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Characterization of Weak I Sequences

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Abstract: In [5], Zhou defined the notion of weak I sequences and characterized such sequences by Koszul cohomology and local cohomology methods. The aim of this paper is to characterize weak I sequences by means of Ext functor.

Key words: Noetherian local ring; Ext functor.
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1. Introduction

Throughout this paper, we use A to denote a commutative Noetherian local ring with unit, m the maximal ideal of A, and I a proper ideal of A. We denote by M a finite d-dimensional A-module. For an A-module N, we use $\mathrm{Ext}^n(-,N)$ to denote the n-th right derived functor of $\mathrm{Hom}(-,N)$. Moreover, we denote $\bigcup_{n\geq 1}(0:_M I^n)$ by $\Gamma_I(M)$. Finally, we use Z^+ to denote the set of positive integers.

Recall that a sequence $x_1,x_2,\ldots,x_r$ contained in I is said to be a weak I sequence with respect to M, if for $1\leq i\leq r$ and a fixed positive integer n

$$(x_1^{n_1},\ldots,x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1},\ldots,x_{i-1}^{n_{i-1}})M : I^n$$

holds for $n_1,n_2,\ldots,n_r$ running through all positive integers[1]. In particular, if $x_1,x_2,\ldots,x_d$ is a weak m sequence with respect to M, then M is a generalized Cohen-Macaulay A-module, and $x_1,x_2,\ldots,x_d$ must be a system of parameters for M[1]. It is well-known that a weak I sequence can be characterized by means of local cohomology and Koszul homology. In this paper, we make use of Ext functor to characterize the properties of weak I sequences and obtain the following results.

Theorem 1.1 *Let I be an ideal of A and M a finite A-module such that $\Gamma_I(M)\neq M$. If there exists an $N>0$ such that $I^N\mathrm{Ext}_A^i(A/I^n,M)=0$, for $n\in Z^+$ and $0\leq i<k$, then there exists a weak I sequences $x_1,x_2,\ldots,x_k$ with respect to M.*

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Theorem 1.2 *Let I be an ideal of A and M a finite A-module with $\Gamma_I(M) \neq M$. Set*

$$r = \inf\{i | I^N \mathrm{Ext}^i(A/I^n, M) \neq 0, N \text{ is an arbitrary fixed number and for all } n\}.$$

Then any maximal weak I sequence with respect to M has the same length r.

2. Basic properties of weak I sequences

In the section, we study some basic properties of weak I sequences by means of Ext functor. First of all, we prove the following lemma.

Lemma 2.1 *Let I be an ideal of the local ring A and M a finite A-module. If there exists a positive integer N such that $I^N \mathrm{Ext}^i_A(A/I^n, M) = 0$ for all $i \geq 0$ and $n \in Z^+$, then $M = \Gamma_I(M)$.*

Proof We use induction on $\dim M = d$. For $d = 0$, the result is trivial. Now assume $d > 0$. Suppose that $M \neq \Gamma_I(M)$, we put $\bar{M} = M/\Gamma_I(M)$. Clearly, $\Gamma_I(\bar{M}) \neq \bar{M}$. Consider the short exact sequence $0 \to \Gamma_I(M) \to M \to \bar{M} \to 0$, then we have the long exact sequence

$$\cdots \to \mathrm{Ext}^i_A(A/I^n, \Gamma_I(M)) \to \mathrm{Ext}^i_A(A/I^n, M) \to \mathrm{Ext}^i_A(A/I^n, \bar{M}) \to \cdots.$$

By the condition, we have $I^N \Gamma_I(M) = 0$, so $I^N \mathrm{Ext}^i_A(A/I^n, \Gamma_I(M)) = 0$. Hence $I^N \mathrm{Ext}^i_A(A/I^n, \Gamma_I(\bar{M})) = 0$ holds for all $i \geq 0$ and $n > 0$. On the other hand, there exists an $\bar{M}$-regular element $x \in I$. Choose a positive integer $s > 2N$ and consider the short exact sequence

$$0 \longrightarrow \bar{M} \xrightarrow{x^s} \bar{M} \longrightarrow \bar{M}/x^s\bar{M} \longrightarrow 0.$$

Then, we have the long exact sequence

$$\cdots \to \mathrm{Ext}^i_A(A/I^n, \bar{M}) \xrightarrow{x^s} \mathrm{Ext}^i_A(A/I^n, \bar{M}) \to \mathrm{Ext}^i_A(A/I^n, \bar{M}/x^s\bar{M}) \to \cdots.$$

From this we obtain $I^{2N}\mathrm{Ext}^i_A(A/I^n, \bar{M}/x^s\bar{M}) = 0$, for all $i \geq 0$ and $n > 0$. In particular, $I^{2N}\Gamma_I(\bar{M}/x^s\bar{M}) = 0$. Since $\dim \bar{M} = d - 1$, by induction we have $\Gamma_I(\bar{M}/x^s\bar{M}) = \bar{M}/x^s\bar{M}$. It follows that $I^{2N}\bar{M} \subseteq x^s\bar{M}$. Hence $I^{2N}\bar{M} \subseteq x^{s-2N}I^{2N}\bar{M}$. By Nakayama Lemma, we have $I^{2N}\bar{M} = 0$, so $\Gamma_I(M) = M$ and this is a contradiction.

Proposition 2.2 *Let I be an ideal of a local ring A and M be a finite A-module with $\Gamma_I(M) \neq M$. If a sequence $x_1, x_2, \ldots, x_s$ contained in I satisfies*

$$(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})M : I^k, \quad 1 \leq i \leq s,$$

where k is a fixed positive integer and $n_1, n_2, \ldots, n_s$ run through all positive integers, then there exists a positive integer $N \in Z^+$ which depends only on k such that $I^N\mathrm{Ext}^i_A(A/I^n, M) = 0$ for $i < s$ and all $n > 0$.

Proof We use induction on $\dim M = d$. For $d = 0$, the result is trivial. Now suppose that $d > 0$ and the result holds for those A-modules M_1 with $\dim M_1 < d$. Set $\bar{M} = M/\Gamma_I(M)$. By the condition $\Gamma_I(M) \neq M$, so $\Gamma_I(\bar{M}) \neq \bar{M}$. Note that there exists an $\bar{M}$-regular element $x_1 \in I$. Choose a positive integer t such that $t > 2k$ and consider the following short exact sequence $0 \longrightarrow \bar{M} \xrightarrow{x_1^t} \bar{M} \longrightarrow \bar{M}/x_1^t\bar{M} \longrightarrow 0$, so we can deduce the long exact sequence

$$\cdots \to \operatorname{Ext}_A^i(A/I^n, \bar{M}) \xrightarrow{x_1^t} \operatorname{Ext}_A^i(A/I^n, \bar{M}) \to \operatorname{Ext}_A^i(A/I^n, \bar{M}/x_1^t\bar{M}) \to \cdots .$$

Clearly, $x_2, x_3, \ldots, x_s$ satisfies

$$(x_2^{n_2}, \ldots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_2^{n_2}, \ldots, x_{i-1}^{n_{i-1}})M : I^{2k}, \quad 2 \le i \le s,$$

where $n_2, \ldots, n_s$ run through all positive integers. Observe that $\dim \bar{M}/x_1^t\bar{M} = d-1$, so by induction, we can choose an integer $N \in Z^+$ such that $I^N \operatorname{Ext}_A^i(A/I^n, \bar{M}/x_1^t\bar{M}) = 0$, $i < s-1$, for all $n, t \in Z^+$. Now for an arbitrary element $a \in \operatorname{Ext}_A^i(A/I^n, \bar{M})$, there exists an integer $t \in Z^+$ such that $x_1^t a = 0$. Hence from the long exact sequence we can see that $I^N a = 0$. This implies $I^N \operatorname{Ext}_A^i(A/I^n, M) = 0$ for $i < s$, and the proof is complete.

3. The length of a maximal weak I sequence

In this section, we will prove the main results Theorems 1.1 and 1.2 of this paper. These results include the condition of the existence of a weak I sequence in terms of Ext functor and characterization of the length of a maximal weak I sequence by means of the Ext functor. Now we first prove an important lemma.

Lemma 3.1 *Let A be a Notherian local ring and I an ideal of A. Let M be a finite A-module with $\Gamma_I(M) \neq M$. Let $x_1, x_2, \ldots, x_r$ be a weak I sequence with respect to M. If there exists an integer s such that*

$$I^s \operatorname{Ext}_A^0(A/I^n, M/(x_1^{n_1}, \ldots, x_r^{n_r})M) = 0$$

holds for all $n_1, n_2, \ldots, n_r \in Z^+$, then there exists an element $x_{r+1} \in I$ such that $x_1, \ldots, x_{r+1}$ is a weak I sequence.

Proof It suffices to construct an element x_{r+1} such that for any $n_1, \ldots, n_r, n_{r+1} \in Z^+$

$$(x_1^{n_1}, \ldots, x_r^{n_r})M : x_{r+1}^{n_{r+1}} \subseteq (x_1^{n_1}, \ldots, x_r^{n_r})M : I^s \tag{3.1}$$

holds.

Put $N = \sum_{i=1}^r n_i$ and choose an element $x_{r+1} \in I$ such that (3.1) holds for $N = r$. This is possible because $I^s \operatorname{Ext}_A^0(A/I^n, M/(x_1^{n_1}, \ldots, x_r^{n_r})M) = 0$. Now we use induction on N to prove that x_{r+1} satisfies (3.1) for all $N \ge r$. Suppose the conclusion holds for those N' with $r \le N' < N$. Note that $x_{i_1}, \ldots, x_{i_r}$ is a weak I sequence with respect to M, where $x_{i_1}, \ldots, x_{i_r}$ is a permutation of $x_1, \ldots, x_r$[5]. Without loss of generality we may assume $n_1 > 1$. For any $a \in ((x_1^{n_1}, \ldots, x_r^{n_r})M : x_{r+1}^{n_{r+1}})$, we may express

$$x_{r+1}^{n_{r+1}} a = x_1^{n_1} a_1 + a_1', \tag{3.2}$$

where $a_1 \in M, a_1' \in (x_2^{n_2}, \ldots, x_r^{n_r})M$. By the induction hypothesis, for any $y \in I^s$, we may write

$$ya = x_1^{n_1-1}a_2 + a_2', \tag{3.3}$$

where $a_2 \in M$, $a_2' \in (x_2^{n_2}, \ldots, x_r^{n_r})M$. From (3.2) and (3.3) we assert that

$$x_1^{n_1-1}(yx_1a - x_{r+1}^{n_{r+1}}a_2) \in (x_2^{n_2}, \ldots, x_r^{n_r})M.$$

On the other hand, $x_2, \ldots, x_r, x_1$ is also a weak I sequence with respect to M. Hence we can find an integer $s' \in Z^+$ such that $y'(yx_1a - x_{r+1}^{n_{r+1}}a_2) \in (x_2^{n_2}, \ldots, x_r^{n_r})M$ holds for any $y' \in I^{s'}$. This implies $x_{r+1}^{n_{r+1}}y'a_2 \in (x_1, x_2^{n_2}, \ldots, x_r^{n_r})M$. By the induction hypothesis, we have $y'y''a_2 \in (x_1, x_2^{n_2}, \ldots, x_r^{n_r})M$ for any $y'' \in I^s$. Hence from (3.3) we obtain $yy'y''a \in (x_1^{n_1}, \ldots, x_r^{n_r})M$. By the arbitrary choice of y, y' and y'', we see that $I^{2s+s'}a \in (x_1^{n_1}, \ldots, x_r^{n_r})M$. But by the assumption, $I^s\mathrm{Ext}_A^0(A/I^n, M/(x_1^{n_1}, \ldots, x_r^{n_r})M) = 0$. Therefore, $a \in (x_1^{n_1}, \ldots, x_r^{n_r})M : I^s$ and this proves the lemma.

The proof of Theorem 1.1 We use induction on $\dim M = d$. Since $\Gamma_I(M) \neq M$, we have $d > 0$. For $d = 1$, we prove that $k \leq 1$ and the result is trivial. In fact, if on the contrary $k > 1$. By the condition we have $I^N\mathrm{Ext}_A^1(A/I^n, M) = 0$ for all $n \in Z^+$. Choose an element $x \in I$ such that x is $\bar{M}(\bar{M} = M/\Gamma_I(M))$-regular. Consider the short exact sequence $0 \to \Gamma_I(M) \to M \to \bar{M} \to 0$ and the long exact sequence

$$\cdots \to \mathrm{Ext}_A^i(A/I^n, \Gamma_I(M)) \to \mathrm{Ext}_A^i(A/I^n, M) \to \mathrm{Ext}_A^i(A/I^n, \bar{M}) \to \cdots.$$

Since $I^N\mathrm{Ext}_A^i(A/I^n, \Gamma_I(M)) = 0$ for all $i \geq 0$ and $I^N\mathrm{Ext}_A^j(A/I^n, M) = 0$ for $j \leq 1$, we have $I^N\mathrm{Ext}_A^j(A/I^n, \bar{M}) = 0$ for $j \leq 1$.

On the other hand, consider the short exact sequence $0 \longrightarrow \bar{M} \xrightarrow{x^t} \bar{M} \longrightarrow \bar{M}/x^t\bar{M} \longrightarrow 0$ and the long exact sequence

$$\cdots \to \mathrm{Ext}_A^i(A/I^n, \bar{M}) \xrightarrow{x^t} \mathrm{Ext}_A^i(A/I^n, \bar{M}) \to \mathrm{Ext}_A^i(A/I^n, \bar{M}/x^t\bar{M}) \to \cdots.$$

We conclude that $I^{2N}\bar{M}/x^t\bar{M} = 0$ for all $t \geq 0$. Choosing $t > 2N$, we have $I^{2N}\bar{M} \subseteq x^{t-2N}I^{2N}\bar{M}$. By Nakayama Lemma, $I^{2N}\bar{M} = 0$. This is a contradiction and thus the result holds for $d = 1$.

Suppose that $d > 1$ and the conclusion holds for integers less than d. As above we can reduce the proof to the case depth $M \geq 1$ and $I^N\mathrm{Ext}_A^i(A/I^n, M/x^tM) = 0$ for $i < k$ and all $n \in Z^+$. By Lemma 3.1 we have x_2 such that x_1, x_2 is a weak I sequence with respect to M and there exists an integer N' with $I^{N'}\mathrm{Ext}_A^i(A/I^n, M/((x_1^{n_1}, x_2^{n_2})M)) = 0$ for all n_1, n_2 and $i < k-1$. Repeat the process, one can construct a weak I sequence $x_1, x_2, \ldots, x_k$ with respect to M.

The proof of Theorem 1.2 Suppose $x_1, \ldots, x_t$ is a maximal weak I sequence with respect to M such that for $1 \leq i \leq t$,

$$(x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \ldots, x_i^{n_{i-1}})M : I^s$$

holds, where s is a fixed positive number and $n_1, n_2, \ldots, n_t$ run through all positive number. By Proposition 2.2, we have $t \leq r$. Now suppose that $t < r$ and consider the short exact sequences

$$0 \to (0 : x_1^{n_1}) \to M \to x_1^{n_1}M \to 0, \quad 0 \to x_1^{n_1}M \to M \to M/x_1^{n_1}M \to 0$$

and the long exact sequences

$$\cdots \to \mathrm{Ext}_A^i(A/I^n, (0 : x_1^{n_1})) \to \mathrm{Ext}_A^i(A/I^n, M) \to \mathrm{Ext}_A^i(A/I^n, x_1^{n_1}M) \to \cdots,$$

$$\cdots \to \mathrm{Ext}_A^i(A/I^n, x_1^{n_1}M) \to \mathrm{Ext}_A^i(A/I^n, M) \to \mathrm{Ext}_A^i(A/I^n, M/x_1^{n_1}M) \to \cdots.$$

Since $I^s\mathrm{Ext}_A^i(A/I^n, (0 : x_1^{n_1})) = 0$ for all $i \geq 0$ and $I^N\mathrm{Ext}_A^i(A/I^n, M) = 0$ for all $i < r$, $I^{s+N}\mathrm{Ext}_A^i(A/I^n, x_1^{n_1}M) = 0$ for $i < r$. It follows that $I^{s+2N}\mathrm{Ext}_A^i(A/I^n, M/x_1^{n_1}M) = 0$ for $i < r - 1$. Repeating the process, one can prove that there exists a positive number N' such that

$$I^{N'}\mathrm{Ext}_A^i(A/I^n, M/(x_1^{n_1}M, \ldots, x_t^{n_t}M)) = 0$$

holds for $i < r - t$ and all $n, n_1, \ldots, n_t \in Z^+$. Note that $r - t > 0$. Thus

$$I^{N'}\mathrm{Ext}_A^0(A/I^n, M/(x_1^{n_1}M, \ldots, x_t^{n_t}M)) = 0$$

holds for all $n, n_1, \ldots, n_t \in Z^+$. By Lemma 3.1, there exists $x_{t+1} \in I$ such that $x_1, x_2, \ldots, x_{t+1}$ is a weak I sequence with respect to M. This is a contradiction and thus $r = t$.

References:

[1] STÜKRAD J, VOGEL W. *Eine verallgemeinerung der Cohen-Macaulay-Ringe und anwendungen aufein problem der multiplizitätstheorie* [J]. J. Math. Kyoto Univ., 1973, **13**: 513–528. (in German)

[2] STÜKRAD J, VOGEL W. *Toward a theorey of Buchsbaum singularities* [J]. Amer. J. Math., Oxford, 1970, **21**: 727–746.

[3] TRUNG N V. *On the associated graded ring of a Buchsbaum ring* [J]. Math. Nachr., 1982, **107**: 209–220.

[4] TRUNG N V. *Toward a theorey of generalized Cohen-Macaulay modules* [J]. Nagoya Math. J., 1986, **102**: 1–49.

[5] ZHOU Cai-jun. *Toward a theory of weak I sequences* [J]. Chinese Ann. Math. Ser. B, 1997, **18**(3): 283–292.

[6] MATSUMRA H. *Commutative Ring Theory* [M]. Cambridge University Press, Cambridge, 1986.

弱 I 序列的刻画

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摘要: 在文献 [5] 中，周才军定义了弱 I 序列，并利用 Koszul 上同调和局部上同调的方法刻画了这种序列. 本文利用 Ext 函子刻画了弱 I 序列.

关键词: Noetherian 局部环； Ext 函子.