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The Structure of ${}^*\tau_x$ and Its Properties

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Abstract: The structure and some properties of ${}^*\tau_x$ are discussed in the nonstandard κ-saturated model in this paper. First, a sufficient and necessary condition of a internal set in ${}^*\tau_x$ is given. Then, in κ-saturation, some properties of ${}^*\tau_x$ are proved. Finally, the approach theorem is easily obtained.

Key words: concurrent; κ-saturation; monad; $*$-finite.
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1. Introduction and preliminaries

Let (X,τ) be an infinite topological space. For any $x \in X$, τ_x denotes the system of neighborhoods of point x. As we all known, the structure of τ_x is clear [1]. By transfer principle, ${}^*\tau_x$ is obtained. What is in ${}^*\tau_x$ and what are the properties of ${}^*\tau_x$? In this paper, we devote to solve these problems.

First of all, some concepts in nonstandard analysis must be recalled.

Definition 1 *Let r be a relation in superstructure $V(S)$. r is called concurrent if whenever $a_1, a_2, \ldots, a_n \in \mathrm{dom}(r)$, there is an element $b \in \mathrm{ran}(r)$ such that $\langle a_i, b\rangle \in r$ for $i = 1, 2, \ldots, n$.*

Definition 2 *Nonstandard model $V({}^*S)$ is called the κ-saturated model of $V(S)$ if an internal relation r in $V({}^*S)$ is concurrent on a subset A of its domain and if $\mathrm{card}(A) < \kappa$ implies the existence of an element y in range of r such that $\langle x, y\rangle$ holds for all $x \in A$.*

In this paper, S denotes a infinite individual set and $X \subseteq S$. Let $V({}^*S)$ be a κ-saturation of $V(S)$ with $\kappa > \mathrm{card}(\tau)$.

Definition 3 *Let (X,τ) be a topological space. For any point $x \in X$, the set $\mu(x) = \cap\{{}^*G : G \in \tau_x\}$ is called the monad of point x.*

Definition 4 *Let (X,τ) be a topological space, $A \subseteq {}^*X$. The set $\mu_\tau(\{A\}) = \cap\{{}^*\Omega : \Omega \subseteq \tau$ and $A \in {}^*\Omega\}$ is called τ-monad of A.*

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Definition 5 *Let $\Lambda \subseteq \mathcal{P}(^*X)$. Then* $^0\Lambda = \{A \in \mathcal{P}(X) : {^*A} \in \Lambda\}$.

2. The structure of $^*\tau_x$

In this section, a sufficient and necessary condition for a internal set in $^*\tau_x$ is obtained.

Let (X,τ) be a topological space. If A is a nonempty subset of $^*X$, let $\tau_{\{A\}} = \{\Omega : \Omega \subseteq \tau, A \in {^*\Omega}\}$. It is obvious that $\mu_\tau(\{A\}) = \cap\{^*\Omega : \Omega \in \tau_{\{A\}}\}$.

Lemma *Let (X,τ) be a topological space. If A is a nonempty internal subset of $^*X$, then there is an element $E \in {^*\tau_{\{A\}}}$ such that $E \subseteq \mu_\tau(\{A\})$.*

Proof Let $r = \{\langle G,H\rangle : G \in {^*\tau_{\{A\}}}, H \in {^*\tau_{\{A\}}}, H \subseteq G\}$. Clearly, r is a internal relation, and $^\sigma\tau_{\{A\}} \subseteq \mathrm{dom}(r) = {^*\tau_{\{A\}}}$. We shall show that r is concurrent on $^\sigma\tau_{\{A\}}$. To see that r is concurrent, let $G_1, G_2, \ldots, G_n \in {^\sigma\tau_{\{A\}}}$. Then $G_i = {^*B_i}$ for each $i = 1,2,\ldots,n$ where $B_i \in \tau_{\{A\}}$. Let $H = \bigcap_{i=1}^n G_i = \bigcap_{i=1}^n {^*B_i} = {^*(\bigcap_{i=1}^n B_i)} \in {^\sigma\tau_{\{A\}}} \subseteq {^*\tau_{\{A\}}}$. Thus $H \subseteq G_i$ for all $i = 1,2,\ldots,n$, that is, there exists $H \in {^*\tau_{\{A\}}}$ such that $\langle G_i, H\rangle \in r$ for each $i = 1,2,\ldots,n$. Hence r is concurrent and $\mathrm{card}(^\sigma\tau_{\{A\}}) = \mathrm{card}(\tau_{\{A\}} < \mathrm{card}(\tau) < \kappa$. Because $V(^*S)$ is the κ-saturation of $V(S)$, there is $E \in {^*\tau_{\{A\}}}$ such that $\langle G, E\rangle \in r$ for each $G \in {^\sigma\tau_{\{A\}}}$, that is, $E \subseteq G$ for each $G \in {^\sigma\tau_{\{A\}}}$. So $E \subseteq \cap\{G : G \in {^\sigma\tau_{\{A\}}}\} = \mu_\tau(\{A\})$. $\square$

Theorem 1 *Let (X,τ) be a topological space and A be a nonempty internal set of $^*X$. Then there exists a point $x \in X$ such that $A \in {^*\tau_x}$, if and only if for every $*$-finite subset $\Lambda \subseteq \mu_\tau(\{A\})$ we have $\cap\{B : B \in \Lambda\} \neq \emptyset$.*

Proof Let A be internal and $A \in {^*\tau_x}$ for some $x \in X$. By the definition, $\mu_\tau(\{A\}) \subseteq {^*\tau_x}$. Since τ_x has the finite intersection property, $^*\tau_x$ has the $*$-finite intersection property. So for every $*$-finite subset $\Lambda \subseteq \mu_\tau(\{A\} \subseteq {^*\tau_x}$, Λ also has the $*$-finite intersection property, that is $\cap\{B : B \in \Lambda\} \neq \emptyset$.

Conversely, from the lemma it follows that there is an element $E \in {^*\tau_{\{A\}}}$ such that $E \subseteq \mu_\tau(\{A\})$. By the hypothesis, the family E of internal sets has the $*$-finite intersection property, that is, there is $E \in {^*\tau_{\{A\}}}$ and E has the $*$-finite intersection property. By the transfer principle, there exists an element $\Omega \in \tau_{\{A\}}$ such that Ω has the finite intersection property. From $\Omega \in \tau_{\{A\}}$ it follows that $A \in {^*\Omega}$, and so A is an element of the extension of the τ_x for some $x \in X$ generated by Ω. This completes the proof of the theorem.

3. Some properties of $^*\tau_x$

In this section, some properties of $^*\tau_x$ are discussed.

Theorem 2 *Let (X,τ) be a topological space. Then for each point $x \in X$, if Λ is an internal subset of $^*\tau_x$ such that $\tau_x =^0 \Lambda$, then there exists an element $E \in \Lambda$ such that $E \subseteq \mu(x)$.*

Proof Let $r = \{\langle A,B\rangle : A \in \Lambda, B \in \Lambda, B \subseteq A\}$. Clearly, r is a internal relation, and $^\sigma\tau_x \subseteq \mathrm{dom}(r) = \Lambda$. We show that r is concurrent on $^\sigma\tau_x$. To this end, let $A_1, A_2, \ldots, A_n \in {^\sigma\tau_x}$. Then

$A_i = {}^*G_i$ for each $i = 1, 2, \ldots, n$ where $G_i \in \tau_x$. Let $B = \bigcap_{i=1}^n A_i = \bigcap_{i=1}^n {}^*G_i = {}^*(\bigcap_{i=1}^n G_i) \in {}^\sigma\tau_x \subseteq \Lambda$. Thus $B \subseteq A_i$ for all $i = 1, 2, \ldots, n$, that is, there exists $B \in \Lambda$ such that $\langle A_i, B\rangle \in r$ for each $i = 1, 2, \ldots, n$. Hence r is concurrent. And $\mathrm{card}({}^\sigma\tau_x) = \mathrm{card}(\tau_x) < \mathrm{card}(\tau) < \kappa$. Because $V({}^*S)$ is the κ-saturation of $V(S)$, there is $E \in \Lambda$ such that $\langle A, E\rangle \in r$ for each $A \in {}^\sigma\tau_x$, that is, $E \subseteq A$ for each $A \in {}^\sigma\tau_x$. So $E \subseteq \cap\{A : A \in \tau_x\} = \mu(x)$. □

Corollary 1 *Let (X, τ) be a topological space. For each point $x \in X$, if Ω is an internal subset of ${}^*\tau_x$ such that $E \in {}^*\tau_x$ and $E \subseteq \mu(x)$ implies $E \in \Omega$, then there exists an element $G \in \tau_x$ such that ${}^*G \in \Omega$.*

Proof Suppose that ${}^*G \notin \Omega$ for any $G \in \tau_x$. Then let $\Lambda = {}^*\tau_x - \Omega$. It is obvious that Λ is an internal subset of ${}^*\tau_x$ and ${}^*G \in \Lambda$ for every $G \in \tau_x$, that is, λ satisfies the conditions of Theorem 2. Hence there exists an element $E \in \Lambda$ such that $E \subseteq \mu(x)$. This contradicts $E \in \Omega$. □

The following corollary is a famous conclusion–Approach Principle in [2]. However, it can be easily obtained in this paper.

Corollary 2 *For each $x \in X$ there is an internal set $D \in {}^*\tau_x$ such that $D \subseteq \mu(x)$.*

Proof We only need to let Λ in Theorem 2 be ${}^*\tau_x$.

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${}^*\tau_x$ 的结构及其性质

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摘要: 在 κ- 饱和的非标准模型中，讨论了 ${}^*\tau_x$ 的结构及其性质. 首先，本文给出了一个内集在 ${}^*\tau_x$ 中的充分必要条件. 其次，对 ${}^*\tau_x$ 的性质做了进一步的讨论. 最后，利用 ${}^*\tau_x$ 的性质，容易地证明了著名的逼近原理.

关键词: 共点关系； κ- 饱和；单子； $*$- 有限.