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A Construction of Orthogroups with Inverse Transversals

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Abstract In this paper, we construct orthogroups with inverse transversals by means of bands with semilattice transversals and Clifford semigroups.

Keywords orthogroup; inverse transversal; band; Clifford semigroup.

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1. Introduction and main result

Recall that an orthogroup is a completely regular and orthodox semigroup; and a subsemigroup S° of a regular semigroup S is called an inverse transversal of S[1], if for any $x \in S$, $|\, V(x) \cap S^\circ| = 1$, where $V(x) = \{x' \in S | xx'x = x, x'xx' = x'\}$.

Orthogroups and orthodox semigroups (with inverse transversals) have been studied by many authors[1,3−7]. In this paper, we discuss the structure of orthogroups with inverse transversals. By virtue of inverse transversals, we give a simpler construction of them, which is different from those in Refs. [4], [6], [7]. It is helpful to understand the global structure of orthogroups.

A Clifford semigroup G can be written in the form $G = [Y; G_\alpha]$, where Y is a semilattice and G_α is a group[5, Theorem IV 2.4]. For $\alpha \in Y$, denote $G_{\geq\alpha} = \bigcup_{\beta\in Y, \beta\geq\alpha} G_\beta$. It is easy to prove that $G_{\geq\alpha}$ is a Clifford subsemigroup of G.

Let $B = (Y; B_\alpha)$ be a band with a semilattice transversal $\widetilde{Y} = \{\widetilde{\alpha} \in B_\alpha |\ \alpha \in Y\}$, where Y is a semilattice and B_α is a rectangular band. Denote $\widetilde{\alpha}B\widetilde{\alpha} = \{\widetilde{\alpha}b\widetilde{\alpha} \in B \mid b \in B\}$, obviously, it is a subband of B. For $\widetilde{\alpha} \in \widetilde{Y}$, an automorphism ξ of $\widetilde{\alpha}B\widetilde{\alpha}$ is called semilattice transversal-preserving automorphism, if $(\widetilde{\beta})\xi = \widetilde{\beta}$, for any $\beta \in Y$ with $\widetilde{\beta} \leq \widetilde{\alpha}$. Denote by $\mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha})$ the group of all semilattice transversal-preserving automorphisms of $\widetilde{\alpha}B\widetilde{\alpha}$.

Let B_α be a rectangular band and G_α be a group. Denote by $B_\alpha \times G_\alpha$ a rectangular group with the multiplication: $(a, g)(b, h) = (ab, gh)$, for $(a, g), (b, h) \in B_\alpha \times G_\alpha$.

Our main result is

Theorem 1.1 *Let $B = (Y; B_\alpha)$ be a band with a semilattice transversal $\widetilde{Y} = \{\widetilde{\alpha} \in B_\alpha | \alpha \in Y\}$,*

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where Y is a semilattice. Let $G=[Y;G_\alpha]$ be a Clifford semigroup. For each $\alpha\in Y$, let

$$\sigma_\alpha : G_{\geq\alpha} \to \mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha});\ h\mapsto \sigma_\alpha(h)$$

be a homomorphism, and denote $b^h=(b)\sigma_\alpha(h)$ for $b\in\widetilde{\alpha}B\widetilde{\alpha}$.

On $S=\bigcup_{\alpha\in Y}(B_\alpha\times G_\alpha)$ define a multiplication by the following: for $x=(a,g)\in B_\alpha\times G_\alpha$, $y=(b,h)\in B_\beta\times G_\beta$, $\alpha,\beta\in Y$,

$$x*y=(a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h b, gh).$$

Then S is an orthogroup with an inverse transversal. Conversely, every orthogroup with inverse transversals is isomorphic to one so constructed.

All other terminologies and notations which are not explained can be found in Refs. [1], [5].

2. The proof of Theorem 1.1

The following lemma will be used for many times.

Lemma 2.1[2] *Let $B=(Y;B_\alpha)$ be a band and $\alpha,\beta\in Y$ with $\alpha\geq\beta$. If $a\in B_\alpha$, $b,c\in B_\beta$, then $bac=bc$.*

I. Proof of the direct part of Theorem 1.1.

We use the notations of Section 1 and assume that the conditions of the statement of Theorem 1.1 are satisfied.

Firstly, let $\beta,\alpha,\gamma\in Y$ with $\beta\geq\alpha\geq\gamma$, $b\in\widetilde{\alpha}B_\gamma\widetilde{\alpha}, h\in G_\beta$. Since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\gamma}\leq\widetilde{\alpha}$. Moreover, since $\sigma_\alpha(h)\in\mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha})$ and $b=b\widetilde{\gamma}b$, it follows that $b^h=(b\widetilde{\gamma}b)^h=b^h\widetilde{\gamma}^h b^h=b^h\widetilde{\gamma}b^h$; similarly, we can get that $\widetilde{\gamma}=\widetilde{\gamma}^h=(\widetilde{\gamma}b\widetilde{\gamma})^h=\widetilde{\gamma}^h b^h\widetilde{\gamma}^h=\widetilde{\gamma}b^h\widetilde{\gamma}$. So $b^h\mathcal{D}\widetilde{\gamma}\mathcal{D}b$. In addition, $b^h=(\widetilde{\alpha}b\widetilde{\alpha})^h=\widetilde{\alpha}^h b^h\widetilde{\alpha}^h=\widetilde{\alpha}b^h\widetilde{\alpha}$. Therefore, we can get that

$$\text{if } b\in\widetilde{\alpha}B_\gamma\widetilde{\alpha},\ \text{then } b^h\in\widetilde{\alpha}B_\gamma\widetilde{\alpha}. \tag{2.1}$$

Secondly, we prove that the multiplication $*$ on S satisfies the associative law. Let $\alpha,\beta,\gamma\in Y$, $x=(a,g)\in B_\alpha\times G_\alpha$, $y=(b,h)\in B_\beta\times G_\beta$, $z=(c,k)\in B_\gamma\times G_\gamma$. For convenience, we denote

$$\alpha\beta=\xi,\ \alpha\beta\gamma=\delta,\ \beta\gamma=\eta.$$

On the one hand,

$$\begin{aligned}
(x*y)*z &= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h b, gh)*z\\
&= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h b(\widetilde{\xi}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, (gh)k)\\
&\quad (\text{since } \widetilde{\alpha}ab\widetilde{\alpha}, \widetilde{\beta}ab\widetilde{\beta}\in B_\xi,\ \ \text{by (2.1)},\ (\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}} \text{ and } (\widetilde{\beta}ab\widetilde{\beta})^h\in B_\xi)\\
&= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h b\widetilde{\xi})(\widetilde{\xi}(\widetilde{\xi}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\xi})(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}\\
&\quad (\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)\\
&\quad (\text{since } \widetilde{Y} \text{ is a semilattice transversal of } B, \widetilde{\alpha}\widetilde{\beta}=\widetilde{\xi})\\
&= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\xi}(\widetilde{\xi}\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)\\
&\quad (\text{since } B_\xi \text{ is a rectangular band, } \widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h b\widetilde{\xi}=\widetilde{\xi}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\xi}=\widetilde{\xi})
\end{aligned}$$

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(by Lemma 2.1, $\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma} = \widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma})^k(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(for $\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma}, \widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma} \in \widetilde{\gamma}B\widetilde{\gamma}$ and $\sigma_\gamma(k) \in Aut^\circ(\widetilde{\gamma}B\widetilde{\gamma})$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi}, \widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma}, \widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma} \in B_\delta$, by (2.1),

$(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}, (\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma})^k, (\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k \in B_\delta$;

and since B_δ is a rectangular band, it follows that

$(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}\widetilde{\gamma})^k(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k$

$= (\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(gh)^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(h^{-1}g^{-1})}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since G is a Clifford semigroup, $(gh)^{-1} = h^{-1}g^{-1}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}((\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{h^{-1}})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since Y is a semilattice, $\xi \leq \alpha, \xi \leq \beta$; and since σ_ξ is a homomorphism,

$\sigma_\xi(h^{-1}g^{-1}) = \sigma_\xi(h^{-1})\sigma_\xi(g^{-1})$, it follows that

$(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{(h^{-1}g^{-1})} = ((\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\xi})^{h^{-1}})^{g^{-1}}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}((\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h(\widetilde{\beta}bc\widetilde{\beta})\widetilde{\xi})^{h^{-1}})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\xi} = \widetilde{\beta}\widetilde{\xi}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}((\widetilde{\beta}ab\widetilde{\beta})^h)^{h^{-1}}(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\xi} \leq \widetilde{\beta}$; and by $\widetilde{\xi}, (\widetilde{\beta}ab\widetilde{\beta})^h$,

$\widetilde{\beta}bc\widetilde{\beta} \in \widetilde{\beta}B\widetilde{\beta}$; also for $\sigma_\beta(h^{-1}) \in Aut^\circ(\widetilde{\beta}B\widetilde{\beta})$, $\widetilde{\xi}^{h^{-1}} = \widetilde{\xi}$, it follows that

$(\widetilde{\xi}(\widetilde{\beta}ab\widetilde{\beta})^h(\widetilde{\beta}bc\widetilde{\beta})\widetilde{\xi})^{h^{-1}} = \widetilde{\xi}((\widetilde{\beta}ab\widetilde{\beta})^h)^{h^{-1}}(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}\widetilde{\beta}ab\widetilde{\beta}(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\sigma_\beta(h)\sigma_\beta(h^{-1}) = \sigma_\beta(hh^{-1}) = \sigma_\beta(\iota_\beta)$ is the identity automorphism of $\mathrm{Aut}^\circ(\widetilde{\beta}B\widetilde{\beta})$, where ι_β is the identity of G_β, it follows that $((\widetilde{\beta}ab\widetilde{\beta})^h)^{h^{-1}} = \widetilde{\beta}ab\widetilde{\beta}$)

$= (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}\widetilde{\beta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

$= (a(\widetilde{\alpha}ab\widetilde{\alpha}\widetilde{\xi}\widetilde{\beta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\xi} \leq \widetilde{\alpha}$; and by $\widetilde{\alpha}ab\widetilde{\alpha}$,

$\widetilde{\xi}\widetilde{\beta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi} \in \widetilde{\alpha}B\widetilde{\alpha}$; also for $\sigma_\alpha(g^{-1}) \in \mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha})$, it follows that

$(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\xi}\widetilde{\beta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}} = (\widetilde{\alpha}ab\widetilde{\alpha}\widetilde{\xi}\widetilde{\beta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}$)

$= (a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\xi})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since B_ξ is a rectangular band, $ab\widetilde{\alpha}\widetilde{\xi}\widetilde{\beta}ab = ab$)

$= (a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^h bc\widetilde{\gamma})^k c, ghk)$

(since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\xi}=\widetilde{\beta}\widetilde{\alpha}$).

On the other hand, similarly,

$$
\begin{aligned}
x*(y*z) &= x*(b(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,hk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\eta})^{hk}\\
&\quad b(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,g(hk))\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\eta}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\eta})\widetilde{\eta})^{hk}\\
&\quad(\widetilde{\eta}b(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta}\widetilde{\eta})^{hk}\widetilde{\eta}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})^{hk}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\alpha}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})^{hk}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})^{hk}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}((\widetilde{\eta}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})^{h})^{k}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}((\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\eta})^{h})^{k}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})^{h}((\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}})^{h}\widetilde{\eta})^{k}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})^{h}(\widetilde{\beta}bc\widetilde{\beta})\widetilde{\eta})^{k}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})^{h}bc\widetilde{\beta}\widetilde{\eta})^{k}(\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})^{h}bc\widetilde{\beta}\widetilde{\eta}\widetilde{\gamma}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\eta}(\widetilde{\beta}ab\widetilde{\beta})^{h}bc\widetilde{\gamma})^{k}c,ghk)\\
&=(a(\widetilde{\alpha}ab(\widetilde{\beta}bc\widetilde{\beta})^{h^{-1}}\widetilde{\alpha})^{g^{-1}}(\widetilde{\gamma}(\widetilde{\beta}ab\widetilde{\beta})^{h}bc\widetilde{\gamma})^{k}c,ghk).
\end{aligned}
$$

So $(x*y)*z=x*(y*z)$. Hence the multiplication $*$ is associative and it follows that S is a semigroup.

Let $\alpha\in Y$, $x=(a,g)$ and $y=(b,h)\in B_\alpha\times G_\alpha$. Since $\widetilde{\alpha}^{g^{-1}}=\widetilde{\alpha}^{h}=\widetilde{\alpha}$, we have $x*y=(a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\alpha}ab\widetilde{\alpha})^{h}b,gh)=(a\widetilde{\alpha}^{g^{-1}}\widetilde{\alpha}^{h}b,gh)=(a\widetilde{\alpha}b,gh)=(ab,gh)$. So the multiplication $*$ on $B_\alpha\times G_\alpha$ coincides with the multiplication of the direct product of B_α and G_α. Therefore, S is completely regular. Since for any $\alpha\in Y$, $B_\alpha\times G_\alpha$ is a rectangular group, by Theorem II.5.3 in Ref. [5], we obtain that S is an orthogroup.

Denote $S^\circ=\bigcup_{\alpha\in Y}G_{\widetilde{\alpha}}$, where $G_{\widetilde{\alpha}}=\{(\widetilde{\alpha},g)\in B_\alpha\times G_\alpha\mid g\in G_\alpha\}$. Let $\alpha,\beta\in Y$, $x=(\widetilde{\alpha},g)\in G_{\widetilde{\alpha}}$ and $y=(\widetilde{\beta},h)\in G_{\widetilde{\beta}}$. Since $\widetilde{Y}$ is a semilattice transversal of B and $\widetilde{\alpha\beta}^{g^{-1}}=\widetilde{\alpha\beta}^{h}=\widetilde{\alpha\beta}$, we get $x*y=(\widetilde{\alpha}(\widetilde{\alpha}\widetilde{\alpha}\widetilde{\beta}\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}\widetilde{\alpha}\widetilde{\beta}\widetilde{\beta})^{h}\widetilde{\beta},gh)=(\widetilde{\alpha}\widetilde{\alpha\beta}^{g^{-1}}\widetilde{\alpha\beta}^{h}\widetilde{\beta},gh)=(\widetilde{\alpha\beta},gh)\in G_{\widetilde{\alpha\beta}}$. So S° is a subsemigroup of S. Let $x=(a,g)\in B_\alpha\times G_\alpha$. It is obvious that $(a,g)(\widetilde{\alpha},g^{-1})(a,g)=(a\widetilde{\alpha}a,gg^{-1}g)=(a,g)$; similarly, $(\widetilde{\alpha},g^{-1})(a,g)(\widetilde{\alpha},g^{-1})=(\widetilde{\alpha},g^{-1})$ holds. So $(\widetilde{\alpha},g^{-1})\in V((a,g))\cap G_{\widetilde{\alpha}}$. Let $(\widetilde{\alpha},h)\in V((a,g))\cap G_{\widetilde{\alpha}}$. Then $(\widetilde{\alpha},h)=(\widetilde{\alpha},h)(a,g)(\widetilde{\alpha},h)=(\widetilde{\alpha}a\widetilde{\alpha},hgh)=(\widetilde{\alpha},hgh)$, and we get $h=hgh$. Similarly, we can prove that $(a,g)=(a,g)(\widetilde{\alpha},h)(a,g)=(a,ghg)$, which implies that $g=ghg$. So $h=g^{-1}$. Hence, $|\,V((a,g))\cap G_{\widetilde{\alpha}}|=1$. Then S° is an inverse transversal of S. Therefore, S is an orthogroup with an inverse transversal S°. The proof of the direct part of

Theorem 1.1 is completed. □

II. Proof of the converse part of Theorem 1.1.

Let $S=(Y;S_\alpha)$ be an orthogroup with an inverse transversal S°, where Y is a semilattice. In view of Theorem II.5.3 and Theorem III.5.2 in Ref. [5], we assume that $S_\alpha = B_\alpha \times G_\alpha$ is a rectangular group, that is, a direct product of a rectangular band B_α and a group G_α with the identity ι_α, for $\alpha\in Y$.

Denote by $E(S)=\bigcup_{\alpha\in Y} E(S_\alpha)$ the set of all the idempotents of S, $e_a=(a,\iota_\alpha)\in E(S_\alpha)$ for $\alpha\in Y, a\in B_\alpha$. Since S is an orthogroup, $E(S)$ is a band. On $B=\bigcup_{\alpha\in Y} B_\alpha$ define a multiplication by the following: for $a,b\in B$, ab is the unique element of B such that $e_ae_b=e_{ab}$ (since S satisfies the identity $x^0y^0=(x^0y^0)^0$ in Theorem II.5.3[5]). This multiplication extends the given one on each of the B_α, and $B\to E(S)$, $a\mapsto e_a$ is an isomorphism of bands.

Since $S^\circ = S\cap S^\circ = \bigcup_{\alpha\in Y}(S_\alpha\cap S^\circ)$ is an inverse transversal of S, it follows that $S_\alpha\cap S^\circ$ is an inverse transversal of S_α. Since S_α is a completely simple semigroup, we have that $S_\alpha\cap S^\circ$ is a group $\mathcal{H}$-class of S_α denoted by $G_{\widetilde{\alpha}}=\{(\widetilde{\alpha},g)\in B_\alpha\times G_\alpha | g\in G_\alpha\}$, whence $S^\circ = \bigcup_{\alpha\in Y} G_{\widetilde{\alpha}}$[7, Proposition 1.2.2, Proposition 1.2.3]. Denote $\widetilde{Y}=\{\widetilde{\alpha}\in B_\alpha |\ \alpha\in Y\}$. Since S is completely regular and S° is an inverse subsemigroup of S, it follows that S° is a Clifford semigroup[5, Lemma IV. 2.3]. So we can denote $S^\circ=[Y;G_{\widetilde{\alpha}},\widetilde{\theta}_{\alpha,\beta}]$. Since S° is a Clifford semigroup, $E(S^\circ)$ is a semilattice. By the definition of B, we obtain that $\widetilde{Y}$ is a semilattice. Let $a\in B$. Then $\widetilde{\alpha}\in V(a)$ and $\widetilde{Y}$ is a semilattice transversal of B.

In view of S°, which is a Clifford semigroup, we have a homomorphism $\theta_{\alpha,\beta}: G_\alpha\to G_\beta; g\mapsto g\theta_{\alpha,\beta}$ for any $\alpha\geq\beta$ such that $(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\beta}=(\widetilde{\beta},g\theta_{\alpha,\beta})$ defines a Clifford semigroup $G=[Y;G_\alpha,\theta_{\alpha,\beta}]$. In fact, let $\alpha,\beta\in Y$ with $\alpha\geq\beta$ and $g,h\in G_\alpha$. Since $\widetilde{\theta}_{\alpha,\beta}$ is a homomorphism, $((\widetilde{\alpha},g)(\widetilde{\alpha},h))\widetilde{\theta}_{\alpha,\beta}=(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\beta}(\widetilde{\alpha},h)\widetilde{\theta}_{\alpha,\beta}$, then $(\widetilde{\beta},(gh)\theta_{\alpha,\beta})=(\widetilde{\beta},g\theta_{\alpha,\beta})(\widetilde{\beta},h\theta_{\alpha,\beta})=(\widetilde{\beta},g\theta_{\alpha,\beta}h\theta_{\alpha,\beta})$, so $(gh)\theta_{\alpha,\beta}=g\theta_{\alpha,\beta}h\theta_{\alpha,\beta}$. Let $(\widetilde{\alpha},g)\in G_{\widetilde{\alpha}}$. Since $(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\alpha}=(\widetilde{\alpha},g)$ and $(\widetilde{\alpha},g\theta_{\alpha,\alpha})=(\widetilde{\alpha},g)$, we have $\theta_{\alpha,\alpha}=1_{G_\alpha}$. In addition, let $\alpha,\beta,\gamma\in Y$ with $\alpha\geq\beta\geq\gamma$ and $(\widetilde{\alpha},g)\in G_{\widetilde{\alpha}}$. Since $(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\beta}\widetilde{\theta}_{\beta,\gamma}=(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\gamma}$ and $(\widetilde{\gamma},g\theta_{\alpha,\beta}\theta_{\beta,\gamma})=(\widetilde{\gamma},g\theta_{\alpha,\gamma})$, we have $\theta_{\alpha,\beta}\theta_{\beta,\gamma}=\theta_{\alpha,\gamma}$. By the multiplication of S°, let $g\in G_\alpha,\ h\in G_\beta$. We can define the multiplication by the following: $gh=g\theta_{\alpha,\alpha\beta}h\theta_{\beta,\alpha\beta}$ such that

$$\begin{aligned}(\widetilde{\alpha\beta},gh)&=(\widetilde{\alpha},g)(\widetilde{\beta},h)=(\widetilde{\alpha},g)\widetilde{\theta}_{\alpha,\alpha\beta}(\widetilde{\beta},h)\widetilde{\theta}_{\beta,\alpha\beta}\\&=(\widetilde{\alpha\beta},g\theta_{\alpha,\alpha\beta})(\widetilde{\alpha\beta},h\theta_{\beta,\alpha\beta})=(\widetilde{\alpha\beta},g\theta_{\alpha,\alpha\beta}h\theta_{\beta,\alpha\beta}).\end{aligned}$$

For $\alpha\in Y$, denote $G_{\geq\alpha}=\bigcup_{\beta\in Y,\beta\geq\alpha} G_\beta$. Let $h\in G_\beta\subseteq G_{\geq\alpha},\ x=(\widetilde{\beta},h)\in G_{\widetilde{\beta}}$. For any $b\in\widetilde{\alpha}B\widetilde{\alpha}$, since $e_ae_b=e_{ab}$ for $e_a,e_b\in E(S)$ and by Lemma 2.1, $b\widetilde{\beta}b=b$, we get that $x^{-1}e_bxx^{-1}e_bx=x^{-1}e_be_{\widetilde{\beta}}e_bx=x^{-1}e_{b\widetilde{\beta}b}x=x^{-1}e_bx$, so $x^{-1}e_bx\in E(S)$. Denote $e_{b^h}=x^{-1}e_bx$. Moreover, since S° is a Clifford semigroup, it follows that $e_{\widetilde{\alpha}b^h\widetilde{\alpha}}=e_{\widetilde{\alpha}}e_{b^h}e_{\widetilde{\alpha}}=e_{\widetilde{\alpha}}x^{-1}e_bxe_{\widetilde{\alpha}}=x^{-1}e_{\widetilde{\alpha}}e_be_{\widetilde{\alpha}}x=x^{-1}e_{\widetilde{\alpha}b\widetilde{\alpha}}x=x^{-1}e_bx=e_{b^h}$, hence $b^h=\widetilde{\alpha}b^h\widetilde{\alpha}\in\widetilde{\alpha}B\widetilde{\alpha}$. In addition, $e_b=e_{\widetilde{\alpha}b\widetilde{\alpha}}=e_{\widetilde{\alpha}}e_be_{\widetilde{\alpha}}=xx^{-1}e_bxx^{-1}=xe_{b^h}x^{-1}=e_{(b^h)^{h^{-1}}}$, so $b=(b^h)^{h^{-1}}$. Let $b,c\in\widetilde{\alpha}B\widetilde{\alpha}$. Since $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\alpha}=\widetilde{\alpha}\widetilde{\beta}$. And since $e_ae_b=e_{ab}$ for $e_a,e_b\in E(S)$, we have $e_{(bc)^h}=x^{-1}e_{bc}x=x^{-1}e_{b\widetilde{\alpha}c}x=x^{-1}e_{b\widetilde{\alpha}\widetilde{\beta}c}x=x^{-1}e_be_{\widetilde{\beta}}e_cx=x^{-1}e_bxx^{-1}e_cx=e_{b^h}e_{c^h}=e_{b^hc^h}$, so $(bc)^h=b^hc^h$.

Thus, from the above, it induces a mapping $\sigma_\alpha : G_{\geq\alpha} \to \mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha}) : h \mapsto \sigma_\alpha(h)$ satisfying $(b)\sigma_\alpha(h) = b^h$ for $b \in \widetilde{\alpha}B\widetilde{\alpha}$.

$\sigma_\alpha(h)$ is semilattice transversal-preserving. In fact, let $\widetilde{\xi} \leq \widetilde{\alpha}$. Since S° is a Clifford semigroup and $\widetilde{Y}$ is a semilattice transversal of B, $\widetilde{\beta}\widetilde{\xi} = \widetilde{\xi}$, we can get that $e_{\widetilde{\xi}^h} = x^{-1}e_{\widetilde{\xi}}x = x^{-1}xe_{\widetilde{\xi}} = e_{\widetilde{\beta}}e_{\widetilde{\xi}} = e_{\widetilde{\beta}\widetilde{\xi}} = e_{\widetilde{\xi}}$, so $\widetilde{\xi}^h = \widetilde{\xi}$.

Moreover, σ_α is a homomorphism. In fact, let $\beta, \gamma \in Y$ and $\beta, \gamma \geq \alpha,\ h \in G_\beta, k \in G_\gamma$. Since G is a Clifford semigroup, we get that $hk \in G_{\beta\gamma} \subseteq G_{\geq\alpha}$. Let $x = (\widetilde{\beta}, h) \in G_{\widetilde{\beta}},\ y = (\widetilde{\gamma}, k) \in G_{\widetilde{\gamma}}$. Since S° is a Clifford semigroup, we have $xy = (\widetilde{\beta\gamma}, hk) \in G_{\widetilde{\beta\gamma}}$. Then for $b \in \widetilde{\alpha}B\widetilde{\alpha}$, $e_{b^{hk}} = (xy)^{-1}e_b(xy) = y^{-1}x^{-1}e_bxy = y^{-1}e_{b^h}y = e_{(b^h)^k}$, so $b^{hk} = (b^h)^k$. Hence $\sigma_\alpha(hk) = \sigma_\alpha(h)\sigma_\alpha(k)$.

Let $x = (a, g) \in S_\alpha$ and $y = (b, h) \in S_\beta$. Since $e_ae_b = e_{ab}$ for $e_a, e_b \in E(S)$, we can obtain that

$$\begin{aligned} xy =& e_a(\widetilde{\alpha}, g)e_ae_b(\widetilde{\beta}, h)e_b = e_a(\widetilde{\alpha}, g)e_{ab}(\widetilde{\beta}, h)e_b = e_a(\widetilde{\alpha}, g)e_{ab\widetilde{\alpha}\widetilde{\beta}ab}(\widetilde{\beta}, h)e_b \\ & \quad \text{(since } B_{\alpha\beta} \text{ is a rectangular band)} \\ =& e_a(\widetilde{\alpha}, g)e_{ab}e_{\widetilde{\alpha}}e_{\widetilde{\beta}}e_{ab}(\widetilde{\beta}, h)e_b = e_a(\widetilde{\alpha}, g)e_{ab}(\widetilde{\alpha}, g^{-1})(\widetilde{\alpha}, g)(\widetilde{\beta}, h)(\widetilde{\beta}, h^{-1})e_{ab}(\widetilde{\beta}, h)e_b \\ =& e_ae_{(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}}(\widetilde{\alpha\beta}, gh)e_{(\widetilde{\beta}ab\widetilde{\beta})^h}e_b \quad \text{(since } S^\circ \text{ is a Clifford semigroup)} \\ =& e_{a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}}(\widetilde{\alpha\beta}, gh)e_{(\widetilde{\beta}ab\widetilde{\beta})^hb} = (a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^hb, gh). \end{aligned}$$

So the multiplication on S coincides with the one defined in the statement of Theorem 1.1. We complete the proof of the converse part of Theorem 1.1. □

So far the proof of Theorem 1.1 is completed.

3. Remarks for some special cases

Recall that an orthogroup S is a regular (normal) orthogroup if $E(S)$ is a regular (normal) band, that is, $E(S)$ satisfies the identity $axya = axaya(axya = ayxa)$.

Remark 3.1 In Theorem 1.1, if $B = (Y; B_\alpha)$ is a regular band with a semilattice transversal $\widetilde{Y} = \{\widetilde{\alpha} \in B_\alpha |\ \alpha \in Y\}$, and the multiplication $*$ is simplified slightly as follows: for $x = (a, g) \in B_\alpha \times G_\alpha,\ y = (b, h) \in B_\beta \times G_\beta,\ x * y = (a(\widetilde{\alpha}b\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}a\widetilde{\beta})^hb, gh)$, then we can get a construction theorem of regular orthogroups with inverse transversals as outlined in the statement of Theorem 1.1, which is similar to the structure theorem of regular orthogroups given by Yamada[5, Theorem V. 2.5].

Remark 3.2 By means of Theorem 1.1, we can get that a normal orthogroup with inverse transversals is a spined product of a normal band with semilattice transversals and a Clifford semigroup.

It is easy to prove that a spined product of a normal band with semilattice transversals and a Clifford semigroup is a normal orthogroup with inverse transversals.

To prove that a normal orthogroup S with an inverse transversal S° is a spined product of a normal band with a semilattice transversal and a Clifford semigroup, it suffices to reexamine the proof of the converse part of Theorem 1.1 for this special case. In fact, for $\alpha \in Y$, the subband

$\widetilde{\alpha}B\widetilde{\alpha}$ of B is a semilattice for B is a normal band. For any $b \in \widetilde{\alpha}B_\beta\widetilde{\alpha} \subseteq \widetilde{\alpha}B\widetilde{\alpha}$, $\xi \in \mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha})$, there exists $c \in \widetilde{\alpha}B_\beta\widetilde{\alpha}$ such that $(c)\xi = b$, and $(b)\xi \in \widetilde{\alpha}B_\beta\widetilde{\alpha}$ since condition (2.1) in the proof of the direct part of Theorem 1.1 also holds. It follows that $(b)\xi = (bcb)\xi = (cbc)\xi = (c)\xi(b)\xi(c)\xi = b(b)\xi b = b$. So ξ is the identity automorphism in $\mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha})$, which implies that $\mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha}) = \{1_{\widetilde{\alpha}B\widetilde{\alpha}}\}$, where $1_{\widetilde{\alpha}B\widetilde{\alpha}}$ is the identity automorphism. Thus, it follows that the homomorphism $\sigma_\alpha : G_{\geq\alpha} \to \mathrm{Aut}^\circ(\widetilde{\alpha}B\widetilde{\alpha}):\ h \mapsto \sigma_\alpha(h) = 1_{\widetilde{\alpha}B\widetilde{\alpha}}$ is trivial.

Moreover, for $x = (a,g) \in B_\alpha \times G_\alpha$, $y = (b,h) \in B_\beta \times G_\beta$, since $(\widetilde{\alpha}ab\widetilde{\alpha})\sigma_\alpha(g^{-1}) = \widetilde{\alpha}ab\widetilde{\alpha}, (\widetilde{\beta}ab\widetilde{\beta})\sigma_\alpha(h) = \widetilde{\beta}ab\widetilde{\beta}$, it follows that $a(\widetilde{\alpha}ab\widetilde{\alpha})^{g^{-1}}(\widetilde{\beta}ab\widetilde{\beta})^h b = a\widetilde{\alpha}ab\widetilde{\alpha}\widetilde{\beta}ab\widetilde{\beta}b = ab$.

References

[1] BLYTH T S, MCFADDEN R. *Regular semigroups with a multiplicative inverse transversal* [J]. Proc. Roy. Soc. Edinburgh Sect. A, 1982, **92**(3-4): 253–270.

[2] CLIFFORD A H. *Semigroups admitting relative inverses* [J]. Ann. of Math. (2), 1941, **42**: 1037–1049.

[3] CLIFFORD A H. *The structure of orthodox unions of groups*[J]. Semigroup Forum, 1972, **3**(4): 283–337.

[4] SONG Guang-tian, ZHANG Jian-gang, LIU Guo-xin. *A construction of orthogroups* [J]. Northeast. Math. J., 2006, **22**(1): 81–88.

[5] PETRICH M, REILLY N. *Completely Regular Semigroups* [M]. John Wiley & Sons, Inc., New York, 1999.

[6] HE Yong, GUO Yu-qi, SHUM K P. *Standard representations of orthodox semigroups* [J]. Comm. Algebra, 2005, **33**(3): 745–761.

[7] ZHU Feng-lin. *Some Studies on Regular Semigroups with Inverse Transversals* [D]. Ph.D. Dissertation, Dept. of Math., University of Science and Technology of China, 2003.