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# On $(m,n)$-Coherent Modules and Preenvelopes

*SONG Xian-mei*[1], *CHEN Jian-long*[2]
(1. Department of Mathematics, Anhui Normal University, Anhui 241000, China;
2. Department of Mathematics, Southeast University, Jiangsu 210096, China)
(E-mail: sxmsgj@yahoo.com.cn)

**Abstract** In this paper, let $m,n$ be two fixed positive integers and $M$ be a right $R$-module, we define $(m,n)$-$M$-flat modules and $(m,n)$-coherent modules. A right $R$-module $F$ is called $(m,n)$-$M$-flat if every homomorphism from an $(n,m)$-presented right $R$-module into $F$ factors through a module in add$M$. A left $S$-module $M$ is called an $(m,n)$-coherent module if $M_R$ is finitely presented, and for any $(n,m)$-presented right $R$-module $K$, $\mathrm{Hom}(K,M)$ is a finitely generated left $S$-module, where $S=\mathrm{End}(M_R)$. We mainly characterize $(m,n)$-coherent modules in terms of preenvelopes (which are monomorphism or epimorphism) of modules. Some properties of $(m,n)$-coherent rings and coherent rings are obtained as corollaries.

**Keywords** $(m,n)$-$M$-flat module; $(m,n)$-coherent module; $(m,n)$-$M$-flat preenvelope.

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## 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unitary. For a right $R$-module $A$, $E(A)$ denotes the injective envelope of $A$ and $i: A \to E(A)$ denotes the inclusion map. Given a right $R$-module $M$, $M^I$ stands for the direct product of copies of $M$ indexed by $I$, and add$M$ indicates the category consisting of all right $R$-modules isomorphic to direct summands of finitely direct sums of copies of $M$. We simplify $\mathrm{Hom}_R(A,B)$ to $\mathrm{Hom}(A,B)$ for right $R$-modules $A,B$.

Let $\mathcal{C}$ be a class of right $R$-modules and $A$ be a right $R$-module. A homomorphism $\varphi: A \to C$ with $C \in \mathcal{C}$ is called a $\mathcal{C}-$preenvelope of $A$ if for any homomorphism $f: A \to C'$ with $C' \in \mathcal{C}$, there is a homomorphism $g: C \to C'$ such that $g\varphi = f$[1]. Moreover, if the only such $g$ is automorphism of $C$ when $C' = C$ and $f = \varphi$, the $\mathcal{C}$-preenvelope $\varphi$ is called a $\mathcal{C}$-envelope of $A$. Following [2], a $\mathcal{C}$-envelope of $A$ $\varphi: A \to C$ has the unique mapping property if for any homomorphism $f: A \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g: C \to C'$ such that $g\varphi = f$.

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Let $m$ and $n$ be two fixed positive integers. Recently, $(m,n)$-flat modules and $(m,n)$-coherent rings were introduced and studied in [3],[4],[5]. A right $R$-module $K$ is said to be $(m,n)$-presented if there exists an exact sequence of right $R$-modules $0 \to L \to R^m \to K \to 0$, where $L$ is $n$-generated[5]. A right $R$-module $A$ is said to be $(m,n)$-flat if $A \otimes_R I \to A \otimes_R R^m$ is a monomorphism for all $n$-generated submodule $I$ of the left $R$-module $R^m$[5]. Moreover, a ring $R$ is said to be left $(m,n)$-coherent if each $n$-generated submodule of the left $R$-module $R^m$ is finitely presented[5]. In this paper, we introduce the concepts of $(m,n)$-$M$-flat modules and $(m,n)$-coherent modules. Let $M$ be a finitely presented right $R$-module with $S = \mathrm{End}(M_R)$ and $m,n$ fixed positive integers, it is showed that ${}_SM$ is $(m,n)$-coherent if and only if every right $R$-module has an $(m,n)$-$M$-flat preenvelope; ${}_SM$ is $(m,n)$-coherent and injective right $R$-modules are $(m,n)$-$M$-flat if and only if every right $R$-module has an $(m,n)$-$M$-flat preenvelope which is a monomorphism; ${}_SM$ is $(m,n)$-coherent and every submodules of $(m,n)$-$M$-flat right $R$-modules are $(m,n)$-$M$-flat if and only if every right $R$-module has an $(m,n)$-$M$-flat preenvelope which is an epimorphism. In particular, some results of left $(m,n)$-coherent rings and left coherent rings are obtained as corollaries.

## 2. $(m,n)$-Coherent modules and preenvelopes

**Definition 2.1** *Let $M$ be a right $R$-module. A right $R$-module $F$ is called $(m,n)$-$M$-flat if every homomorphism from a $(n,m)$-presented right $R$-module into $F$ factors through a module in* add$M$*, i.e., for any $(n,m)$-presented right $R$-module $K$ and any homomorphism $f: K \to F$, there exist a module $X$ in* add$M$ *and homomorphisms $g: K \to X$, $h: X \to F$ such that $f = hg$.*

**Remark** (1) By [5], a right $R$-module $F$ is $(m,n)$-flat if and only if for every homomorphism from $(n,m)$-presented right $R$-module into $F$ factors through a free module. Hence $(m,n)$-$R$-flat modules are just $(m,n)$-flat right $R$-modules.

(2) It is easy to see that $F$ is an $(m,n)$-$M$-flat right $R$-module if and only if for any $(n,m)$-presented right $R$-module $K$, any homomorphism $f: K \to F$, there exist a positive integer $s$ and homomorphisms $g: K \to M^s$, $h: M^s \to F$ such taht $f = hg$.

(3) By definition, the class of $(m,n)$-$M$-flat right $R$-modules is closed under direct summands, finitely direct sums.

(4) If $X$ is in add$M$, then $X$ is $(m,n)$-$M$-flat; if $X$ is $(n,m)$-presented, and $X$ is $(m,n)$-$M$-flat, then $X$ is in add$M$.

(5) If $M$ is a projective right $R$-module, $F$ is an $(m,n)$-$M$-flat right $R$-module, then $F$ is $(m,n)$-flat.

**Definition 2.2** *For a right $R$-module $M$, $S$ denotes the ring $\mathrm{End}(M_R)$. ${}_SM$ is called an $(m,n)$-coherent module if $M_R$ is finitely presented, and for any $(n,m)$-presented right $R$-module $K$, $\mathrm{Hom}(K,M)$ is a finitely generated left $S$-module.*

Let $M$ be a right $R$-module, $S = \mathrm{End}(M_R)$ and $A, B$ be right $R$-modules. We use $\sigma_{A,B}$ denote the homomorphism $\mathrm{Hom}(M,A) \otimes_S \mathrm{Hom}(B,M) \to \mathrm{Hom}(B,A)$ given by $\sigma_{A,B}(f \otimes g) = fg$

where $f \in \mathrm{Hom}(M,A)$, $g \in \mathrm{Hom}(B,M)$. It is easy to see that if $C$ is a direct summand of $B$, and $\sigma_{A,B}$ is an isomorphism, then $\sigma_{A,C}$ is also an isomorphism. Hence if $B \in \mathrm{add}M$, then $\sigma_{A,B}$ is an isomorphism. Moreover, it is clear that if $A \in \mathrm{add}M$, then $\sigma_{A,B}$ is also an isomorphism.

**Proposition 2.1** *Let $M$ be a right $R$-module. The following statements are equivalent:*

*(1) $F$ is an $(m,n)$-$M$-flat right $R$-module;*

*(2) For any $(n,m)$-presented right $R$-module $K$, $\sigma_{F,K}$ is an epimorphism.*

**Proof** (1) $\Rightarrow$ (2). Let $K$ be an $(n,m)$-presented right $R$-module. For any $f \in \mathrm{Hom}(K,F)$, by the definition of $(m,n)$-$M$-flatness, there exist a positive integer $k$ and homomorphisms $g : K \to M^k$ and $h : M^k \to F$ such that $f = hg$. Let $p_i : M^k \to M$ and $\lambda_i : M \to M^k$ denote the ith canonical projection and canonical injection respectively. Put $g_i = p_i g$, $h_i = h\lambda_i$, then $\sum_{i=1}^{k}(h_i \otimes g_i) \in \mathrm{Hom}(M,F) \otimes_S \mathrm{Hom}(K,M)$ and $\sigma_{F,K}(\sum_{i=1}^{k}(h_i \otimes g_i)) = \sum_{i=1}^{k} h_i g_i = \sum_{i=1}^{k} h\lambda_i p_i g = f$. Hence $\sigma_{F,K}$ is an epimorphism.

(2) $\Rightarrow$ (1). Let $K$ be an $(n,m)$-presented right $R$-module, and $f \in \mathrm{Hom}(K,F)$. By hypothesis, $f = \sigma_{F,K}(\sum_{i=1}^{k}(h_i \otimes g_i))$ for some $h_i \in \mathrm{Hom}(M,F)$, $g_i \in \mathrm{Hom}(K,M)$. Put $X = M^k$. We define $g : K \to M^k$ by $g(x) = (g_1(x), \ldots, g_k(x))$ for every $x \in K$ and $h : M^k \to F$ by $h(m_1,\ldots,m_k) = \sum_{i=1}^{k} h_i(m_i)$ for every $(m_1,\ldots,m_k) \in M^k$. Then $f = hg$ and so $F$ is $(m,n)$-$M$-flat.

**Proposition 2.2** *Let $M$ be a pure projective right $R$-module. Then every pure submodule of $(m,n)$-$M$-flat right $R$-module is $(m,n)$-$M$-flat.*

**Proof** Let $A$ be a pure submodule of $(m,n)$-$M$-flat module $B$, and $K$ be an $(n,m)$-presented right $R$-module. Let $j : A \to B$ denote the inclusion map, and $\pi : B \to B/A$ denote a canonical epimorphism. For any $f \in \mathrm{Hom}(K,A)$, then there exist a module $X$ in $\mathrm{add}M$ and homomorphisms $g : K \to X$, $h : X \to B$ such that $jf = hg$. Note that there is a pure projective right $R$-module $Y$ such that $\alpha : Y \to B$ is a pure epimorphism by [6]. Since $M$ is pure projective, so $X$ is pure projective, and there is a homomorphism $\beta : X \to Y$ such that $\alpha\beta = h$. Put a pullback of $\alpha : Y \to B$ and $j : A \to B$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & K & \xrightarrow{g} & X & & & & \\
 & & & & \downarrow{\scriptstyle\beta} & & & & \\
0 & \longrightarrow & U & \xrightarrow{\lambda} & Y & \xrightarrow{\pi\alpha} & B/A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha} & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{\pi} & B/A & \longrightarrow & 0
\end{array}$$

Note that $\pi\alpha\beta g = \pi h g = \pi j f = 0$, it follows that $\beta g(K) \subseteq \mathrm{Ker}(\pi\alpha) = \mathrm{Im}\lambda$. But $\lambda$ is a monomorphism, hence there is a submodule $V$ of $U$ such that for any $y \in K$, there is unique $x \in V$ satisfying $\beta g(y) = \lambda(x)$. It is easy to see that $\lambda$ is a pure monomorphism since $j$ is a pure monomorphism and $\alpha$ is a pure epimorphism. Thus since $K$ is finitely generated, $V$ is finitely generated. By [7], there exists a homomorphism $\gamma : Y \to U$ such that $\gamma\lambda(x) = x$ for any $x \in V$.

Let $k=\delta\gamma\beta : X\to A$. Then foy any $y\in K$,

$$kg(y)=\delta\gamma\beta g(y)=\delta\gamma\lambda(x)=\delta(x)=\alpha\lambda(x)=hg(y)=f(y).$$

Therefore $A$ is $(m,n)$-$M$-flat.

**Lemma 2.3**[8] *Let $M,A$ be right $R$-modules and $S=\mathrm{End}(M_R)$. Then $A$ has an $\mathrm{add}M$-preenvelope if and only if $\mathrm{Hom}(A,M)$ is a finitely generated left $S$-module.*

**Proposition 2.4** *Let $M$ be a right $R$-module and $S=\mathrm{End}(M_R)$. The following statements are equivalent:*

*(1) ${}_SM$ is an $(m,n)$-coherent module;*

*(2) $M_R$ is finitely presented and every $(n,m)$-presented right $R$-module has an $\mathrm{add}M$-preenvelope;*

*(3) $M_R$ is finitely presented and every $(n,m)$-presented right $R$-module has an $(m,n)$-$M$-flat preenvelope.*

**Proof** By Lemma 2.3, (1) $\Leftrightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). For any $(n,m)$-presented right $R$-module $K$, it has an $\mathrm{add}M$-preenvelope $f:K\to X$ with $X\in\mathrm{add}M$. For any $(m,n)$-$M$-flat right $R$-module $F$ and any homomorphism $g:K\to F$, there exist $X_1\in\mathrm{add}M$ and homomorphisms $g_1:K\to X_1$, $g_2:X_1\to F$ such that $g=g_2g_1$. Note that $f$ is an $\mathrm{add}M$-preenvelope, it follows that there is a homomorphism $h:X\to X_1$ such that $hf=g_1$. Hence $g_2hf=g_2g_1=g$, that is, $f$ is an $(m,n)$-$M$-flat preenvelope.

(3) $\Rightarrow$ (2). For any $(n,m)$-presented right $R$-module $K$, it has an $(m,n)$-$M$-flat preenvelope $f:K\to F$. Hence there exist a right $R$-module $X\in\mathrm{add}M$ and homomorphisms $g:K\to X$, $h:X\to F$ such that $f=hg$. It is easy to see that $g:K\to X$ is an $\mathrm{add}M$-preenvelope of $K$.

**Theorem 2.5** *Let $M$ be a finitely presented right $R$-module and $S=\mathrm{End}(M_R)$. The following statements are equivalent:*

*(1) ${}_SM$ is an $(m,n)$-coherent module;*

*(2) Every right $R$-module has an $(m,n)$-$M$-flat preenvelope;*

*(3) Every $(n,m)$-presented right $R$-module has an $\mathrm{add}M$-preenvelope;*

*(4) Every $(n,m)$-presented right $R$-module has an $(m,n)$-$M$-flat preenvelope;*

*(5) $\prod_{i\in I}M$ is an $(m,n)$-$M$-flat right $R$-module for any index set $I$;*

*(6) The direct products of $(m,n)$-$M$-flat right $R$-modules is $(m,n)$-$M$-flat;*

*(7) $\mathrm{Hom}_S(P,M)$ is an $(m,n)$-$M$-flat right $R$-module for any projective left $S$-module $P$.*

**Proof** (2) $\Rightarrow$ (1), (6) $\Rightarrow$ (5) are trivial and by Proposition 2.4, (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (6). Let $\{F_i\}_{i\in I}$ be a family of $(m,n)$-$M$-flat right $R$-modules and $K$ be an $(n,m)$-presented right $R$-module. We denote by $p_i:\prod_{i\in I}F_i\to F_i$ the ith canonical projection. For any $f\in\mathrm{Hom}(K,\prod_{i\in I}F_i)$, $p_if$ factors through $X_i$ with $X_i\in\mathrm{add}M$, that is, there are homomorphisms $g_i:K\to X_i$ and $h_i:X_i\to F_i$ such that $p_if=h_ig_i$. Note that ${}_SM$ is $(m,n)$-coherent, it follows that $K$ has an $\mathrm{add}M$-preenvelope $g:K\to X$. Hence there is a homomorphism

$k_i : X \to X_i$ such that $k_i g = g_i$. By the universal property of direct products, there is a homomorphism $h : X \to \prod_{i\in I} F_i$ such that $p_i h = h_i k_i$, then $p_i h g = h_i k_i g = h_i g_i = p_i f$. Therefore $hg = f$.

(6) $\Rightarrow$ (2). Let $A$ be a right $R$-module with Card$A \le \aleph$. For any $(m,n)$-$M$-flat module $F$ and any homomorphism $f : A \to F$, Card$f(A) \le \aleph$. Let $\aleph_\alpha = \max\{\text{Card}R, \aleph\}$. Then by [9], there is a pure submodule $S$ of $F$ such that $f(A) \subseteq S$ and Card$S \le \aleph_\alpha$. By Proposition 2.2, $S$ is $(m,n)$-$M$-flat. Let $(\varphi_i)_{i\in I}$ give all such homomorphisms $\varphi_i : A \to S_i$ with Card$S_i \le \aleph_\alpha$. So any homomorphism $A \to F$ has a factorization $A \to S_j \to F$ for some $j \in I$. Hence $A \to \prod_{i\in I} S_i$ is an $(m,n)$-$M$-flat preenvelope since $\prod_{i\in I} S_i$ is $(m,n)$-$M$-flat by (6).

(5) $\Rightarrow$ (1). We need to prove for any $(n,m)$-presented right $R$-module $A$, Hom$(A,M)$ is a finitely generated left $S$-module. Let $p_i : M^I \to M$ denote the ith canonical projection. By Lemma 3.2.21 in [9], this is equivalent to prove that for any index set $I$, $\tau : \text{Hom}(M, M^I) \otimes_S \text{Hom}(A,M) \to \text{Hom}(A,M)^I$ given by $\tau(g\otimes f) = (p_i g f)$ is an epimorphism, where $f \in \text{Hom}(A,M)$, $g \in \text{Hom}(M, M^I)$. Note that $\sigma_{M^I,A} : \text{Hom}(M, M^I) \otimes \text{Hom}(A,M) \to \text{Hom}(A, M^I)$ is an epimorphism since $M^I$ is $(m,n)$-$M$-flat, $\theta : \text{Hom}(A, M^I) \to \text{Hom}(A,M)^I$ given by $\theta(f) = (p_i f)_{i\in I}$ is an isomorphism and $\tau = \theta\sigma_{M^I,A}$, it follows that $\tau$ is an epimorphism. Therefore ${}_SM$ is $(m,n)$-coherent.

(5) $\Rightarrow$ (7). Since $P$ is a projective left $S$-module, Hom$(P,M)$ is isomorphic to a direct summand of $M^I$ for some index set $I$. By hypothesis, $M^I$ is $(m,n)$-$M$-flat, therefore Hom$(P,M)$ is $(m,n)$-$M$-flat.

(7) $\Rightarrow$ (5). Put ${}_SP = S^{(I)}$ for any index set $I$.

In Ref. [5], it was proved that $R$ is a left $(m,n)$-coherent ring if and only if $R^I$ is an $(m,n)$-flat right $R$-module for any index set $I$. Hence by Theorem 2.5, when $M_R = R_R$, it is easy to see that ${}_RR$ is $(m,n)$-coherent if and only if $R$ is left $(m,n)$-coherent ring. So the equivalence (1)–(3) and (6)–(9) of Theorem 3.1 in Ref. [3] is just Corollary 2.6.

**Corollary 2.6** *The following statements are equivalent:*

*(1) $R$ is a left $(m,n)$-coherent ring;*

*(2) Every right $R$-module has an $(m,n)$-flat preenvelope;*

*(3) Every $(n,m)$-presented right $R$-module has a finitely generated projective preenvelope;*

*(4) Every $(n,m)$-presented right $R$-module has an $(m,n)$-flat preenvelope;*

*(5) $\prod_{i\in I} R$ is $(m,n)$-flat right $R$-module for any index set $I$;*

*(6) The direct products of $(m,n)$-flat right $R$-modules is $(m,n)$-flat;*

*(7) Hom$(P,R)$ is an $(m,n)$-flat right $R$-module for any projective left $R$-module.*

Let $M$ be a right $R$-module $S = \text{End}(M_R)$. We denote Hom$(A,M)$ by $A^*$, where $A^*$ is a left $S$-module. Put $A^{**} = \text{Hom}_S(A^*, M)$. Define $\delta_A : A \to A^{**}$ by $\delta_A(a)(f) = f(a)$ for any $a \in A$ and $f \in \text{Hom}(A,M)$. It is clear that if $A \in \text{add}M$, then $\delta_A$ is an isomorphism.

**Proposition 2.7** *Let ${}_SM$ be $(m,n)$-coherent and $K$ be an $(n,m)$-presented right $R$-module. Then $K$ has an* add$M$*-envelope if and only if $K^*$ has a projective cover.*

**Proof** Let $f : K \to X$ be an add$M$-envelope of $K$. Then $f^* : X^* \to K^*$ is an epimorphism and $X^*$ is a finitely generated projective left $S$-module. For any $h \in \mathrm{Hom}(X^*, X^*)$ such that $f^*h = f^*$, since $f^{**}\delta_K = \delta_X f$, it follows that $\delta_X^{-1}h^*\delta_X f = \delta_X^{-1}h^*f^{**}\delta_K = \delta_X^{-1}(f^*h)^*\delta_K = \delta_X^{-1}f^{**}\delta_K = f$. Note that $f$ is an add$M$-envelope of $K$, hence $\delta_X^{-1}h^*\delta_X$ is an isomorphism, and so $h = \delta_{X^*}^{-1}h^{**}\delta_{X^*}$ is also an isomorphism. This proves that $f^*$ is a projective cover of $K^*$. Conversely, let $h : P \to K^*$ be a projective cover of $K^*$. Note that ${}_SM$ is $(m, n)$-coherent, $K^*$ is a finitely generated left $S$-module and so $P$ is finitely generated. It is clear that $P^* \in \mathrm{add}M$. Let $f = h^*\delta_K : K \to P^*$. For any $\alpha : K \to X$ with $X \in \mathrm{add}M$, $X^*$ is a projective left $S$-module, hence there is a homomorphism $\beta : X^* \to P$ such that $h\beta = \alpha^*$. Put $\gamma = \delta_X^{-1}\beta^* : P^* \to X$, then we have $\gamma f = \delta_X^{-1}\beta^*f = \delta_X^{-1}\beta^*h^*\delta_K = \delta_X^{-1}(h\beta)^*\delta_K = \delta_X^{-1}\alpha^{**}\delta_K = \alpha$. Therefore $f$ is an add$M$-preenvelope of $K$. If $\varphi : P^* \to P^*$ satisfies $\varphi f = f$. Note that $h^{**}\delta_P = \delta_{K^*}h$ and $\delta_K^*\delta_{K^*} = 1_{K^*}$, it follows that $h = \delta_K^*h^{**}\delta_P$. Hence we have $h(\delta_P^{-1}\varphi^*\delta_P) = \delta_K^*h^{**}\varphi^*\delta_P = (\varphi h^*\delta_K)^*\delta_P = (\varphi f)^*\delta_P = f^*\delta_P = \delta_K^*h^{**}\delta_P = h$. Since $h$ is a projective cover of $K^*$, $\delta_P^{-1}\varphi^*\delta_P$ is an isomorphism. And so $\varphi = \delta_{P^*}^{-1}\varphi^{**}\delta_{P^*}$ is also an isomorphism. This proves that $f$ is an add$M$-envelope.

**Corollary 2.8** *Let $R$ be a left $(m, n)$-coherent ring and $K$ be an $(n, m)$-presented right $R$-module. Then $K$ has a projective envelope if and only if* $\mathrm{Hom}(K, R)$ *has a projective cover.*

## 3. Preenvelopes which are monomorphism

In this section, we mainly investigate when every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism.

**Theorem 3.1** *Let $M$ be a finitely presented right $R$-module, $S = \mathrm{End}(M_R)$.*

*(1) ${}_SM$ is $(m, n)$-coherent, every injective right $R$-module is $(m, n)$-$M$-flat.*

*(2) ${}_SM$ is $(m, n)$-coherent, injective envelope of every $(n, m)$-presented right $R$-module is $(m, n)$-$M$-flat.*

*(3) ${}_SM$ is $(m, n)$-coherent, injective envelope of every simple right $R$-module is $(m, n)$-$M$-flat.*

*(4) ${}_SM$ is $(m, n)$-coherent, every $(n, m)$-presented right $R$-module is cogenerated by $M_R$.*

*(5) ${}_SM$ is $(m, n)$-coherent, every $(n, m)$-presented right $R$-module may be embedded in a module in* add$M$.

*(6) Every right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism.*

*(7) Every $(n, m)$-presented right $R$-module has an $(m, n)$-$M$-flat preenvelope which is a monomorphism.*

*(8) Every $(n, m)$-presented right $R$-module has an* add$M$*-preenvelope which is a monomorphism.*

**Proof** $(1) \Rightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (6), (6) \Rightarrow (7), (8) \Rightarrow (5)$ are trivial.

$(2) \Rightarrow (1)$. Let $E$ be an injective right $R$-module and $K$ be an $(n, m)$-presented right $R$-module. For any homomorphism $f : K \to E$, by hypothesis of (2), $E(K)$ is an $(m, n)$-$M$-flat. Hence there are a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$, $h : X \to E(K)$

such that $i = hg$. Note that $E$ is injective, there exists a homomorphism $k : E(K) \to E$ such that $f = ki = khg$. Therefore $E$ is $(m,n)$-$M$-flat.

$(3) \Rightarrow (4)$. Let $K$ be an $(n,m)$-presented right $R$-module. By [10], we need to prove that for any nonzero $x \in K$, there is a homomorphism $\varphi : K \to M$ such that $\varphi(x) \neq 0$. In fact, there exists a maximal submodule $L$ of $xR$, then $E(xR/L)$ is $(m,n)$-$M$-flat. We let $\lambda : xR \to K$ denote an inclusion map and $\pi : xR \to xR/L$ be the canonical epimorphism. By the injectivity of $E(xR/L)$, there exists a homomorphism $f : K \to E(xR/L)$ such that $f\lambda = i\pi$. Note that $E(xR/L)$ is $(m,n) - M$-flat, it follows that there are a positive integer $n$ and homomorphisms $g : K \to M^n$, $h : M^n \to E(xR/L)$ such that $hg = f$. Since $x \neq 0$, we have $i\pi(x) \neq 0$, thus $g(x) \neq 0$. Suppose $g(x) = (m_1, \ldots, m_i, \ldots, m_n)$. Then there is a nonzero element $m_i \in M$ for some $1 \leq i \leq n$. Hence $p_i g(x) \neq 0$. Therefore (4) holds.

$(4) \Rightarrow (5)$. Let $K$ be an $(n,m)$-presented right $R$-module. By hypothesis, there is a monomorphism $\lambda : K \to M^I$ for some index set $I$. Note that ${}_SM$ is $(m,n)$-coherent, hence $M^I$ is $(m,n)$-$M$-flat. It follows that there are a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$, $h : X \to M^I$ such that $\lambda = hg$. Since $\lambda$ is monomorphism, $g$ is a monomorphism, i.e., (5) holds.

$(5) \Rightarrow (1)$. Let $E$ be an injective right $R$-module, $K$ an $(n,m)$-presented right $R$-module and $f : K \to E$ a homomorphism. By hypothesis, there is a monomorphism $\lambda : K \to X$ with $X \in$ add$M$. Note that $E$ is injective, there is a homomorphism $g : X \to E$ such that $f = g\lambda$. Hence $E$ is $(m,n)$-$M$-flat.

$(7) \Rightarrow (8)$. For any $(n,m)$-presented right $R$-module $K$, by hypothesis, there is an $(m,n)$-$M$-flat preenvelope $f : K \to F$ which is a monomorphism. Then there exist a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$, $h : X \to F$ such that $f = hg$. It is easy to see that $g$ is an add$M$-preenvelope which is a monomorphism. Therefore (8) holds.

**Corollary 3.2** *The following statements are equivalent:*

*(1) $R$ is a left $(m,n)$-coherent ring, every injective right $R$-module is $(m,n)$-flat;*

*(2) $R$ is a left $(m,n)$-coherent ring, injective envelope of every $(n,m)$-presented right $R$-module is $(m,n)$-flat;*

*(3) $R$ is a left $(m,n)$-coherent ring, injective envelope of every simple right $R$-module is $(m,n)$-flat;*

*(4) $R$ is a left $(m,n)$-coherent ring, every $(n,m)$-presented right $R$-module is torsionless;*

*(5) $R$ is a left $(m,n)$-coherent ring, every $(n,m)$-presented right $R$-module may be embedded in a finitely generated projective right $R$-module;*

*(6) Every right $R$-module has an $(m,n)$-flat preenvelope which is monomorphism;*

*(7) Every $(n,m)$-presented right $R$-module has an $(m,n)$-flat preenvelope which is monomorphism;*

*(8) Every $(n,m)$-presented right $R$-module has a finitely generated projective preenvelope which is monomorphism.*

**Proposition 3.3** *Let $M$ be an injective right $R$-module. The following statements are equivalent:*

*(1) Every $(n,m)$-presented right $R$-module has an $(m,n)$-$M$-flat envelope which is a monomorphism;*

*(2) Injective envelope of every $(n,m)$-presented right $R$-module is in* add$M$*;*

*(3) Every injective right $R$-module is $(m,n)$-$M$-flat;*

*(4) For any $(n,m)$-presented right $R$-module $K$, $(m,n)$-$M$-flat envelope of $K$ exists, and coincides with its injective envelope.*

**Proof** (1) $\Rightarrow$ (2). For any $(n,m)$-presented right $R$-module $K$, by hypothesis, $K$ has an $(m,n)$-$M$-flat envelope $f : K \to F$ which is a monomorphism. Hence there are a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$, $h : X \to F$ such that $f = hg$. Note that $g$ is a monomorphism since $f$ is a monomorphism and $X$ is injective since $M$ is injective, it follows that there are $\alpha : X \to E(K)$ and $\beta : E(K) \to X$ such that $\alpha g = i, \beta i = g$. Then $\alpha\beta = 1$ since $E(K)$ is an injective envelope, i.e., $E(K)$ is isomorphic to a direct summand of $X$. Therefore $E(K)$ is in add$M$.

(2) $\Rightarrow$ (3). Let $E$ be an injective right $R$-module and $K$ be an $(n,m)$-presented right $R$-module. By hypothesis, $E(K)$ is in add$M$. For any homomorphism $f : K \to E$, there is a homomorphism $g : E(K) \to E$ such that $f = gi$. This shows that $f$ factors through a right $R$-module in add$M$ and hence $E$ is $(m,n)$-$M$-flat.

(3) $\Rightarrow$ (4). Let $K$ be an $(n,m)$-presented right $R$-module. By hypothesis, $E(K)$ is $(m,n)$-$M$-flat. For any $(m,n)$-$M$-flat right $R$-module $F$, and any homomorphism $f : K \to F$, there are a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$ and $h : X \to F$ such that $f = hg$. Note that $X$ is injective since $M$ is injective, it follows that there is a homomorphism $k : E(K) \to X$ with $ki = g$. Hence $hki = hg = f$. Therefore $i$ is an $(m,n)$-$M$-flat preenvelope. But $E(k)$ is an injective envelope of $K$, hence $i$ is an $(m,n)$-$M$-flat envelope.

(4) $\Rightarrow$ (1) is trivial.

## 4. Preenvelopes which are epimorphisms

In this section, we consider when every right $R$-module has an $(m,n)$-$M$-flat preenvelope which is an epimorphism.

**Theorem 4.1** *Let $M$ be a finitely presented right $R$-module and $S = \mathrm{End}(M_R)$. The following statements are equivalent:*

*(1) ${}_SM$ is $(m,n)$-coherent, and a submodule of $(m,n)$-$M$-flat right $R$-module is $(m,n)$-$M$-flat;*

*(2) Every $(n,m)$-presented right $R$-module has an $(m,n)$-$M$-flat preenvelope which is an epimorphism;*

*(3) Every right $R$-module has an $(m,n)$-$M$-flat preenvelope which is an epimorphism;*

*(4) Every right $R$-module has an $(m,n)$-$M$-flat envelope which is an epimorphism;*

*(5) Every $(n,m)$-presented right $R$-module has an* add$M$*-preenvelope which is an epimorphism.*

**Proof** (1) $\Rightarrow$ (3). By Theorem 2.5, every right $R$-module $A$ has an $(m,n)$-$M$-flat preenvelope

$f : A \to F$. Let $F_1 = \mathrm{Im}f$. Then $F_1$ is $(m,n)-M$-flat by (1). Hence $f : M \to F_1$ is an $(m,n)$-$M$-flat preenvelope which is an epimorphism.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (5). Let $K$ be an $(n,m)$-presented right $R$-module. Then $K$ has an $(m,n)$-$M$-flat preenvelope $f : K \to F$ which is an epimorphism. Since $F$ is $(m,n)$-$M$-flat, there exist a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$ and $h : X \to F$ such that $f = hg$. But $X$ is $(m,n)$-$M$-flat, hence there exists $k : F \to X$ with $kf = g$. Thus $f = hkf$, and so $hk = 1_F$ since $f$ is epimorphism. Therefore $F \in \mathrm{add}M$ and (5) holds.

(5) $\Rightarrow$ (1). By Proposition 2.4, ${}_SM$ is $(m,n)$-coherent. Let $N$ be $(m,n)$-$M$-flat and $N_1$ a submodule of $N$. For any $(n,m)$-presented right $R$-module $K$, and any homomorphism $f : K \to N_1$, let $\lambda : N_1 \to N$ be the inclusion map. Then there exist a right $R$-module $X$ in add$M$ and homomorphisms $g : K \to X$ and $h : X \to N$ such that $\lambda f = hg$. Note that $K$ has an add$M$-preenvelope $k : K \to Y$ which is an epimorphism, then there is a homomorphism $\alpha : Y \to X$ with $\alpha k = g$. Thus $\lambda f = h\alpha k$, whence $\mathrm{Ker}k \subseteq \mathrm{Ker}f$. Define $\beta : Y \to N_1$ by $\beta(k(x)) = f(x)$ for any $x \in K$. It is clear that $\beta$ is well-defined and $f = \beta k$. Therefore $N_1$ is $(m,n)$-$M$-flat and (1) holds.

(1) $\Leftrightarrow$ (4). By Remark in Section 2, the class of $(m,n)$-$M$-flat right $R$-modules is closed under direct summands. Then from Theorem 2 in Ref. [11], it follows that every right $R$-module has an $(m,n)$-$M$-flat envelope if and only if the class of $(m,n)$-$M$-flat right $R$-modules is closed under direct products and submodules. Note that $M_R$ is a a finitely presented right $R$-module, hence the class of $(m,n)$-$M$-flat right $R$-modules is closed under direct products if and only if ${}_SM$ is an $(m,n)$-coherent module by Theorem 2.5. Therefore (1) $\Leftrightarrow$ (4).

**Corollary 4.2** *The following statements are equivalent:*

*(1) $R$ is a left $(m,n)$-coherent ring, and a submodule of $(m,n)$-flat right $R$-module is $(m,n)$-flat;*

*(2) Every $(n,m)$-presented right $R$-module has an $(m,n)$-flat preenvelope which is an epimorphism;*

*(3) Every right $R$-module has an $(m,n)$-flat preenvelope which is an epimorphism;*

*(4) Every right $R$-module has an $(m,n)$-flat envelope which is an epimorphism;*

*(5) Every $(n,m)$-presented right $R$-module has a finitely generated projective preenvelope which is an epimorphism.*

**Proposition 4.3** *Let $M_R$ be a finitely presented right $R$-module with $S = \mathrm{End}(M_R)$. The following statements are equivalent:*

*(1) ${}_SM$ is $(m,n)$-coherent, for any $(n,m)$-presented right $R$-module $K$, $K^*$ is a projective left $S$-module and $\delta_K$ is an epimorphism;*

*(2) Every $(n,m)$-presented right $R$-module has an* add$M$*-envelope which is an epimorphism;*

*(3) Every $(n,m)$-presented right $R$-module has an* add$M$*-envelope with the unique mapping property and $\delta_K$ is an epimorphism.*

**Proof** (1) $\Rightarrow$ (2). By (1), $K^*$ is a finitely generated projective left $S$-module, hence $K^{**} \in \text{add}M$. Next we prove that $\delta_K : K \to K^{**}$ is an add$M$-envelope of $K$. For any $X \in \text{add}M$, and any homomorphism $f : K \to X$, then $\delta_X$ is an isomorphism and $\delta_X f = f^{**}\delta_K$. Hence $f = \delta_X^{-1} f^{**}\delta_K$. This proves that $\delta_K$ is an add$M$-preenvelope. If $\alpha \in \text{Hom}(K^{**}, K^{**})$ with $\alpha\delta_K = \delta_K$, then $\delta_K^*\alpha^* = \delta_K^*$. Note that $K^*$ is finitely generated projective, $\delta_{K^*}$ is an isomorphism, and so $\delta_K^*$ is an isomorphism since $\delta_K^*\delta_{K^*} = 1_{K^*}$. It follows that $\delta_{K^*}^{-1}\alpha^*\delta_{K^*} = 1_{K^*}$, whence $\alpha = \delta_{K^{**}}^{-1}\alpha^{**}\delta_{K^{**}}$ is an isomorphism. Therefore $K$ has an add$M$-envelope $\delta_K$ which is an epimorphism and (2) holds.

(2) $\Rightarrow$ (3). For any $(n, m)$-presented right $R$-module $K$, there is an add$M$-envelope $f : K \to X$ which is an epimorphism. Hence we have the following exact sequence $0 \to \text{Hom}(X, L) \to \text{Hom}(K, L) \to 0$ for any $L \in \text{add}M$. Therefore $f$ has the unique mapping property. Put $L = M$. Then $f^*$ is an isomorphism, and so $f^{**}$ is also isomorphism. Since $f^{**}\delta_K = \delta_X f$, it is clear that $\delta_K$ is an epimorphism.

(3) $\Rightarrow$ (1). For any $(n, m)$-presented right $R$-module $K$, by (3), there is an add$M$-preenvelope $f : K \to X$ and $X^* \cong K^*$. Hence $K^*$ is a finitely generated projective left $S$-module. Therefore $_SM$ is $(m, n)$-coherent, $K^*$ is a projective left $S$-module, and (1) holds.

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