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A Class of Exceptional Sets in Oppenheim Series Expansion

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Abstract: In this paper, we investigate the Hausdorff dimension of a class of exceptional sets occurring in Oppenheim series expansion. As an application, we get the exact Hausdorff dimension of the set in Lüroth series expansion. Moreover, we give an estimate of such dimensional number.

Key words: Oppenheim series expansion; Lüroth series expansion; Hausdorff dimension.
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1. Introduction

Let $a_n(j)$ and $b_n(j)$, for $n \geq 1$, be two sequences of positive integer-valued functions of the positive integer $j \geq 1$. The algorithm $0 < x \leq 1$, $x = x_1$, and, for any $n \geq 1$, with positive integers $d_n(x)$,

$$\frac{1}{d_n(x)} < x_n \leq \frac{1}{d_n(x)-1}, \quad x_n = \frac{1}{d_n(x)} + \frac{a_n(d_n(x))}{b_n(d_n(x))} \cdot x_{n+1} \tag{1}$$

leads to the series expansion

$$x = \frac{1}{d_1(x)} + \sum_{n=1}^{\infty} \frac{a_1(d_1(x)) \cdots a_n(d_n(x))}{b_1(d_1(x)) \cdots b_n(d_n(x))} \frac{1}{d_{n+1}(x)}, \tag{2}$$

which is called the Oppenheim series expansion of x. Set

$$h_n(j) = \frac{a_n(j)}{b_n(j)} j(j-1), \quad j \geq 2. \tag{3}$$

If $h_n(j)$ is integer-valued $(n \geq 1, j \geq 2)$, Equality (2) is termed the restricted Oppenheim series expansion of x. Here and in what follows, we always assume h_j is integer-valued, for all $j \geq 1$.

The algorithm (1) implies

$$d_1(x) \geq 2,\ d_{n+1}(x) \geq h_n(d_n(x)) + 1,\ \text{for any } n \geq 1. \tag{4}$$

On the other hand, any $\{d_n,\ n \geq 1\}$ of integer sequence satisfying Inequality (4) is an Oppenheim admissible sequence, that is, there exists a unique $x \in (0,1]$ such that $d_n(x) = d_n$ for any $n \geq 1$. The representation (2) under (1) is unique.

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The growth rates of digits in the Oppenheim series expansions are of interest and the metric properties have been investigated by some authors[1−8].

We use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_H$ to denote the Hausdorff dimension and 'cl' the closure of a subset of $(0,1]$, respectively.

2. Hausdorff dimension of B_α

In this section, we give our main result. Let

$$B_m=\{x\in(0,1):\frac{d_n(x)}{h_{n-1}(d_{n-1}(x))}>m, n\geq 1\},\quad \text{for } m\geq 2, \tag{5}$$

where we set $h_0=1$.

Theorem 2.1 *Assume that $l\leq h_j<L$, for j ultimately, then*

$$\inf_{l\leq a\leq L} S(a)\leq \dim_H B_m\leq \sup_{l\leq a\leq L} S(a), \tag{6}$$

where $S(a)$, for any integer $a\geq 1$, is defined as

$$S(a):\sum_{b>ma}(\frac{a}{b(b-1)})^{S(a)}=1. \tag{7}$$

Proof Assume that $l\leq h_j\leq L$, for all $j\geq t_0$. Here and what follows, we often make use of symbolic space defined as follows: For any $k\geq 1$, let

$$D_k=\{\sigma=(\sigma_1,\ldots,\sigma_k)\in\mathbf{N}^k, \frac{\sigma_j}{h_{j-1}(\sigma_{j-1})}>m \text{ for } 1\leq j\leq k\},$$

and define

$$D^*=\bigcup_{k=0}^{\infty} D_k\ (D_0:=\emptyset).$$

For any $k\geq 1$, and $\sigma=(\sigma_1,\ldots,\sigma_k)\in D_k$, let J_σ and I_σ denote the following closed subintervals of $(0,1]$, respectively.

$$J_\sigma=\bigcup_{d>mh_k(d_k)}\text{cl}\{x\in(0,1], d_1(x)=\sigma_1,\ldots,d_k(x)=\sigma_k, d_{k+1}(x)=d\},$$

$$I_\sigma=\text{cl}\{x\in(0,1], d_1(x)=\sigma_1, d_2(x)=\sigma_2,\ldots,d_k(x)=\sigma_k\}.$$

Each J_σ is called an n-th order interval. It is obvious that

$$B_m=\bigcap_{k=1}^{+\infty}\bigcup_{\sigma\in D_k} J_\sigma.$$

From the proof of Theorem 6.1 in [6], we have, $k\geq 1$, for any $\sigma\in D_k$, I_σ is an interval with endpoints

$$A_\sigma=\sum_{i=1}^{k-1}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{i-1}(\sigma_{i-1})}{b_{i-1}(\sigma_{i-1})}\frac{1}{\sigma_i}+\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\frac{1}{\sigma_k},$$

$$B_\sigma=\sum_{i=1}^{k-1}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{i-1}(\sigma_{i-1})}{b_{i-1}(\sigma_{i-1})}\frac{1}{\sigma_i}+\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\frac{1}{\sigma_k-1}.$$

As a result, we have

$$|I_\sigma|=\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\frac{1}{\sigma_k(\sigma_k-1)},\tag{8}$$

$$\begin{aligned}|J_\sigma|&=\sum_{d>mh_k(\sigma_k)}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\frac{1}{d(d-1)}\\&=\frac{1}{m}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\frac{1}{\sigma_k(\sigma_k-1)}.\end{aligned}\tag{9}$$

For the upper bound, for any $k\geq 1$, $\bigcup_{\sigma\in D_k} I_\sigma$ is a natural covering of B_m. Thus, for any $s>\sup_{1\leq a\leq L}S(a)$, by the definition of $S(a)$, we have

$$\begin{aligned}H^s(B_m)&\leq\liminf_{n\to+\infty}\sum_{\sigma\in D_{n+1}}|I_\sigma|^s\\&=\liminf_{n\to+\infty}\sum_{\sigma\in D_{n+1}}(\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{a_2(\sigma_2)}{b_2(\sigma_2)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\frac{1}{\sigma_{k+1}(\sigma_{k+1}-1)})^s\\&=\liminf_{n\to+\infty}\sum_{\sigma\in D_n}|I_\sigma|^s\cdot\sum_{\sigma_{k+1}>mh_k(\sigma_k)}(\frac{h_k(\sigma_k)}{\sigma_{k+1}(\sigma_{k+1}-1)})^s\\&\leq\liminf_{n\to+\infty}\sum_{\sigma\in D_n}|I_\sigma|^s\leq\cdots\leq\sum_{\sigma\in D_{t_0}}|I_\sigma|^s<+\infty.\end{aligned}\tag{10}$$

This indicates $\dim_H B_m\leq\sup\limits_{1\leq a\leq L}S(a)$.

To get the lower bound, we consider a sequence of subset of B_m: for any $\alpha\in N:\alpha>1, B_m(\alpha)=\{x\in(0,1]:m<d_j/h_{j-1}(d_{j-1})\leq\alpha m, j\geq 1\}$.

We claim that: $\dim_H B_m(\alpha)\geq\inf_{l\leq a\leq L}S(a,\alpha)$, where $S(a,\alpha)$is defined as follows

$$\sum_{ma<b\leq\alpha ma}(\frac{a}{b(b-1)})^{S(a,\alpha)}=1.\tag{11}$$

Since $H^s(E)=H^s_\wp(E)$ in R^1, where $H^s_\wp(E)$ denotes in the evaluation of Hausdorff measure of E, and any cover of E is restricted to a collection of open intervals. It is natural that $B_m(\alpha)$ is a closed set, then by Heine-Borel Theorem, any open covering system U, consisting of an enumerable number of open intervals, can be replaced by a finite number of open intervals; Furthermore, these intervals may be closed by the addition of their endpoints, and finally these intervals may be altered to have their endpoints in B_m, without at any stage destroying the property that U is a covering system of B_m and increasing $\sum_{U_i\in \mathrm{U}}|U_i|^s$.

Let G be an interval in U, of positive length. G is contained in $I_0=[0,1]$ and is not contained in an I_σ, $\sigma\in D_k$, for k sufficient large. Therefore, there exists the largest value of k, say n, for which G belongs to some I_σ, $\sigma\in D_n$. We see then that there exists numbers $n;\sigma_1,\ldots,\sigma_n;d,l$ with $d\neq l$, such that G is contained in $I_{\sigma_1\cdots\sigma_n}$, and

$$GI_{\sigma_1\cdots\sigma_n d}\neq\emptyset,\ GI_{\sigma_1\cdots\sigma_n l}\neq\emptyset.$$

And if the endpoints of G are in B_m, then

$$GJ_{\sigma_1\cdots\sigma_n d}\neq\emptyset,\ GJ_{\sigma_1\cdots\sigma_n l}\neq\emptyset.$$

Therefore, $|G|$ is greater than or equal to the gap between $J_{\sigma_1\cdots\sigma_n k}$ and $J_{\sigma_1\cdots\sigma_n l}$, Thus, by (9), we know that for $d>l$,

$$\begin{aligned}|G|&\geq(1-\frac{1}{m})\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_n(\sigma_n)}{b_n(\sigma_n)}\frac{1}{d(d-1)}\\&\geq\frac{1}{\alpha^2m^3}\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_n(\sigma_n)}{b_n(\sigma_n)}\frac{1}{h_n^2(\sigma_n)}\\&\geq\frac{1}{\alpha^2m^3L}|I_{\sigma_1\cdots\sigma_n}|.\end{aligned}\tag{12}$$

Let Ω be the finite set of intervals I_σ corresponding in the above way to intervals G of U. Of course, $\bigcup_{I\in\Omega}I$ is a covering of B_m.

Let $K_1=\max\{k:\sigma\in D_k, I_\sigma\in\Omega\}$, $K_2=\min\{k:\sigma\in D_k, I_\sigma\in\Omega\}$ and define, $W_{K_1}=\{I_\sigma\in\Omega:\sigma\in D_{K_1}\}$. By the definition of K_1, we know that if $I_{\sigma_1,\ldots,\sigma_{K_1-1}*j}\in W_{K_1}$, then for any $mh_{K_1-1}(\sigma_{K_1-1})<j\leq\alpha mh_{K_1-1}(\sigma_{K_1-1})$, $I_{\sigma_1,\ldots,\sigma_{K_1-1}*j}\in W_{K_1}$. For any $s<\inf_{1\leq a\leq L}S(a,\alpha)$, we have

$$\begin{aligned}&\sum_{mh_{K_1-1}(\sigma_{K_1-1})<j\leq\alpha mh_{K_1-1}(\sigma_{K_1-1})}|I_{\sigma_1,\ldots,\sigma_{K_1-1}*j}|^s\\&=|I_{\sigma_1,\ldots,\sigma_{K_1-1}}|^s\sum_{mh_{K_1-1}(\sigma_{K_1-1})<j\leq\alpha mh_{K_1-1}(\sigma_{K_1-1})}(\frac{h_{K_1-1}(\sigma_{K_1-1})}{j(j-1)})^s\\&\geq|I_{\sigma_1,\ldots,\sigma_{K_1-1}}|^s.\end{aligned}$$

The argument above shows that we can decrease the basic interval covering to a new one with lower degree (here, the degree of I_σ is defined as the length of σ as a word) and without increasing the sum. As a result, we can replace the covering Ω by a new covering Ω^* in which all basic interval are of the same order and, moreover,

$$\sum_{I_\sigma\in\Omega}|I_\sigma|^s\geq\sum_{\sigma\in D_{k_2}}|I_\sigma|^s.$$

At the same time, since

$$\begin{aligned}\sum_{\sigma\in D_{k+1}}|I_\sigma|^s&=\sum_{\sigma\in D_k}\sum_{mh_{K_1-1}(\sigma_{K_1-1})<j\leq\alpha mh_k(\sigma_k)}|I_{\sigma*j}|^s\\&\geq\sum_{\sigma\in D_k}|I_\sigma|^s\geq\cdots\geq\sum_{\sigma\in D_{t_0}}|I_\sigma|^s,\end{aligned}$$

we have

$$\sum_{G\in\mathrm{U}}|G|^s\geq\frac{1}{\alpha^2m^3L}\sum_{\sigma\in\Omega}|I_\sigma|^s\geq\frac{1}{\alpha^2m^3L}\sum_{\sigma\in D_{k_2}}|I_\sigma|^s\geq\frac{1}{\alpha^2m^3L}\sum_{\sigma\in D_{t_0}}|I_\sigma|^s.$$

Since s is arbitrarily, we have $\dim_H B_m(\alpha) \geq \inf_{l\leq a\leq L} S(a,\alpha)$. This can finish the proof of Theorem 2.2, according to the following lemma:

Lemma 2.2 *$S(a,\alpha)$ is increasing, as to α, and*

$$\lim_{\alpha\to+\infty} S(a,\alpha) = S(a). \tag{13}$$

Proof The fact that $S(a,\alpha)$ is increasing, as to α, is quite natural, and $S(a) > S(a,\alpha)$ for all $\alpha > 1$. And for the other side, we notice firstly that $S(a,\alpha) \geq \frac{1}{2}$ for α sufficiently large, and we have

$$\sum_{ma<j<\alpha ma} (\frac{a}{j(j-1)})^{\frac{1}{2}} \geq \sum_{ma<j<\alpha ma} \frac{a^{\frac{1}{2}}}{j} \geq a^{\frac{1}{2}} \log\alpha > 1.$$

Now, we show that, for any $\epsilon > 0, S(a) < S(a,\alpha)+\epsilon$ ultimately. Since for α sufficiently large,

$$\begin{aligned}&\sum_{j>ma} (\frac{a}{j(j-1)})^{S(a,\alpha)+\epsilon}\\ &= \sum_{ma<j<\alpha ma} (\frac{a}{j(j-1)})^{S(a,\alpha)+\epsilon} + \sum_{j>\alpha am} (\frac{a}{j(j-1)})^{S(a,\alpha)+\epsilon}\\ &\leq (\frac{1}{am^2})^{\epsilon} + \sum_{j>\alpha am} (\frac{a}{j(j-1)})^{S(a,\alpha)+\epsilon} < 1.\end{aligned}$$

This implies $S(a) < S(a,\alpha)+\epsilon$, for α sufficiently large. As a result, we have

$$\lim_{\alpha\to+\infty} S(a,\alpha) = S(a).$$

Since $\dim_H B_m \geq \dim_H B_m(\alpha)$, for any $\alpha > 1$, then

$$\dim_H B_m \geq \sup_{\alpha>1} \inf_{l\leq a\leq L} S(a,\alpha) = \inf_{l\leq a\leq L} S(a).$$

This completes the proof of Theorem 2.2.

3. Hausdorff dimension of a set in Lüroth series expansion

Form Theorem 2.2, we get the following corollary.

Corollary 3.1 *For Lüroth series expansion, the set $B_m = \{x \in (0,1) : d_n(x) > m, n \geq 1\}$ is of Hausdorff dimension $S(1)$. Moreover,* for *$m \geq K^K \geq 17$, we have*

$$\frac{1}{2} + \frac{\log K}{2\log(m+2)} \leq S(1) \leq \frac{1}{2} + \frac{\log\log(m-1)}{2\log(m-1)}.$$

Proof From the proof Theorme 2.2, the upper bound s of $\dim_H B_m$ satisfies:

$$\sum_{b>m} (\frac{1}{b(b-1)})^s \leq 1. \tag{14}$$

Since

$$\sum_{b>m}(\frac{1}{b(b-1)})^s < \sum_{b=m+1}^{\infty}(\frac{1}{b-1})^{2s} < \int_m^{\infty}(x-1)^{-2s}\mathrm{d}x = \frac{1}{2s-1}(\frac{1}{m-1})^{2s-1},$$

the velue of s satisfies Inequality (14) if

$$\frac{1}{2s-1}(\frac{1}{m-1})^{2s-1} \leq 1,$$

that is

$$(2s-1)(m-1)^{2s-1} \geq 1. \tag{15}$$

Now, when

$$2s-1 = \frac{\log\log(m-1)}{\log(m-1)}, \tag{16}$$

Inequality (15) can be written in the form

$$\log\log(m-1) \geq 1$$

or

$$m \geq 1 + e^e > 17. \tag{17}$$

Thus, the upper bound s of $\dim_H B_m$ is obtained in Equality (16) with m satisfying Inequality (17), i.e.,

$$\dim_H B_m \leq \frac{1}{2} + \frac{\log\log(m-1)}{2\log(m-1)}$$

is implied by condition (17).

From the proof Theorme 2.2, the lower bound s of $\dim_H B_m$ has the property:

$$\sum_{b>m}(\frac{1}{b(b-1)})^s \geq 1. \tag{18}$$

Since

$$\sum_{b>m}(\frac{1}{b(b-1)})^s > \sum_{b=m+1}^{\infty}(\frac{1}{b})^{2s} > \int_{m+2}^{\infty}x^{-2s}\mathrm{d}x = \frac{1}{2s-1}(\frac{1}{m+2})^{2s-1},$$

the velue of s satisfies Inequality (18) if

$$(2s-1)(m+2)^{2s-1} \leq 1$$

or

$$m+2 \leq (\frac{1}{2s-1})^{\frac{1}{2s-1}}.$$

Thus $\dim_H B_m \geq s_0$, where

$$m+2 = (\frac{1}{2s_0-1})^{\frac{1}{2s_0-1}}. \tag{19}$$

Notice that x^x is, for $x \geq 1$, a strictly increasing function of x. For any constant $K > 2$ and

$$m+2 > K^K, \tag{20}$$

there exists a real number $c > 0$, such that

$$m + 2 = (K + c)^{K+c}. \tag{21}$$

Therefore,

$$\log_K(m + 2) = (K + c)\log_k(K + c) \geq (K + c).$$

It follows that

$$\log_K(m + 2) \geq K + c. \tag{22}$$

On the other hand, by Equalities (19) and (21), we have

$$\frac{1}{2s_0 - 1} = K + c \tag{23}$$

and by (22) and (23), we have $\log_K(m + 2) \geq \frac{1}{2s_0-1}$. So $s_0 \geq \frac{1}{2} + \frac{\log K}{2\log(m+2)}$, where m satisfies condition (20). Hence we have proved

$$\dim_H B_m \geq \frac{1}{2} + \frac{\log K}{2\log(m + 2)},$$

for any $m \geq K^K - 2$.

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一类 Oppenheim 展式的例外关系集

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摘要: 本文研究了 Oppenheim 展式中一类例外关系集的 Hausdorff 维数。作为其应用，我们得到了 Lüroth 级数展式中一些集合的 Hausdorff 维数的确切值，并给出了这些确切值的一个估计式

关键词: Oppenheim 级数展式; Lüroth 级数展式; Hausdorff 维数.