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Extended Cesáro Operators from BMOA Spaces to Bloch-Type

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Abstract: In this paper the extended Cesáro operator T_g is characterized between the α-Bloch spaces B_α and the BMOA space on the unit disk. Some necessary and sufficient conditions are given for which T_g is a bounded operator or a compact operator from BMOA to B_α.

Key words: extended Cesáro operators; Bloch spaces; BMOA spaces.
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1. Introduction

Let $D=\{z:|z|<1\}$ be the open unit disk in the complex plane $\mathbf{C}$, and $H(D)$ denote the set of all analytic functions on D. For $f\in H(D)$ with Taylor expansion $f(z)=\sum_{i=0}^{\infty}a_iz^i$, the Cesáro operator acting on f is

$$C[f](z)=\sum_{i=0}^{\infty}(\frac{1}{i+1}\sum_{k=0}^{i}a_k)z^k.$$

It is well known that $C[\cdot]$ act as a bounded operator on various spaces of analytic functions[4,6,8] including the Hardy and Bergman spaces. But it is not bounded on the Bloch space[9]. Now we consider the extended Cesáro operator T_g,

$$T_g(f)(z)=\int_0^z f(t)g'(t)\mathrm{d}t$$

acting on function f analytic on D. When $g(z)=z$ or $g(z)=\log(\frac{1}{1-z})$, T_g is the integral operator or the Cesáro operator respectively. It is interesting to provide a function theoretic characterization when g induces a bounded or compact extended Cesáro operator on various spaces (see [1,2] for more information). Problems of this kind were studied recently for Cesáro operator on the Bloch spaces[5], the BMOA space[7], between Bloch-type spaces and Dirichlet-type spaces[12], to mention only a few related works. Our work here is to characterize this operator which is bounded or compact from BMOA to α-Bloch spaces.

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Definition 1 *$f \in H(D)$ belongs to the BMOA space if*

$$\|f\|_{\mathrm{BMOA}} = \sup_{a\in D}\{\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)\mathrm{d}m(z)\}^{\frac{1}{2}} < \infty,$$

where $\varphi_a(z) = (z-a)/(1-\overline{a}z)$ is a Möbius transformation of D, and $\mathrm{d}m(z)$ is the Euclidean area element on D. The expression $\|f\|_{\mathrm{BMOA}}$ defines a semi-norm while the natural norm is given by

$$\|f\|_* = |f(0)| + \|f\|_{BMOA}. \tag{1}$$

It makes BMOA into a Banach space. Let VMOA denote the subspace of BMOA consisting of those $f \in$ BMOA for which

$$\lim_{|a|\to 1}\int_D |f'(z)|^2(1-|\varphi_a(z)|^2)\mathrm{d}m(z) = 0.$$

*The space BMOA with the norm (1) is isometric to the second dual VMOA**[14].*

Here we mention two simple facts about BMOA, which we will use the later. First, there exists constant C, for any $f \in$ BMOA, such that

$$|f(z)| \le C\|f\|_{\mathrm{BMOA}} \log\frac{2}{1-|z|^2}. \tag{2}$$

Second, for any $a \in D$, $f_a(z) = \log\frac{2}{1-\overline{a}z} \in$ VMOA and $\|f_a\|_{\mathrm{BMOA}} \le \|\log\frac{2}{1-z}\|_{\mathrm{BMOA}} < \infty$. For more information about BMOA, see Ref.[14].

Definition 2 *For $f \in H(D)$, let*

$$\|f\|_{B_\alpha} = \sup\{(1-|z|^2)^\alpha|f'(z)| : z \in D\}, \quad 0 < \alpha < +\infty.$$

As in [13], the α-Bloch space B_α consists of all $f \in H(D)$ satisfying $\|f\|_{B_\alpha} < +\infty$ and the little α-Bloch space B_α^0 consists of all $f \in H(D)$ satisfying $\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0$. It is well known that with the norm $\|f\|_\alpha = |f(0)| + \|f\|_{B_\alpha}$. B_α is a Banach space and B_α^0 is a closed subspace of B_α. For more information about B_α, see Ref.[13].

In what follows C will stand for positive constants not depending on the functions being considered, but whose value may change from line to line. The expression "two quantities A and B are equivalent", denoted by $A \approx B$, means $\frac{1}{C}A \le B \le CA$.

2. Main theorems

Lemma 1[3] *If $f \in B_1^0$, then $\lim_{|z|\to 1}\frac{|f(z)|}{\log\frac{1}{1-|z|^2}} = 0$.*

Theorem 1 *For $g \in H(D)$ and $\forall\alpha > 0$, then the following statements are equivalent.*

(a) $T_g :$ *BMOA* $\longrightarrow B_\alpha$ *is bounded;*

(b) $T_g :$ *VMOA* $\longrightarrow B_\alpha^0$ *is bounded;*

(c) $T_g :$ *VMOA* $\longrightarrow B_\alpha$ *is bounded;*

(d) $\sup_{z\in D}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| < \infty.$

In each case $\|T_g\| \approx \sup_{z\in D}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)|$.

Proof Suppose T_g is bounded from BMOA (or VMOA) to B_α. For and $a \in D$, take the test function $f_a(z) = \log \frac{2}{1-\bar{a}z} \in$ VMOA. We know that $\|f_a\|_* \le \|\log \frac{2}{1-z}\|_* \le C < \infty$, so

$$\sup_{z\in D}(1-|z|^2)^\alpha |f_a(z)g'(z)| \le \|T_g(f_a)\|_\alpha \le \|T_g\|\|f_a\|_* \le C < \infty.$$

Setting $z = a$, we obtain

$$(1-|a|^2)^\alpha \log \frac{2}{1-|a|^2}|g'(a)| \le \|T_g\|\|\log \frac{2}{1-z}\|_* \le C < \infty. \tag{3}$$

This gives the implications (a)$\Longrightarrow$ (d) and (c)$\Longrightarrow$ (d).

Now we suppose g satisfies (d).

Step 1. For $f \in$ BMOA, by (2), we know

$$\begin{aligned}\sup_{z\in D}(1-|z|^2)^\alpha |(T_g f)'(z)| &= \sup_{z\in D}(1-|z|^2)^\alpha |g'(z)||f(z)| \\ &\le C\|f\|_* \sup_{z\in D}(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|g'(z)|.\end{aligned}$$

This, together with $(T_g f)(0) = 0$, gives

$$\|T_g\| \le C \sup_{z\in D}(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|g'(z)|. \tag{4}$$

The implication (d)$\Longrightarrow$ (a) is proved.

The estimate

$$\|T_g\| \approx \sup_{z\in D}(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|g'(z)|$$

comes from (3) and (4).

Step 2. For $f \in$ VMOA, $f \in B_1^0$ by [9, Theorem 3]. We have

$$\begin{aligned}(1-|z|^2)^\alpha |(T_g f)'(z)| &\le \sup_{z\in D}(1-|z|^2)^\alpha \log \frac{2}{1-|z|^2}|g'(z)| \frac{|f(z)|}{\log \frac{2}{1-|z|^2}} \\ &\le C \frac{|f(z)|}{\log \frac{2}{1-|z|^2}} \longrightarrow 0 (|z| \to 1)\end{aligned}$$

by Lemma 1. So this together with the first step result implies that (d)$\Longrightarrow$ (b).

(b)$\Longrightarrow$ (c) is obvious. The proof is completed. □

We now turn to the compactness of T_g. Recall that if X, Y are Banach spaces and $T: X \longrightarrow Y$ is a linear operator, T is said to be compact if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a convergent subsequence. T is weakly compact, if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a weakly convergent subsequence. Every compact operator is weakly compact operator. In our case, a useful characterization of weak compactness is Gantmacher's Theorem[11]: T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$ where T^{**} is the second adjoint of T and X is identified with its image under the natural embedding into its second dual X^{**}.

Lemma 2 *Let T_g be a bounded operator from BMOA to B_α. Then T_g is compact if and only* *if for any bounded sequence $\{f_n\}$ in BMOA which converges to* 0 *uniformly on compact subsets of D, we have $\|T_g(f_n)\|_\alpha \to 0$ as $n \to \infty$.*

Proof The result can be proved by (2) and Montel Theorem. The details are omitted here.

Lemma 3 *Suppose $\alpha > 0$. Then a closed set U in B_α^0 is compact if and only if it is bounded and satisfies $\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)^\alpha|f'(z)| = 0$. The proof is similar to Ref.[15, Lemma 1].*

Theorem 2 *For $g \in H(D)$ and $\forall \alpha > 0$, the following statements are equivalent.*

(a) T_g : BMOA $\longrightarrow B_\alpha$ is compact;

(b) T_g : VMOA $\longrightarrow B_\alpha^0$ is compact;

(c) T_g : VMOA $\longrightarrow B_\alpha^0$ is compact;

(d) $\lim_{|z|\to 1}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| = 0$.

Proof (d)$\Rightarrow$(a). By Theorem 1, T_g is a bounded operator from BMOA to B_α. Let $K = \sup_z (1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| < \infty$. Suppose that $\{f_n\}$ be a bounded sequence in BMOA which converges to 0 uniformly on compact subsets of D. Let $M = \sup_n \|f_n\|_* < +\infty$. Given $\varepsilon > 0$, there is $0 < r < 1$ such that $|z| > r$ implies

$$(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| < \frac{\varepsilon}{M}. \tag{5}$$

We know that the sequence $\{f_n\}$ converges uniformly on $|z| \le r$. It follows that for large enough n,

$$\sup_{|z|\le r}|f_n(z)| < \frac{\log 2}{K}\varepsilon. \tag{6}$$

So, by (2), (5) and (6),

$$\begin{aligned}
\|T_g f_n\|_\alpha = \|T_g f_n\|_{B_\alpha} &= \sup_z (1-|z|^2)^\alpha |f_n(z)g'(z)| \\
&\le \sup_{|z|\le r}(1-|z|^2)^\alpha|f_n(z)g'(z)| + \sup_{|z|>r}(1-|z|^2)^\alpha|f_n(z)g'(z)| \\
&\le \varepsilon + C\sup_{|z|>r}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}\|f_n\|_*|g'(z)| \\
&\le \varepsilon + CM\sup_{|z|>r}(1-|z|^2)^\alpha \log\frac{2}{1-|z|^2}|g'(z)| < (C+1)\varepsilon
\end{aligned}$$

if n is sufficiently large. Then $\|T_g f_n\|_\alpha \longrightarrow 0$ as n tends to ∞. Hence T_g is compact from BMOA to B_α by Lemma 2.

(a)$\Rightarrow$(b). It is obvious.

(b)$\Rightarrow$(c). For any bounded sequence $\{f_n\}$ in VMOA, by the compactness of T_g from VMOA to B_α, we have some $\{f_{n_k}\}$ and $f_0 \in B_\alpha$ such that $\|T_g(f_{n_k}) - f_0\|_\alpha \longrightarrow 0$, $k \longrightarrow \infty$. However, by Theorem 1, we know that $T_g(f_{n_k}) \in B_\alpha^0$, so $f_0 \in B_\alpha^0$. It shows that T_g is compact from VMOA to B_α^0.

(c)$\Rightarrow$(d). By Lemma 3, T_g is compact from VMOA to B_α^0 if and only if T_g is bounded and

$$\lim_{|z|\to 1}\sup_{\|f\|_*\le 1}(1-|z|^2)^\alpha|(T_g(f))'(z)| = 0.$$

On the other hand, we have

$$(1-|z|^2)^\alpha|(T_g(f))'(z)| = (1-|z|^2)^\alpha \log(\frac{2}{1-|z|^2})|g'(z)| \times \frac{|f(z)|}{\log\frac{2}{1-|z|^2}},$$

and

$$\sup_{\|f\|_*\le 1}\frac{|f(z)|}{\log\frac{2}{1-|z|^2}} \ge C > 0.$$

Hence T_g is compact from VMOA to B^0_α if and only if

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \log(\frac{2}{1-|z|^2})|g'(z)| = 0.$$

The proof is completed. □

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BMOA 空间与 Bloch 型空间之间的加权 Cesáro 算子

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摘要: 本文研究单位圆盘上的 BMOA 空间和 α-Bloch 型空间之间的加权 Cesáro 算子，给出了 T_g 是 BMOA 空间到 B_α 空间的有界算子或紧算子的充分必要条件.

关键词: 加权 Cesáro 算子； Bloch 空间； BMOA 空间.