第27卷第3期 数 学 研 究 与 评 论 Vol.27, No.3
2007年8月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Aug., 2007

Article ID: 1000-341X(2007)03-0539-08 **Document code**: A

A Novel Neural Network for Linear Complementarity Problems

LI Yang, JIN Li, ZHANG Li-wei

(Department of Applied Mathematics, Dalian University of Technology, Liaoning 116024, China)
(E-mail: lwzhang@online.ln.cn)

Abstract: In this paper, we present a neural network for solving linear complementarity problem in real time. It possesses a very simple structure for implementation in hardware. In the theoretical aspect, this network is different from the existing networks which use the penalty functions or Lagrangians. We prove that the proposed neural network converges globally to the solution set of the problem starting from any initial point. In addition, the stability of the related differential equation system is analyzed and five numerical examples are given to verify the validity of the neural network.

Key words: neural network; linear complementarity; convergence; stability.
MSC(2000): 90C25; 49J15
CLC number: O175.1; O221.1

1. Introduction

Given a matrix $M \in R^{n\times n}$ and a vector $q \in R^n$, the problem of finding vectors x and y satisfying

$$y - Mx = q$$

and

$$x \geq 0, \quad y \geq 0, \quad x^{\mathrm{T}}y = 0 \tag{1.1}$$

(or possibly showing that no such solution exists) is called the linear complementarity problem, which is denoted by LCP(M, q).

The problem (1.1) is a fundamental problem in mathematical programming and numerical optimization[1]. The most distinguishing feature of LCP from the general optimization problems is that there is no objective function to minimize. Linear and quadratic programming problems, bimatrix game problems, and equilibrium problems can be reformulated as linear complementarity problems. Moreover, many algorithms for solving nonlinear optimization problems require handling linear complementarity problems[2].

It is well known that one promising approach to solve the optimization problems in real time is to employ artificial neural networks based on the circuit implementation. Since 1980s, Several neural networks have been shown to be efficient for solving linear programming problems[3−7].

Received date: 2005-05-20; **Accepted date**: 2006-07-03
Foundation item: the State Foundation of Ph.D Units of China (20020141013); the National Natural Science Foundation of China (10471015).

Then how to use a neural network method to solve the linear complementarity problem in real time becomes an important issue in this area. In this paper, we proposed a recurrent neural network for solving LCP. Compared with existing networks in the literature, this neural network is much simpler and more suitable for implementation in hardware. It contains only the variables of the original problem. When the problem has many solutions, we prove that, for any initial point, the trajectory of the network does converge to an exact solution of the problem.

This paper is divided into four sections. In Section 2, preliminary information is introduced to facilitate later discussions. In Section 3, we propose a neural network and prove that the proposed neural network is stable in the sense of Lyapunov and globally converges to an exact optimal solution. In Section 4, we report five numerical examples to show the efficiency of the neural network.

2. A transformation

Let $\Phi(x) = \frac{1}{2}(\max(x^{\mathrm{T}}Mx + q^{\mathrm{T}}x, 0))^2$, $x \in R^n$. Then Φ is a differentiable convex function under the condition that M is positive semidefinite. Moreover, $\nabla\Phi(x)$ is locally Lipschitz continuous.

We can easily prove the following results.

Lemma 2.1 *The point x^* is the solution of (1.1) if x^* is the solution of the following equation*

$$\Psi(x) = \begin{bmatrix} x^{\mathrm{T}}(Mx+q) \\ \sum_{i=1}^{n} \min\{0, x_i\}^2 \\ \sum_{i=1}^{n} \min\{0, (Mx+q)_i\}^2 \end{bmatrix} = 0.$$

The energy function for the LCP is defined as follows:

$$E(x) = \Phi(x) + \sum_{i=1}^{n} \min\{0, x_i\}^2 + \sum_{i=1}^{n} \min\{0, (Mx+q)_i\}^2.$$

In this paper, we just consider the situation when M is semidefinitely positive (denoted by $M \succeq 0$). It is easy to see that $E(x) = 0$ is equivalent to $\Psi(x) = 0$.

Theorem 2.1 *When $M \succeq 0$, $E(x)$ is a non-negative, continuously differentiable and convex function.*

Proof Clearly, $E(x)$ is a non-negative, and continuously differentiable function. As Φ is convex and $\varphi(x) = \sum_{i=1}^{n} \min\{0, x_i\}^2$ is also convex, we only need to prove the convexity of $\sum_{i=1}^{n} \min\{0, (Mx+q)_i\}^2 = \varphi(Mx+q)$. For any $\mu \in (0\ 1)$, since $M \succeq 0$, we have

$$\begin{aligned} \varphi(M(\mu x_1 + (1-\mu)x_2) + q) &= \varphi(M(\mu x_1 + (1-\mu)x_2) + \mu q + (1-\mu)q) \\ &= \varphi(\mu(Mx_1 + q) + (1-\mu)(Mx_2 + q)) \\ &\le \mu\varphi(Mx_1 + q) + (1-\mu)\varphi(Mx_2 + q), \end{aligned}$$

which implies that $E(x)$ is a convex function. □

Theorem 2.2 Let $x^* \in R^n$. Then the following statements are equalent.

(a) $E(x^*) = 0$.

(b) x^* is the solution of the LCP.

Proof As $E(x^*) = 0$ is equivalent to $\Psi(x^*) = 0$, we obtain the conclusion. □

3. The neural network

Based on the above results, a neural network for LCP can then be constructed as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla E(x), \quad x(0) = x^0, \tag{3.1}$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^{\mathrm{T}}(Mx+q)(2Mx+q) - 2\sum_{i=1}^{n}\min\{0, x_i\} - 2\sum_{i=1}^{n}\min\{0, M^{\mathrm{T}}(Mx+q)_i\}, \quad x(0) = x^0. \tag{3.2}$$

The architecture of neural network (3.1) is illustrated by Fig.1.

In order to discuss the stability of the neural network in (3.1) or (3.2), we first prove the following two theorems.

Theorem 3.1 *The initial value problem of the system of differential equations in (3.1) has a unique solution.*

Proof By the definition of the LCP, it is obvious that the right-hand side of (3.2) is locally Lipschitz continuous. Thus for any $x^0 \in I\!R^n$, the initial value problem has a unique solution $x(t)$ with $x(0) = x^0$ by the existence and uniqueness theorem of the initial value problem of a system of differential equations. □

Theorem 3.2 *Let $A = \{x \in R^n | \nabla E(x) = 0\}$ be the set of equilibrium points of (3.1) and B be the set of optimal solutions of LCP. Then $A = B$.*

Proof Suppose that $\overline{x} \in A$, that is $\nabla E(\overline{x}) = 0$. And assume x^* is any solution of LCP. Then $E(x^*) = 0$ by Theorem 2.3. Since $E(x)$ is a differentiable convex function by Theorem 2.2, from the necessary and sufficient conditions for convex functions, we have

$$E(\overline{x}) + (x^* - \overline{x})^{\mathrm{T}}\nabla E(\overline{x}) \leq E(x^*) = 0.$$

Hence,

$$E(\overline{x}) \leq (\overline{x} - x^*)^{\mathrm{T}}\nabla E(\overline{x}), \tag{3.3}$$

which implies $E(\overline{x}) \leq 0$ by $\nabla E(\overline{x}) = 0$. Nevertheless, $E(\overline{x}) \geq 0$, and we immediately obtain $E(\overline{x}) = 0$. Thus $\overline{x} \in B$ by Theorem 2.3 and $A \subseteq B$.

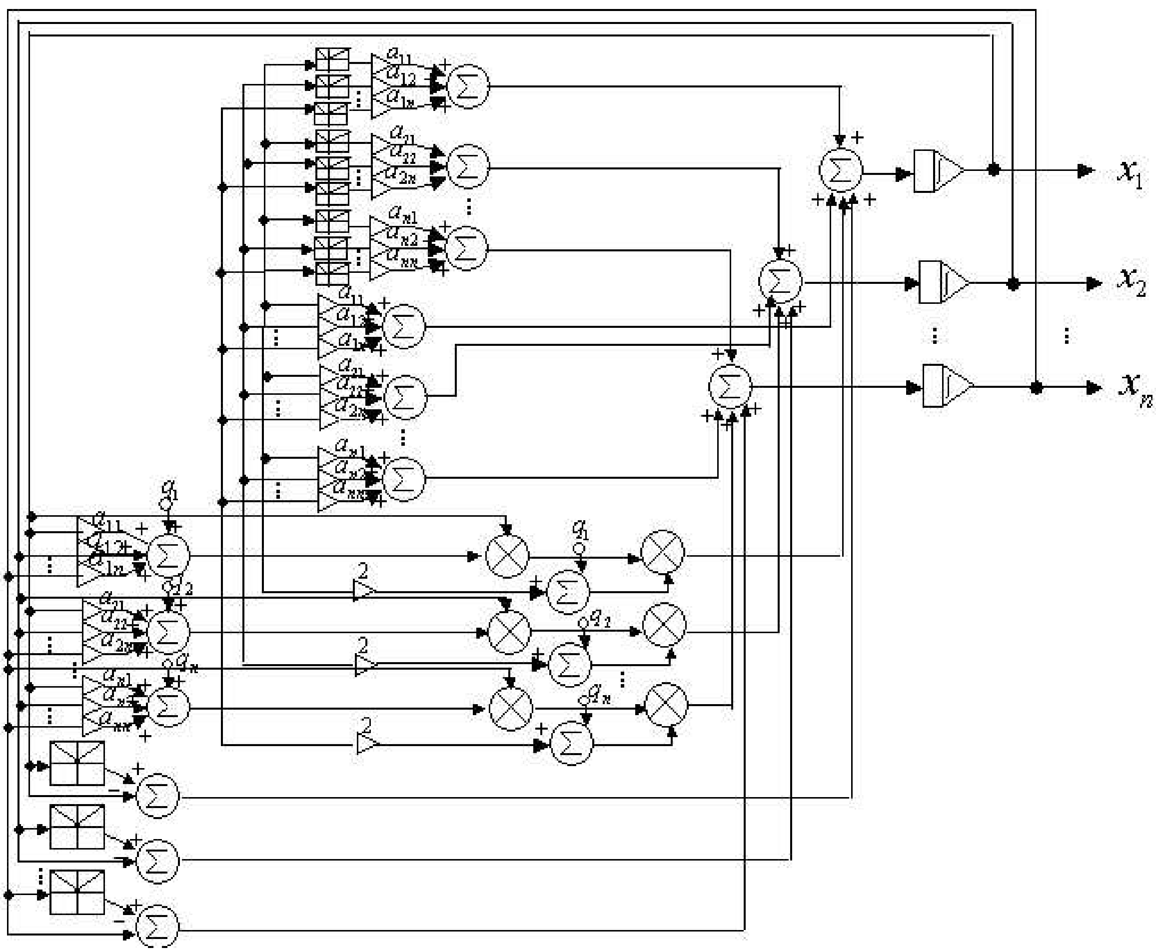

Figure 1: The architecture of (3.1)

On the other hand, for any $\overline{x} \in B$, $\overline{x}^{\mathrm{T}}(M\overline{x}+q)=0$ and $\overline{x} \geq 0$. We have $\overline{x}_i - |\overline{x}_i| = 0$, $i=1,2,\ldots,n$, and $M^{\mathrm{T}}(M\overline{x}+q)_i - M^{\mathrm{T}}|(M\overline{x}+q)_i| = 0$. Thus, $\nabla E(\overline{x})=0$, implying $\overline{x} \in A$ and $B \subseteq A$. $\Box$

Now we are in a position to prove the stability of System (3.1).

Theorem 3.3 *Suppose that LCP has an optimal solution x^*, then x^* is a globally asymptotically stable equilibrium point of system(3.1).*

Proof Suppose that the initial point x^0 is arbitrarily given ($x^0 \in \mathbb{R}^n$) and $x(t)=x(t;t_0;x^0)$ is the solution of the initial value problem of system of differential equation in (3.1). Define

$$H(x)=\frac{1}{2}\|x-x^*\|^2(\geq 0),$$

where x^* is the optimal solution of LCP and $H(x^*)=0$. Obviously, $H(x)$ is a positively unbound function ($H(x) \to +\infty$ when $x \to +\infty$). In view of the convexity of $E(x)$, we have $E(x)+(x^*-x)^{\mathrm{T}}\nabla E(x) \le E(x^*)$, which implies $(x^*-x)^{\mathrm{T}}\nabla E(x) \le -E(x)$. Therefore, for $x \ne x^*$,

$$\begin{aligned}\frac{\mathrm{d}H(x(t))}{\mathrm{d}t} &= \nabla H(x)^{\mathrm{T}}\frac{\mathrm{d}x}{\mathrm{d}t} = (x-x^*)^{\mathrm{T}}(-\nabla E(x)) \\ &= (x^*-x)^{\mathrm{T}}(\nabla E(x)) \le -E(x) < 0. \end{aligned} \tag{3.4}$$

Thus, when $x \ne x^*$, along the trajectory $x = x(t)$, $\dot{H}[x(t)]$ is negative definite. Therefore, x^* is globally asymptotically stable by the Lyapunov stability theorem[8,9]. □.

The follow theorem deals with the case where the solution set has infinitely many points.

Theorem 3.4 *Suppose that LCP has infinitely many solutions. Then for any $x^0 (x^0 \in I\!R^n)$, the trajectory $x = x(t, x^0)$ $(t \ge 0)$ corresponding to the neural network in (3.1) converges to an optimal solution of LCP.*

Proof Suppose that x^* is an equilibrium point of the network in (3.1), namely, $\nabla E(x^*) = 0$. By the proof of Theorem 3.3, along the trajectory $x = x(t, x^0)(t \ge 0, \forall x^0)$, we have $\dot{H}(x(t)) \le 0$ $(t \ge 0)$ and $H(x(t, x^0))$ is monotone nonincreasing, namely

$$\|x(t, x^0) - x^*\| \le \|x^0 - x^*\| \quad (t \ge 0),$$

yielding

$$\|x(t, x^0)\| \le \|x^*\| + \|x^0 - x^*\|.$$

Therefore, $\gamma^+(x^0) = \{x(t, x^0)|t \ge 0\}$ is bounded. Take strictly monotonely increasing sequence $\{\bar{t}_n\}$, $0 \le \bar{t}_1 \le \bar{t}_2 \le \cdots \le \bar{t}_i \le \bar{t}_{i+1} \le \cdots \le \bar{t}_n \to +\infty$, then $\{x(\bar{t}_n, x^0)\}$ is a bounded sequence composed of infinitely many points. Thus there exists a limit point $\bar{x}$, that is, there exists a subsequence $\{t_n\} \subseteq \{\bar{t}_n\}$, $t_n \to +\infty$ such that

$$\lim_{n\to+\infty} x(t_n, x^0) = \bar{x}. \tag{3.5}$$

Hence $\bar{x}$ is an ω-limit point of trajectory $\gamma^+(x^0)$. Moreover, along trajectory $x = x(t, x^0)(t \ge 0)$, one has

$$\dot{E}(x) = \frac{\mathrm{d}}{\mathrm{d}t}E[x(t, x^0)] = \nabla E(x)^{\mathrm{T}}\frac{\mathrm{d}x}{\mathrm{d}t} = -\|\nabla E(x)\|^2 \le 0.$$

Hence $E(x)$ is a Lyapunov function of the network in (3.1) on $I\!R^n$, and $\gamma^+(x^0) = \{x(t, x^0)|t \ge 0\}$ is a bounded trajectory of R^n. By LaSalle invariance principle[8], we have $\nabla E(\bar{x}) = 0$ and $\overline{x}$ is an optimal solution of LCP. By (3.5), for any $\varepsilon \ge 0$, there is a nature number N such that

$$\|x(t_n, x^0) - \bar{x}\| \le \varepsilon, \quad n \ge N.$$

Similar to the decreasing property of $H(x)$ in the proof of Theorem 3.3, we can easily prove that $\|x - \overline{x}\|^2$ is monotonely nonincreasing along trajectory $x = x(t, x^0)$ $(t \ge 0)$. Therefore, when $t \ge 0$ and $t \ge t_N$, one has

$$\|x(t, x^0) - \bar{x}\| \le \|x(t_N, x^0) - \bar{x}\| \le \varepsilon.$$

Thus $\lim_{t\to+\infty} x(t,x^0)=\bar{x}$ and the theorem is proved. □

4. Simulation experiments

In order to demonstrate the efficiency of the proposed neural network, in this section, we present five simulation examples. The ordinary differential equation solver engaged is ode45s.

Example 1[10] Consider the LCP(M,q) with

$$M_{10\times 10}=\begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$q_{10\times 1}=(-1,-1,-1,\dots,-1)^{\mathrm{T}}$. The optimal solution of the LCP is $x^*=(0,0,0,0,0,0,0,0,0,1)^{\mathrm{T}}$. Fig.2 shows the trajectories with the initial points $(0,0,0,0,0,0,0,0,0,0)^{\mathrm{T}}$, $(1,1,1,1,1,1,1,1,1,1)^{\mathrm{T}}$ and $(2,2,2,2,2,2,2,2,2,2)^{\mathrm{T}}$, respectively. The trajectories all converge to the theoretical optimal solution $x^*=(0,0,0,0,0,0,0,0,0,1)^{\mathrm{T}}$, see Fig.2.

Example 2 Consider the LCP(M,q) with

$$M=\begin{pmatrix} 1 & 0.5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

$q=(-1,-1,-1)^{\mathrm{T}}$. The optimal solution of the LCP is $x^*=(1,0,0)^{\mathrm{T}}$. Fig.3 shows the trajectories with the initial points $(0,0,0)^{\mathrm{T}}$, $(1,1,1)^{\mathrm{T}}$, $(2,2,2)^{\mathrm{T}}$, $(1,2,1)^{\mathrm{T}}$ and $(2,3,1)^{\mathrm{T}}$, respectively. The trajectories all converge to the theoretical optimal solution $x^*=(0.97968,0.041227,-0.000898)^{\mathrm{T}}$, see Fig.3.

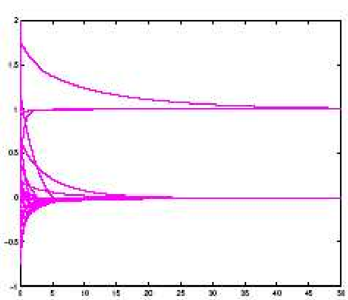

Figure 2: Convergent behavior of the continuous-time system

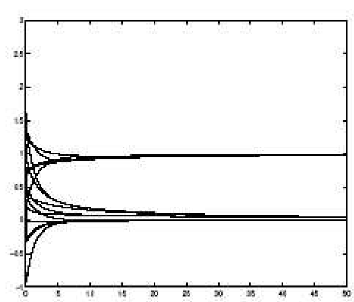

Figure 3: Convergent behavior of the continuous-time system

Example 3 Consider the LCP(M, q) with

$$M = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$$

$q = (-1, 0)^{\mathrm{T}}$. The optimal solution of the LCP is $x^* = (3, 1)^{\mathrm{T}}$. Fig.4 shows the trajectories with the initial points $(0, 0)^{\mathrm{T}}$, $(1, 1)^{\mathrm{T}}$, $(2, 2)^{\mathrm{T}}$, $(1, 2)^{\mathrm{T}}$ and $(2, 3)^{\mathrm{T}}$, respectively. The trajectories all converge to the theoretical optimal solution $x^* = (2.996979, 0.99875083)^{\mathrm{T}}$, see Fig.4.

Example 4 Consider the LCP(M,q) with

$$M = \begin{pmatrix} 3 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & -1 & -1 & 5 \end{pmatrix}$$

$q = (-2, 3, -4, 5)^{\mathrm{T}}$. The optimal solution of the LCP is $x^* = (1, 0, 1, 0)^{\mathrm{T}}$. Fig.5 shows the trajectories with the initial points $(0, 0, 0, 0)^{\mathrm{T}}$, $(1, 1, 1, 1)^{\mathrm{T}}$, $(2, 2, 0, 0)^{\mathrm{T}}$ and $(1, 2, 0, 0)^{\mathrm{T}}$, respectively. The trajectories all converge to the theoretical optimal solution $x^* = (1.0002, 0.0086, 1.0009, -0.0021)^{\mathrm{T}}$, see Fig.5.

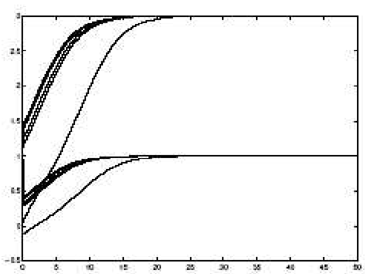

Figure 4: Convergent behavior of the continuous-time system

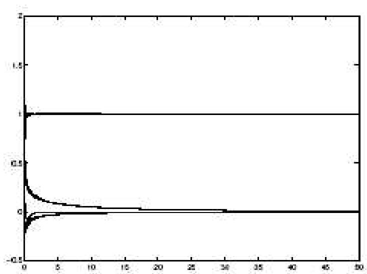

Figure 5: Convergent behavior of the continuous-time system

Example 5 Consider the LCP(M, q) with

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0.5 \end{pmatrix}$$

$q = (-1, -1, -1, -1)^{\mathrm{T}}$. The optimal solution of the LCP is $x^* = (1, 0, 1, 0)^{\mathrm{T}}$. Fig.6 shows the trajectories with the initial points $(0, 0, 0, 0)^{\mathrm{T}}$, $(1, 0, 1, 0)^{\mathrm{T}}$, $(1, 0.1, 1, 0.1)^{\mathrm{T}}$ and $(1, 0, 1, 0.1)^{\mathrm{T}}$, respectively. The trajectories all converge to the theoretical optimal solution $x^* = (1, 0, 1, 0)^{\mathrm{T}}$, see Fig.6.

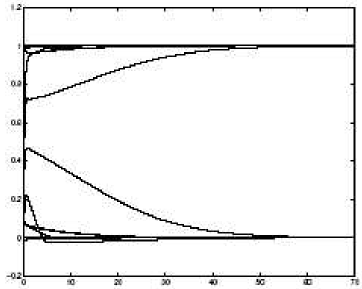

Figure 6: Convergent behavior of the continuous-time system

References:

[1] COTTLE R W, PANG J S, STONE R E. *The Linear Complementarity Problem* [M]. Academic Press, Boston, MA, 1992.
[2] COTTLE R W, PANG J S, STONE R E. *Review of the linear complemtntarity problem* [J]. Linear Algebra Appl., 1996, **235**: 275–276.
[3] BOUZERDORM A, PATTISON T R. *Neural networks for quadratic optimization with bound constraints* [J]. IEEE Transactions on Neural Networks, 1993, **4**(2): 293–304.
[4] KENNEDY M P, CHUA L O. *Neural networks for linear and nonlinear programming* [J]. IEEE Transactions on Circuits and Systems, 1993, **35**(5): 554–562.
[5] MAA C Y, SHANBLATT M V. *Linear and quadratic programming neural network analysis* [J]. IEEE Transactions on Neural Networks, 1992, **3**(4): 580–594.
[6] WANG Jia-song, XIAO Jian-hua. *The least element algorithm for a class of linear complementarity problem* [J]. Mathematic Numerica Cinica, 1992, 14(2): 167-172.
[7] XIA You-shen, WANG Jun. *A general methodology for designing globally convergent optimization neural networks* [J]. IEEE Transactions on Neural Networks, 1998, 9(6): 1311-1343.
[8] HALE J K. *Ordinary Differential Equations* [M]. New York, Wiley-Inter-Science, 1969.
[9] LA Scalle J, Lefschctz S, *Stability by Lyapunov's Direct Method with Applications* [M]. New York: Academic, 1961.
[10] LIAO Li-Zhi, QI Hou-duo. *A neural network for the linear complementarity problem* [J]. Math. Comput. Modelling, 1999, **29**(3): 9–18.

一个解决线性互补问题的新型神经网络

李 阳， 金 丽， 张立卫

(大连理工大学应用数学系，辽宁 大连 116024)

摘要: 本文构造了一个新型的解决线性互补问题的神经网络. 不同于那些运用罚函数和拉格朗日函数的神经网络，它的结构简单，易于计算. 我们证明了该神经网络的全局收敛性和稳定性，并给出数值实验检验其有效性.

关键词: 神经网络；线性互补；收敛性；稳定性.