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Global Exponential Stability of Fuzzy Cellular Neural Networks with Impulses and Infinite Delays

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Abstract In this paper, the global exponential stability of fuzzy cellular neural networks with impulses and infinite delays is investigated. Based on an impulsive delayed integro-differential inequality and the properties of fuzzy logic operation and M-matrix, an easily verified sufficient condition is obtained. Moreover, the exponential convergent rate for the fuzzy cellular neural networks with impulses and infinite delays is also given. An example is given to illustrate the effectiveness of our theoretical result.

Keywords global exponential stability; fuzzy cellular neural networks; impulses; infinite delays; integro-differential inequality.

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1. Introduction

Yang et al.[1,2] introduced fuzzy cellular neural networks (FCNN), which integrates fuzzy logic into the structure of traditional CNN and maintains local connectedness among cells. Unlike previous CNN structures, FCNN has fuzzy logic between its template and input and/or output besides the "sum of product" operation, which allows us to combine the low information processing capability of CNN's with the high level information processing capability, such as image understanding of fuzzy systems. FCNN is a useful paradigm for image processing problems and Euclidean distance transformation. Also, FCNN has inherent connection to mathematical morphology, which is a cornerstone in image processing and pattern recognition. To guarantee that the performance of FCNN is what we wanted, it is important to study its equilibrium points and the stability of those equilibrium points. Yang et al.[3] investigated the existence and stability of equilibrium point of FCNN. And then the delay effect on the stability of equilibrium point of FCNN has been studied by some researchers. Chen et al.[4] considered the stability of FCNN with time-varying delays and Liu et al.[5] studied the stability of FCNN with constant

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time delays and time-varying delays. Huang et al.[6] investigated the stability of FCNN with infinite delays. However, besides the delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time. For example, some biological systems such as biological neural networks and bursting rhythm models in pathology, as well as frequency-modulated signal processing systems, and flying motions, are characterized by abrupt changes of states at certain time instants. Their study is assuming a greater importance[7,8]. Xu et al.[9] established an impulsive delay differential inequality and studied the stability of neural networks with impulses. Guan et al.[10] investigated the impulsive synchronization for Takagi-Sugeno fuzzy model.

In this paper, we will study the stability of the FCNN with impulses and infinite delays:

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i x_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij}\mu_j + I_i + \\ \qquad \displaystyle\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t)) + \bigwedge_{j=1}^{n} \gamma_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(x_j(s))\mathrm{d}s + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \\ \qquad \displaystyle\bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t)) + \bigvee_{j=1}^{n} \theta_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(x_j(s))\mathrm{d}s + \bigvee_{j=1}^{n} H_{ij}\mu_j,\ t \neq t_k, \\ x_i(t_0^+) = \phi_i \in R, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = J_{ik}(x_1(t_k^-), \ldots, x_n(t_k^-)) + h_{ik},\ k = 1, 2, \ldots, \end{cases} \tag{1}$$

where $i = 1, \ldots, n$, $a_i > 0$, α_{ij} and γ_{ij} are elements of fuzzy feedback MIN template; β_{ij} and θ_{ij} are elements of fuzzy feedback MAX template; T_{ij} and H_{ij} are elements of fuzzy feed-forward MIN template and fuzzy feed-forward MAX template, respectively; b_{ij} are elements of feed-forward template; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively; x_i, μ_i, I_i denote state, input and bias of the ith neurons, respectively; h_{ik} are impulsive constant input; f_i are the activation functions; $k_{ij}(s) \geq 0$ are the feedback kernels, defined on $[0, \infty)$, satisfying

$$\text{(H)}: \quad \int_0^{\infty} e^{\lambda_0 s} k_{ij}(s)\mathrm{d}s < \infty, \quad i, j = 1, \ldots, n, \tag{2}$$

where λ_0 is a positive constant; t_k are the impulsive moments; $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$.

2. Preliminaries

In what follows, we will introduce some notations and basic definitions.

Let R^n be the space of n-dimensional real column vectors, and let $R^{m\times n}$ denote the set of $m \times n$ real matrices. E denotes an $n \times n$ unit matrix. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B$ $(A > B)$ means that each pair of corresponding elements of A and B satisfies the inequality " $\geq (>)$". Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

$C[X, Y]$ denotes the space of continuous mappings from a topological space X to a topological space Y. Especially, let $C \triangleq C[(-\infty, 0], R^n]$.

$$PC[I, R^n] \triangleq \{\varphi : I \to R^n \,|\varphi(t^+) = \varphi(t) \text{ for } t \in I,\ \varphi(t^-) \text{ exists for } t \in (t_0, \infty),$$

$\varphi(t^-) = \varphi(t)$ for all but points $t_k \in (t_0, \infty)\}$, where $I \subset R$ is an interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the left limit and right limit of scalar function $\varphi(t)$, respectively. Especially, let $PC = PC((-\infty, 0), R^n)$.

For $x \in R^n$, $A \in R^{n\times n}$, we define

$$[x]^+ = (|x_1|, \ldots, |x_n|)^{\mathrm{T}}, \quad [A]^+ = (|a_{ij}|)_{n\times n},$$

and introduce the corresponding norm for them as follows

$$||x|| = \max_{1\le i\le n}\{|x_i|\}, \quad ||A|| = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|.$$

Definition 1 *For any given $t_0 \in R$, $\phi_i \in PC$, a function $x_i(t) \in PC[(-\infty, +\infty), R]$ is called a solution of (1) through (t_0, ϕ_i), if $x_i(t)$ satisfies (1) for $t \ge t_0$, denoted by $x_i(t, t_0, \phi)$ or simply by $x_i(t)$ if no confusion arises. Especially, a point $x^* = (x_1^*, \ldots, x_n^*)^{\mathrm{T}}$ is called an equilibrium of (1), if $x_i(t) = x_i^*$ is a solution of (1) for $i = 1, \ldots, n$.*

Throughout this paper, we assume that for any $\phi_i \in PC$, there exists at least one solution of (1). Let $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ be any solution of (1). As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $x(t)$ we define that $x(t_k^+) \equiv x(t_k)$.

Let x^* be an equilibrium point of (1). And set $y(t) = x(t) - x^* = (x_1(t) - x_1^*, \ldots, x_n(t) - x_n^*)^{\mathrm{T}}$. Substituting them into (1), we can get

$$\begin{cases} \dfrac{\mathrm{d}y_i}{\mathrm{d}t} = -a_i y_i(t) + \bigwedge_{j=1}^{n}\alpha_{ij}g_j(y_j(t)) + \bigwedge_{j=1}^{n}\gamma_{ij}\displaystyle\int_{-\infty}^{t}k_{ij}(t-s)g_j(y_j(s))\mathrm{d}s + \\ \qquad \bigvee_{j=1}^{n}\beta_{ij}g_j(y_j(t)) + \bigvee_{j=1}^{n}\theta_{ij}\displaystyle\int_{-\infty}^{t}k_{ij}(t-s)g_j(y_j(s))\mathrm{d}s \\ y_i(t_0^+) = \varphi_i \in R, \\ \Delta y_i(t_k) = \psi_{ik}(y_1(t_k^-), \ldots, y_n(t_k^-)),\ k = 1, 2, \ldots, \end{cases} \tag{3}$$

where $\psi_{ik}(y_1(t_k^-), \ldots, y_n(t_k^-)) = J_{ik}(y_1(t_k^-) + x_1^*, \ldots, y_n(t_k^-) + x_n^*) - J_{ik}(x_1^*, \ldots, x_n^*)$, $g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $\varphi_i = \phi_i - x_i^*$.

It is clear that the stability of the zero solution of (3) is equivalent to the stability of the equilibrium point x^* of (1). Therefore, we may mainly discuss the stability of the zero solution of (3).

Definition 2 *The zero solution of (3) is said to be globally exponentially stable if for any solution $y(t)$ with the initial condition $\varphi = (\varphi_1, \ldots, \varphi_n)^{\mathrm{T}} \in PC$, there exist a constant $\lambda > 0$ and a vector $z > 0$ such that*

$$[y(t)]^+ \le z e^{-\lambda(t-t_0)}, \quad t \ge t_0. \tag{4}$$

Definition 3[11] *For $A=(a_{ij})\in R^{m\times n}$ and $B=(b_{ij})\in R^{m\times n}$, define $A\circ B$ as follows:*

$$A\circ B \triangleq \begin{pmatrix} a_{11}b_{11} & \cdots & a_{1n}b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1}b_{m1} & \cdots & a_{mn}b_{mn} \end{pmatrix}. \tag{5}$$

Then, $A\circ B$ is called the Hadamard product or Schur product of A and B.

Definition 4[12] *Let the matrix $D=(d_{ij})_{n\times n}$ have nonpositive off-diagonal elements (i.e., $d_{ij}\le 0, i\neq j$). Then each of the following conditions is equivalent to the statement "D is a nonsingular M-matrix".*

(i) All the leading principle minors of D are positive;

(ii) $D=C-M$ and $\rho(C^{-1}M)<1$, where $M\ge 0$, $C=\mathrm{diag}\{c_1,\dots,c_n\}$ and $\rho(\cdot)$ is the spectral radius of the matrix $(\cdot)$;

(iii) The diagonal elements of D are all positive and there exists a positive vector d such that $Dd>0$ or $D^Td>0$.

For a nonsingular M-matrix D, we denote

$$\Omega_M(D)\triangleq\{z\in R^n|Dz>0, z>0\}.$$

From (iii) of the definition of M-matrix, we have the following lemma.

Lemma 1 *$\Omega_M(D)$ is nonempty and for any $z_1,\ z_2\in\Omega_M(D)$, we have*

$$k_1z_1+k_2z_2\in\Omega_M(D),\ \ \forall\ k_1,\ k_2>0.$$

Lemma 2[2] *For any $a_{ij}\in R$, $x_j,\ y_j\in R$, $i,j=1,\dots,n$, we have the following estimations:*

$$\Big|\bigwedge_{j=1}^n a_{ij}x_j-\bigwedge_{j=1}^n a_{ij}y_j\Big|\le\sum_{j=1}^n|a_{ij}|\,|x_j-y_j|, \tag{6}$$

and

$$\Big|\bigvee_{j=1}^n a_{ij}x_j-\bigvee_{j=1}^n a_{ij}y_j\Big|\le\sum_{j=1}^n|a_{ij}|\,|x_j-y_j|. \tag{7}$$

3. Global exponential stability

In this section, we will first establish an impulsive delayed integro-differential inequality and then give some sufficient conditions on the global exponential stability of equilibrium point for system (3).

Theorem 1 *Let $0\le u(t)=(u_1(t),\dots,u_n(t))^{\mathrm T}\in PC([t_0,\infty),R^n)$ satisfy the following impulsive delayed integro-differential inequality*

$$\begin{cases} D^+u(t)\le Pu(t)+\displaystyle\int_0^\infty Q(s)u(t-s)\mathrm{d}s, \quad t\neq t_k,\ t\ge t_0, \\ u(t)\le W_ku(t^-), \quad t=t_k,\ k=1,2,\dots, \\ u(t)=\phi(t), \quad -\infty<t\le t_0, \end{cases} \tag{8}$$

where $P = (p_{ij})_{n\times n}$ *with* $p_{ij} \geq 0$ *for* $i \neq j$, $W_k = (w_{ij}^k)_{n\times n} \geq 0$, $\phi(t) \in PC$, $Q(t) = (q_{ij}(t))_{n\times n} \geq 0$ *for any* $t \geq t_0$ *and satisfies*

$$(\mathrm{H}_0): \quad \int_0^\infty e^{\lambda_1 s} Q(s)\mathrm{d}s < \infty,$$

in which λ_1 *is a positive constant.*

Write $Q = (\int_0^\infty q_{ij}(s)\mathrm{d}s)_{n\times n}$. *If* $D = -(P+Q)$ *is a nonsingular M-matrix, then there exists a positive vector* $z = (z_1, \ldots, z_n)^{\mathrm{T}} \in \Omega_M(D)$ *such that*

$$u(t) \leq \delta_1 \cdots \delta_{k-1} z e^{-\lambda(t-t_0)},\ t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, \tag{9}$$

where $\delta_k = \max\{1, ||W_k||\}$ *and the positive constant* $\lambda \leq \lambda_1$ *is determined by the following inequality*

$$[\lambda E + P + \int_0^\infty Q(s)e^{\lambda s}\mathrm{d}s]z < 0, \quad \textit{for the given} \quad z \in \Omega_M(D). \tag{10}$$

Proof Since D is a nonsingular M-matrix, by Lemma 1, there exists a positive vector $z \in \Omega_M(D)$ such that $Dz > 0$ or $(P+Q)z < 0$. By using continuity and hypothesis (H_0), we know that (l0) has at least one positive solution $\lambda \leq \lambda_1$, i.e.

$$\lambda z_i + \sum_{j=1}^n \left(p_{ij} + \int_0^\infty q_{ij}(s)e^{\lambda s}\mathrm{d}s \right) z_j < 0. \tag{11}$$

Since $\phi(t) \in PC$ is bounded, by Lemma 1, we can always choose a sufficiently large $z \in \Omega_M(D)$ such that

$$u(t) \leq z e^{-\lambda(t-t_0)}, \quad t \in (-\infty, t_0]. \tag{12}$$

Now we shall prove that

$$u(t) \leq z e^{-\lambda(t-t_0)}, \quad t \in [t_0, t_1). \tag{13}$$

In order to prove (13), we first prove for any $l > 1$

$$u_i(t) < l z_i e^{-\lambda(t-t_0)} \triangleq v_i(t),\ t \in [t_0, t_1),\ i = 1, \ldots, n. \tag{14}$$

If (14) is not true, then from the fact that $u(t)$ is continuous in $[t_0, t_1)$, there must be a $t^* \in [t_0, t_1)$ and some integer m such that

$$u_m(t^*) = v_m(t^*), \quad D^+ u_m(t^*) \geq v'_m(t^*), \tag{15}$$

$$u_i(t) \leq v_i(t),\ t \in (-\infty, t^*],\ i = 1, \ldots, n. \tag{16}$$

Hence, by (8), (11), the equality of (15), (16) and $p_{ij} \geq 0$ for $i \neq j$, $q_{ij}(t) \geq 0$, we have

$$\begin{aligned} D^+ u_m(t^*) &\leq \sum_{j=1}^n \left(p_{mj} u_j(t^*) + \int_0^\infty q_{mj}(s) u_j(t^* - s)\mathrm{d}s \right) \\ &\leq \sum_{j=1}^n \left(p_{mj} + \int_0^\infty q_{mj}(s)e^{\lambda s}\mathrm{d}s \right) l z_j e^{-\lambda(t^*-t_0)} \\ &< -\lambda z_m l e^{-\lambda(t^*-t_0)} \end{aligned}$$

$$= v'_m(t^*),$$

which contradicts the inequality of (15), so (l4) holds for all $t \in [t_0, t_1)$. Let $l \to 1$. Then (13) holds for $t \in [t_0, t_1)$.

Using the discrete part of (8), (13), the fact that $W_1 z \leq ||W_1||z$, and the definition of δ, we obtain that

$$u(t_1) \leq W_1 u(t_1^-) \leq W_1 z e^{-\lambda(t_1-t_0)} \leq ||W_1|| z e^{-\lambda(t_1-t_0)} \leq \delta_1 z e^{-\lambda(t_1-t_0)}.$$

So we have

$$u(t) \leq \delta_1 z e^{-\lambda(t-t_0)}, \quad t \in (-\infty, t_1]. \tag{17}$$

By an argument similar to (13), we can use (17) to derive that

$$u(t) \leq \delta_1 z e^{-\lambda(t-t_0)}, \quad t \in [t_1, t_2). \tag{18}$$

So, by simple induction, we conclude that

$$u(t) \leq \delta_1 \cdots \delta_{k-1} z e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots. \tag{19}$$

The proof is completed. □

Now we will state and prove our main results. For convenience, the following notations will be used.

$$A_0 = \mathrm{diag}\{a_1, \ldots, a_n\}, \quad A = (|\alpha_{ij}| + |\beta_{ij}|)_{n\times n},$$

$$B = (|\gamma_{ij}| + |\theta_{ij}|)_{n\times n}, \quad K(s) = (k_{ij}(s))_{n\times n}.$$

Theorem 2 *Assume that the hypothesis (H) holds. Furthermore, suppose the following:*

(H1) For any $y_j \in R$, $j = 1, \ldots, n$, there exist nonnegative constants L_j such that

$$|g_j(y_j)| \leq L_j |y_j|. \tag{20}$$

(H2) For any $y_j \in R$, there exist nonnegative constants w_{ij}^k, $k = 1, 2, \ldots, i, j = 1, \ldots, n$ such that

$$|\psi_{ik}(y_1, \ldots, y_n)| \leq \sum_{j=1}^{n} w_{ij}^k |y_j|. \tag{21}$$

(H3) Let $L = \mathrm{diag}\{L_1, \ldots, L_n\}$, $Q(s) = B \circ K(s)L$, $P = -A_0 + AL$, $Q = \int_0^\infty Q(s)\mathrm{d}s$ and $D = -(P+Q)$ be a nonsingular M-matrix.

(H4) Let

$$\gamma_k = \max\{1, ||W_k||\}, \quad W_k = (w_{ij}^k)_{n\times n}. \tag{22}$$

And there exists a positive constant η such that

$$\frac{\ln \gamma_k}{t_k - t_{k-1}} \leq \eta < \lambda,\ k = 1, 2, \ldots, \tag{23}$$

where the positive constant $\lambda \leq \lambda_0$ is determined by the following inequality

$$[\lambda E + P + B \circ \int_0^\infty K(s) e^{\lambda s} \mathrm{d}s L] z < 0, \quad \text{for } z \in \Omega_M(D). \tag{24}$$

Then the zero solution of (3) is globally exponentially stable, i.e., the equilibrium point x^* of (1) is globally exponentially stable. The convergent rate is equal to $\lambda - \eta$.

Proof Obviously, the zero solution of (3) is an equilibrium point. The uniqueness of the equilibrium point follows from the global exponential stability of the equilibrium.

Since D is a nonsingular M-matrix, by Lemma 1, there exists a positive vector $z \in \Omega_M(D)$ such that $Dz > 0$ or $(P+Q)z < 0$. By using continuity and hypothesis (H), we obtain that (24) has at least one positive solution $\lambda \le \lambda_0$.

Then calculating the upper right derivative $D^+|y_i(t)|$ along the solution of (3), we have

$$
\begin{aligned}
D^+|y_i(t)| =& \mathrm{sgn}(y_i(t))\frac{\mathrm{d}y_i}{\mathrm{d}t} \\
\le& -a_i|y_i(t)| + |\bigwedge_{j=1}^{n} \alpha_{ij} g_j(y_j(t))| + |\bigwedge_{j=1}^{n} \gamma_{ij} \int_{-\infty}^{t} k_{ij}(t-s) g_j(y_j(s))\mathrm{d}s| + \\
& |\bigvee_{j=1}^{n} \beta_{ij} g_j(y_j(t))| + |\bigvee_{j=1}^{n} \theta_{ij} \int_{-\infty}^{t} k_{ij}(t-s) g_j(y_j(s))\mathrm{d}s| \\
\le& -a_i|y_i(t)| + \sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|)\, |g_j(y_j(t))| + \\
& \sum_{j=1}^{n} (|\gamma_{ij}| + |\theta_{ij}|)\, |\int_{-\infty}^{t} k_{ij}(t-s) g_j(y_j(s))\mathrm{d}s| \\
\le& -a_i|y_i(t)| + \sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|)\, L_j|y_j(t)| + \\
& \sum_{j=1}^{n} (|\gamma_{ij}| + |\theta_{ij}|) \int_{0}^{\infty} k_{ij}(s) L_j |y_j(t-s)|\mathrm{d}s, \qquad (25)
\end{aligned}
$$

where $\mathrm{sgn}(\cdot)$ is the sign function, the second inequality is due to Lemma 2 and the third inequality is due to condition (H1). So by (25), we have

$$
\begin{aligned}
D^+[y(t)]^+ \le& -A_0[y(t)]^+ + AL[y(t)]^+ + \int_0^{\infty} B \circ K(s) L [y(t-s)]^+ \mathrm{d}s \\
=& P[y(t)]^+ + \int_0^{\infty} Q(s)[y(t-s)]^+ \mathrm{d}s,\ t \ge t_0. \qquad (26)
\end{aligned}
$$

Using the discrete part of (3) and Condition (H2), we have

$$[y(t_k)]^+ \le W_k[y(t_k^-)],\ k = 1, 2, \ldots. \qquad (27)$$

So, conditions (H3), (22), (26), (26) and (27) imply that all the conditions of Theorem 1 are satisfied. Hence we conclude that

$$[y(t)]^+ \le \gamma_1 \cdots \gamma_{k-1} z e^{-\lambda(t-t_0)},\ t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots. \qquad (28)$$

From (28), we have $\gamma_k \le e^{\eta(t_k - t_{k-1})}$, $k = 1, 2, \ldots$, so

$$
\begin{aligned}
\gamma_1 \cdots \gamma_{k-1} \le& e^{\eta(t_1 - t_0)} \cdots e^{\eta(t_{k-1} - t_{k-2})} = e^{\eta(t_{k-1} - t_0)} \\
\le& e^{\eta(t - t_0)},\ t \in [t_{k-1}, t_k), k = 1, 2, \ldots. \qquad (29)
\end{aligned}
$$

So, combining (28) and (29), we derive that

$$[y(t)]^+ \leq ze^{-(\lambda-\eta)(t-t_0)},\ t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, \tag{30}$$

which implies that the conclusions of Theorem 2 hold. The proof is completed. □

Remark 1 If there are no impulses in (1), that is, (1) degenerates to the following form:

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i x_i(t) + \displaystyle\sum_{j=1}^{n} b_{ij}\mu_j + I_i + \\ \quad \displaystyle\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t)) + \bigwedge_{j=1}^{n} \gamma_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(x_j(s))\mathrm{d}s + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \\ \quad \displaystyle\bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t)) + \bigvee_{j=1}^{n} \theta_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(x_j(s))\mathrm{d}s + \bigvee_{j=1}^{n} H_{ij}\mu_j, \\ x_i(t_0) = \phi_i \in R,\ i = 1, \ldots, n, \end{cases} \tag{31}$$

then, by using Theorem 2, we can obtain the global exponential stability of equilibrium point x^* of (31).

Corollary 1 *Assume that hypothesis (H) holds. Furthermore, suppose that*

(H5) For any $x_j,\ y_j \in R$, $j = 1, \ldots, n$, there exist nonnegative constants L_j such that

$$|f_j(x_j) - f_j(y_j)| \leq L_j |x_j - y_j|; \tag{32}$$

(H6) $D = -(P+Q)$ is a nonsingular M-matrix, where $P,\ Q$ are the same as those in condition (H3).

Then (31) has only one equilibrium point x^*, which is globally exponentially stable.

We can similarly prove the existence and uniqueness of the equilibrium point x^* by imitating the proofs of Theorems 1 and 2 in [3]. The global exponential stability of the equilibrium point follows from Theorem 2.

Remark 2 By the definition of nonsingular M-matrix, we know that condition (H6) is equivalent to $\rho(A_0^{-1}(AL+B \circ KL)) < 1$, where $\rho(\cdot)$ denotes the spectral radius. So Corollary 1 is equivalent to Theorem 1 in [6]. Note that we drop the condition that f_j is a bounded function. Hence, Corollary 1 improves Theorem 1 in [6], and Theorem 1 extends Theorem 1 in [6] to impulsive systems.

Remark 3 In general, it is always assumed that there is an equilibrium point for the impulsive systems to study their stability. However, Corollary 1 shows that there is a unique equilibrium point x^* of the continuous part of the system (1) under the conditions (H5) and (H6). In many cases, x^* may not be a solution of the discrete part of the system (1) without the external impulsive input, that is, the entire system (1) may have no equilibrium point. In order to guarantee that the entire system (1) has an equilibrium point, we introduce the external impulsive input h_{ik} so that x^* is also an equilibrium point of the discrete part of the system (1).

4. An illustrative example

In this section, we will give an example to further illustrate the global exponential stability of FCNN (1).

Example Consider system (1) with the following parameters and activation functions ($n = 2$, $i, j = 1, 2$).

$$a_1 = -10,\ a_2 = -8,\ \alpha_{11} = 1,\ \alpha_{12} = 2,\ \alpha_{21} = 3,\ \alpha_{22} = 1,\ \gamma_{ij} = 1,$$

$$\theta_{11} = 1,\ \theta_{12} = 2,\ \theta_{21} = 2,\ \theta_{22} = 3,\ k_{ij}(t) = e^{-(i+j)t},\ b_{ij} = 1,$$

$$T_{ij} = 1,\ H_{ij} = 1,\ \mu_j = 1,\ I_1 = \frac{7}{6},\ I_2 = \frac{25}{6},\ f_j(x_j(t)) = |x_j(t)|.$$

And

$$x_1(t_k) = J_{1k}(x_1(t_k^-), x_2(t_k^-)) + h_{1k}, \quad x_2(t_k) = J_{2k}(x_1(t_k^-), x_2(t_k^-)) + h_{2k}, \tag{33}$$

where $t_0 = 0$, $t_k = t_{k-1} + 0.5k$, for $k = 1, 2, \ldots$.

One can check that all the properties given in (H) are satisfied, provided that $0 < \lambda_0 < 2$. In this example, we may let $\lambda_0 = 1$. And there exist positive constants $L_1 = L_2 = 1$ such that condition (H1) holds. By the given parameters, we have

$$A_0 = \begin{pmatrix} 10 & 0 \\ 0 & 8 \end{pmatrix},\ A = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix},\ B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix},\ L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ K(t) = \begin{pmatrix} e^{-2t} & e^{-3t} \\ e^{-3t} & e^{-4t} \end{pmatrix}.$$

So we obtain

$$P = \begin{pmatrix} -8 & 3 \\ 5 & -5 \end{pmatrix},\ Q(t) = \begin{pmatrix} 2e^{-2t} & 3e^{-3t} \\ 3e^{-3t} & 4e^{-4t} \end{pmatrix}, Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 7 & -4 \\ -6 & 4 \end{pmatrix},$$

where $Q(t) = B \circ K(t)L$, $P = -A_0 + AL$, $Q = \int_0^\infty Q(s)\mathrm{d}s$ and $D = -(P + Q)$. We can easily observe that D is a nonsingular M-matrix.

(I) If $J_{ik}(x_1, x_2) = x_i$ and $h_{ik} = 0$ for $i = 1, 2$ and $k = 1, 2, \ldots$, then system (1) becomes FCNN without impulses. By Corollary 1, system (1) has exactly one globally exponentially stable equilibrium $(1, 2)^{\mathrm{T}}$.

Remark 4 Clearly, the activation functions f_j, $j = 1, 2$, do not satisfy the assumption on bound, so the exponential stability criteria in [6] cannot be applied here. Hence Corollary 1 is less conservative than Theorem 1 proposed by [6].

(II) Next we consider the case where

$$J_{1k} = 0.2e^{0.05k}x_1 - 0.6e^{0.05k}x_2, \quad J_{2k} = -0.8e^{0.05k}x_1 + 0.2e^{0.05k}x_2,$$

$$h_{1k} = 1 + e^{0.05k}, \quad h_{2k} = 2 + 0.4e^{0.05k}.$$

We can verify that point $(1, 2)^{\mathrm{T}}$ is also an equilibrium point of the FCNN (1) with impulses (33), and the parameters of conditions (H2) and (H4) are as follows:

$$W_k = e^{0.05k} \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.2 \end{pmatrix},\ ||W_k|| = e^{0.05k} > 1,\ k = 1, 2, \ldots,$$

$$\Omega_M(D) = \{(z_1, z_2)^{\mathrm{T}} > 0 \,|\, \frac{3}{2}z_1 < z_2 < \frac{7}{4}z_1\}.$$

Let $z = (3,5)^{\mathrm{T}} \in \Omega_M(D)$ and $\lambda = 0.15 < \lambda_0$ which satisfies the inequality

$$(\lambda E + P + B \circ \int_0^{\infty} K(s)e^{\lambda s}\mathrm{d}sL)z = (-0.0436, -0.8973)^{\mathrm{T}} < (0,0)^{\mathrm{T}}.$$

And we can obtain that for $k = 1, 2, \ldots$

$$\gamma_k = e^{0.05k} = \max\{1, e^{0.05k}\}, \quad \frac{\ln \gamma_k}{t_k - t_{k-1}} \le \frac{\ln e^{0.05k}}{0.5k} = 0.1 < \lambda.$$

Clearly, all conditions of Theorem 2 are satisfied, so the equilibrium $(1,2)^{\mathrm{T}}$ is globally exponentially stable and the exponentially convergent rate is equal to 0.05.

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