}^{(\lambda)}(\,\omega+\frac{\sigma\pi}{2})\,\mathcal{P}^{(\mu)}(\,\omega+\frac{\sigma\pi}{2})^*\exp\{4ik\omega\}\,\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\delta_{\lambda,\,\mu}\,\mathbf{I}_s\cdot\exp\{4ik\omega\}\,\mathrm{d}\omega=\delta_{\lambda,\,\mu}\,\delta_{0,\,k}\mathbf{I}_s.\end{aligned}$$

Theorem 3 *For any $m,\ n\in Z_+$ and $k\in Z$, we have*

$$\langle\,\mathbf{\Psi}_m(\cdot)\,,\ \mathbf{\Psi}_n(\cdot-k)\,\rangle=\delta_{m,\,n}\,\delta_{0,\,k}\mathbf{I}_s. \tag{21}$$

Proof For $m=n$, (21) follows by Theorem 1. Without loss of generality, we suppose $m>n$ for the case of $m\neq n$. Rewrite $m,\ n$ as $m=4[m/4]+\lambda_1,\ \ n=4[n/4]+\mu_1$ where $\lambda_1,\ \mu_1\in\{\,0,1,2,3\}$.

(i). If $[m/4]=[n/4]$, then $\lambda_1\neq\mu_1$. Formula (21) follows from (14),(16) and (19), since

$$\begin{aligned}\langle\mathbf{\Psi}_m\,(\cdot)\mathbf{\Psi}_n(\cdot-k)\rangle&=\frac{2}{\pi}\int_R P^{(\lambda_1)}(\omega)\widehat{\mathbf{\Psi}}_{[\frac{m}{4}]}(\omega)\,\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega)^*P^{(\mu_1)}(\omega)^*\cdot\exp\{4ik\omega\}\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{2\pi}P^{(\lambda_1)}(\omega)\{\sum_{l\in Z}\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega+2l\pi)\widehat{\mathbf{\Psi}}_{[\frac{n}{4}]}(\omega+2l\pi)^*\,\}P^{(\mu_1)}(\omega)^*\cdot e^{4ik\omega}\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sum_{\sigma=0}^{3}\mathcal{P}^{(\lambda_1)}(\,\omega+\frac{\sigma\pi}{2})\,\mathcal{P}^{(\mu_1)}(\,\omega+\frac{\sigma\pi}{2})^*\cdot\exp\{4ik\omega\}\,\mathrm{d}\omega\\&=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\delta_{\lambda_1,\,\mu_1}\,\mathbf{I}_s\cdot\exp\{4ik\omega\}\,\mathrm{d}\omega=\mathbf{O}.\end{aligned}$$

(ii). If $[\frac{m}{4}]\neq[\frac{n}{4}]$, then set $[m/4]=4[\,[m/4]/4]+\lambda_2,\ [n/4]=4[\,[n/4]/4]+\mu_2,\ \ \lambda_2,\ \mu_2\in\{\,0,1,2,3\}$. If $[\,[m/4]/4]=[\,[n/4]/4]$ Formula (21) follows similar to the case (i); If $[\,[m/4]/4]\neq[\,[n/4]/4]$, then we order $[\,[m/4]/4]=4[\,[\,[m/4]4]/4]+\lambda_3,\ [\,[n/4]/4]=4[\,[\,[n/4]4]/4]+\mu_3,\ \ \lambda_3,\ \mu_3\in\{\,0,1,2,3\}$ once more. Thus, after taking finite times steps (denoted by κ), denoting by $\lambda_\kappa=\overbrace{[[\cdots[}^{\kappa}m/4]\cdots]/4],\ \mu_\kappa=\overbrace{[\,[\cdots[}^{\kappa}n/4]\cdots]/4],$ we obtain $\lambda_\kappa,\ \mu_\kappa\in\{0,1,2,3\}$ and

$$\star\ \ \lambda_\kappa=\mu_\kappa=1,\quad\text{or}\quad\lambda_\kappa=\mu_\kappa=2,\quad\text{or}\quad\lambda_\kappa=\mu_\kappa=3;$$

$$\star\star\ \ \lambda_\kappa=1,\mu_\kappa=0,\lambda_\kappa=2,\mu_\kappa=0,\quad\text{or}\quad\lambda_\kappa=2,\mu_\kappa=1\ \ \text{or}\quad\lambda_\kappa=3,\mu_\kappa=0,$$

$$\lambda_\kappa = 3,\ \ \mu_\kappa = 1, \ \ \text{or} \ \ \lambda_\kappa = 3,\ \ \mu_\kappa = 2.$$

For the case $\star$, (21) fillows similarly as the case (i). As for the case $\star\star$, we get from (6) and (7) that $\sum_{l\in Z} \widehat{\mathbf{\Psi}}_{\lambda_\kappa}(\omega+2l\pi)\, \widehat{\mathbf{\Psi}}_{\mu_\kappa}(\omega+2l\pi)^* = \mathbf{O}$, $\omega \in R$. Thus

$$\begin{aligned}
&\langle\, \mathbf{\Psi}_m(\cdot)\,,\ \mathbf{\Psi}_n(\cdot-k)\,\rangle = \frac{1}{2\pi}\int_R \widehat{\mathbf{\Psi}}_m(\omega)\ \widehat{\mathbf{\Psi}}_n(\omega)^* \cdot \exp\{ik\omega\}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\cdot\int_R P^{(\lambda_1)}(\frac{\omega}{4})\ \widehat{\mathbf{\Psi}}_{[m/4]}(\frac{\omega}{4})\ \widehat{\mathbf{\Psi}}_{[n/4]}(\frac{\omega}{4})^* P^{(\mu_1)}(\frac{\omega}{4})^* \cdot \exp\{ik\omega\}\,\mathrm{d}\omega = \cdots \\
&= \frac{1}{2\pi}\int_R \prod_{\sigma=1}^{\kappa} P^{(\lambda_\sigma)}(\frac{\omega}{4^\sigma})\,\widehat{\mathbf{\Psi}}_{\lambda_\kappa}(\frac{\omega}{4^\kappa})\,\widehat{\mathbf{\Psi}}_{\mu_\kappa}(\frac{\omega}{4^\kappa})^*\,(\prod_{\sigma=1}^{\kappa} P^{(\mu_\sigma)}(\frac{\omega}{4^\sigma}))^* \cdot \exp\{ik\omega\}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_0^{4^{\kappa+1}\pi} \prod_{\sigma=1}^{\kappa} P^{(\lambda_\sigma)}(\frac{\omega}{4^\sigma})(\sum_{l\in Z}\widehat{\mathbf{\Psi}}_{\lambda_\kappa}(\frac{\omega}{4^\kappa}+2l\pi)\ \widehat{\mathbf{\Psi}}_{\mu_\kappa}(\frac{\omega}{4^\kappa}+2l\pi)^*\,)(\prod_{\sigma=1}^{\kappa} P^{(\mu_\sigma)}(\frac{\omega}{4^\sigma}))^* \cdot e^{ik\omega}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_0^{4^{\kappa+1}\pi} \prod_{\sigma=1}^{\kappa} P^{(\lambda_\sigma)}(\frac{\omega}{4^\sigma})\cdot\mathbf{O}\cdot(\prod_{\sigma=1}^{\kappa} P^{(\mu_\sigma)}(\frac{\omega}{4^\sigma}))^* \cdot \exp\{ik\omega\}\,\mathrm{d}\omega = \mathbf{O}.
\end{aligned}$$

In a word, for any $m,\, n \in Z_+$ and $k \in Z$, Formula (21) holds. □

Lemma 3 *If* $\{\,\mathbf{\Psi}_n(t),\ n=0,1,2\ldots\}$ *are multiple vector-valued wavelet packets with respect to the orthogonal multiple vector-valued scaling function* $\Upsilon(t)$, *then for every* $n \in Z+$, *we have*

$$\mathbf{\Psi}_n(4t-k) = \frac{1}{4}\sum_{\sigma=0}^{3}\sum_{l\in Z}(P^{(\sigma)}_{k-4l})^*\mathbf{\Psi}_{4n+\sigma}(t-l),\ \ k\in Z. \tag{22}$$

Proof

$$\begin{aligned}
&\frac{1}{4}\sum_{\sigma=0}^{3}\sum_{l\in Z}(P^{(\sigma)}_{k-4l})^*\mathbf{\Psi}_{4n+\sigma}(t-l) = \sum_{\sigma=0}^{3}\sum_{l\in Z}(P^{(\sigma)}_{k-4l})^*\sum_{j\in Z}P^{(\sigma)}_{j}\,\mathbf{\Psi}_n(4t-4l-j) \\
&= \sum_{\sigma=0}^{3}\sum_{l\in Z}\sum_{m\in Z}(P^{(\sigma)}_{k-4l})^* P^{(\sigma)}_{m-4l}\,\mathbf{\Psi}_n(4t-m) = \sum_{m\in Z}\{\sum_{\sigma=0}^{3}\sum_{l\in Z}(P^{(\sigma)}_{k-4l})^*\,P^{(\sigma)}_{m-4l}\,\}\mathbf{\Psi}_n(4t-m) \\
&= \sum_{m\in Z}\delta_{k,\,m}\mathbf{I}_s\,\mathbf{\Psi}_n(4t-m) = \mathbf{\Psi}_n(4t-k).
\end{aligned}$$

4. Orthogonal vector-valued wavelet bases of $L^2(R, C^{s\times s})$

In this section we shall construct orthogonal vector-valued wavelet bases of $L^2(R,\, C^{s\times s})$ by using the multiple vector-valued wavelet packets.

We start from introducing a dilation operator $(D\hbar)(t) = \hbar(\,4t)$ where $\hbar(t) \in L^2(R,\, C^{s\times s})$. For $\forall\ \mathbf{\Omega} \subset L^2(R,\, C^{s\times s})$ and $\forall\ n \in Z_+$, denote $D\mathbf{\Omega}$ by $D\mathbf{\Omega} = \{D\hbar : \hbar \in \mathbf{\Omega}\}$, and define

$$\mathbf{\Omega}_n = \{\,\hbar(t):\ \hbar(t) = \sum_{k\in Z} Q_k\Psi_n(t-k),\ \{Q_k\}_{k\in Z} \in \ell^2(Z)^{s\times s}\}. \tag{23}$$

Then $\mathbf{\Omega}_0 = \mathbf{V_0},\ \mathbf{\Omega}_1 \bigoplus \mathbf{\Omega}_2 \bigoplus \mathbf{\Omega}_3 = \mathbf{W}_0$, where $\bigoplus$ denotes the orthogonal direct sum.

For $X, Y \subset R$, set $\sigma X = \{\sigma \cdot x : \sigma \in R,\ x \in X\}$, $X + Y = \{x + y : x \in X, y \in Y\}$. For a fix positive integer τ, denote by $\widetilde{E}_\tau = \sum_{\imath=0}^{\imath=\tau} 4^\imath \{0, 1, 2, 3\}$, $E_\tau = \widetilde{E}_\tau \backslash \widetilde{E}_{\tau-1}$.

Lemma 4 *For arbitrary $n \in Z_+$, the space $D\mathbf{\Omega}_n$ can be orthogonally decomposed into spaces $\mathbf{\Omega}_{4n+\lambda}$, $\lambda = 0, 1, 2, 3$, i.e., $D\mathbf{\Omega}_n = \bigoplus_{\lambda=0}^{3} \mathbf{\Omega}_{4n+\lambda}$.*

Proof By Formulas (13) and (23), we have $\bigoplus_{\lambda=0}^{3} \mathbf{\Omega}_{4n+\lambda} \subset \mathbf{\Omega}_n$ for $\forall\ n \in Z_+$.

On the other hand, $\mathbf{\Omega}_{4n}$, $\mathbf{\Omega}_{4n+1}$, $\mathbf{\Omega}_{4n+2}$ and $\mathbf{\Omega}_{4n+3}$ are orthogonal each other according to Theorem 2. By Lemma 3, we get

$$\mathbf{\Psi}_n(4t - k) = \frac{1}{4} \sum_{\sigma=0}^{3} \sum_{l \in Z} (P_{k-4l}^{(\sigma)})^* \mathbf{\Psi}_{4n+\sigma}(t - l), \quad k \in Z.$$

Therefore, the basis of the space $\mathbf{\Omega}_n$ can be linearly represented by the basis of the space $\mathbf{\Omega}_{4n+\lambda}$, $\lambda = 0, 1, 2, 3$. Thus, $\mathbf{\Omega}_n \subset \bigoplus_{\lambda=0}^{3} \mathbf{\Omega}_{4n+\lambda}$. This leads to $D\mathbf{\Omega}_n = \bigoplus_{\lambda=0}^{3} \mathbf{\Omega}_{4n+\lambda}$. □

Theorem 4 *The family of multiple vector-valued functions $\{\mathbf{\Psi}_n(\cdot - k),\ n \in E_\tau,\ k \in Z\}$ forms an orthogonal basis of $D^\tau \mathbf{W}_0$. In particular, the set $\{\mathbf{\Psi}_n(\cdot - k),\ n \in Z_+,\ k \in Z\}$ constitutes an orthogonal basis of $L^2(R, C^{s\times s})$.*

Proof By Lemma 4, we get that $D\mathbf{\Omega}_0 = \mathbf{\Omega}_0 \bigoplus_{\lambda=0}^{3} \mathbf{\Omega}_\lambda$, i.e., $D\mathbf{V}_0 = \mathbf{V}_0 \bigoplus \mathbf{W}_0$.

It can inductively be proved by using Theorem 2 and Lemma 4 that

$$D^\tau \mathbf{V}_0 = \bigoplus_{n \in \widetilde{E}_\tau} \mathbf{\Omega}_n, \quad \text{and} \quad D^\tau \mathbf{V}_0 \bigoplus D^\tau \mathbf{W}_0 = D^{\tau+1} \mathbf{V}_0, \quad \text{i.e.,} \quad D^\tau \mathbf{W}_0 = \bigoplus_{n \in E_\tau} \mathbf{\Omega}_n.$$

Therefore, the set $\{\mathbf{\Psi}_n(\cdot - k),\ n \in E_\tau,\ k \in Z\}$ forms an orthogonal basis of $D^\tau \mathbf{W}_0$. Moreover

$$L^2(R, C^{s\times s}) = \mathbf{V}_0 \bigoplus (\bigoplus_{0 \le \tau} D^\tau \mathbf{W}_0) = \mathbf{\Omega}_0 \bigoplus (\bigoplus_{0 \le \tau} (\bigoplus_{n \in E_\tau} \mathbf{\Omega}_n)) = \bigoplus_{n \in Z_+} \mathbf{\Omega}_n.$$

That is, the set $\{\mathbf{\Psi}_n(\cdot - k),\ n \in Z_+,\ k \in Z\}$ forms an orthogonal basis of $L^2(R, C^{s\times s})$. □

Theorem 5 *For each $\tau \in Z_+ \backslash \{0\}$, the family of multiple vector-valued functions $\{\mathbf{\Psi}_n(4^j t - k),\ n \in E_\tau,\ j \in Z,\ k \in Z\}$ forms an orthogonal basis of $L^2(R, C^{s\times s})$.*

Proof Note that by Theorem 4, $\{\mathbf{\Psi}_n(t - k),\ n \in E_\tau,\ k \in Z\}$ forms an orthogonal basis of $D^\tau \mathbf{W}_0$. Then for each $j \in Z$, $\{\mathbf{\Psi}_n(4^j t - k),\ n \in E_\tau,\ k \in Z\}$ forms an orthogonal basis of $D^j D^\tau \mathbf{W}_0$. Hence, For each $\tau \in Z_+ \backslash \{0\}$,

$$\bigoplus_{j \in Z} D^j D^\tau \mathbf{W}_0 = \bigoplus_{j \in Z} D^{j+\tau} \mathbf{W}_0 = \bigoplus_{j \in Z} D^j \mathbf{W}_0.$$

Thus, the family $\{\mathbf{\Psi}_n(4^j t - k),\ n \in E_\tau,\ j, k \in Z\}$ forms an orthogonal basis of $L^2(R, C^{s\times s})$. □

Acknowledgement The authors greatly thank the anonymous referees and the editors for their valuable suggestions and comments which make the article more readable.

References:

[1] HE Hai-bo, CHENG Shi-jie. *Home network power-line communication signal processing based on wavelet packet analysis* [J]. IEEE Trans. Power Delivery, 2005, **20**(3): 1879–1885.
[2] ZHANG Ning, WU Xiao-lin. *Lossless compression of color mosaic images* [J]. IEEE Trans Image Processing, 2006, **15**(16): 1379–1388.
[3] DENG Hai, LING Hao. *Fast solution of electromagnetic integral equations using adaptive wavelet packet transform* [J]. IEEE Trans. Antennas and Propagation, 1999, **47**(4): 674–682.
[4] YANG Shou-zhi, CHENG Zheng-xing. *Orthonormal multi-wavelet packets with scale* $= a$ [J]. Math. Appl. (Wuhan), 2000, **13**(1): 61–65. (in Chinese)
[5] XIA Xiang-gen, SUTER B W. *Vector-valued wavelets and vector filter banks* [J]. IEEE Trans. Signal Processing, 1996, **44**(3): 508–518.
[6] EFROMOVICH S, LAKEY J, PEREYRA M C. et al. *Data-driven and optimal denoising of a signal and recovery of its derivative using multiwavelets* [J]. IEEE Trans. Signal Process, 2004, **52**(3): 628–635.
[7] XIA Xiang-gen. *A new prefilter design of prefilters for discrete multiwavelet transforms* [J]. IEEE Trans. Signal Processing, 1998, **46**(6): 1558–1570.
[8] DAUBECHIES I. *Ten Lectures on Wavelets* [M]. Academic: New York, 1992.
[9] XIA Xiang-gen, SUTER B W. *FIR paraunitary filter banks given several analysis filters: Factorizations and constructions*[J]. IEEE Trans. Signal Processing, 1996, **44**(3): 720–723.

多重向量值正交小波包

陈清江 [1,2], 程正兴 [1], 李学志 [3]
(1. 西安建筑科技大学理学院，陕西 西安 710055; 2. 西安交通大学理学院，陕西 西安 710049;
3. 信阳师范学院数学系，河南 信阳 464000)

摘要: 本文引入多重向量值小波包的概念，提供一类多重向量值正交小波包的构造方法，并运用积分变换和算子理论，讨论了多重向量值正交小波包的性质. 利用多重向量值正交小波包，构造了空间 $L^2(R, C^{s\times s})$ 的新的正交基.

关键词: 向量值多分辨分析；多重向量值尺度函数；多重向量值小波包；加细方程.