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f-Projective and f-Injective Modules

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Abstract Let R be a ring. A right R-module M is called f-projective if $\mathrm{Ext}^1(M,N)=0$ for any f-injective right R-module N. We prove that ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is a complete cotorsion theory, where $\mathcal{F}$-proj ($\mathcal{F}$-inj) denotes the class of all f-projective (f-injective) right R-modules. Semihereditary rings, von Neumann regular rings and coherent rings are characterized in terms of f-projective modules and f-injective modules.

Keywords f-projective module; f-injective module; finitely presented cyclic module; (pre)envelope; (pre)cover.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. We use M_R to indicate a right R-module, $\mathrm{FI}(M_R)$ stands for the f-injective envelope of M_R, the character module M^+ is defined by $M^+=\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})$, and $\mathrm{pd}(M_R)$ denotes the projective dimension of M_R. $\mathrm{Hom}(M,N)$ ($\mathrm{Ext}^n(M,N)$) means $\mathrm{Hom}_R(M,N)$ ($\mathrm{Ext}^n_R(M,N)$) for an integer $n\geq 1$. General background material can be found in Ref. [1, 6, 12, 15].

A module M_R is called f-injective (or $\aleph_0$-injective; coflat)[3,4,8] if $\mathrm{Ext}^1_R(R/I,M)=0$ for any finitely generated right ideal I of R. f-injective modules have been studied in many papers such as Ref. [2-4, 8]. In Section two of this paper, we first introduce the notion of f-projective modules, and then give some equivalent characterizations of these modules when R is a self f-injective ring. For instance, it is shown that if R is self f-injective, then M is f-projective if and only if M is a cokernel of an f-injective preenvelope $K\to F$ with F projective. We also prove that ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is a complete cotorsion theory, where $\mathcal{F}$-proj ($\mathcal{F}$-inj) denotes the class of all f-projective (f-injective) right R-modules. In Section three, some new characterizations of semihereditary rings, von Neumann regular rings and coherent rings are given. For example, it is proven that R is a right semihereditary ring if and only if every (quotient module of any injective) right R-module M has a monic $\mathcal{F}$-inj-cover if and only if $\mathrm{pd}(M)\leq 1$ for every f-projective right R-module M if and only if ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary and every f-projective right R-module

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has a monic $\mathcal{F}$-inj-cover; R is a von Neumann regular ring if and only if every cotorsion right R-module is f-injective if and only if every f-projective right R-module is projective if and only if every f-projective right R-module is flat if and only if every right R-module has an $\mathcal{F}$-inj-envelope with the unique mapping property if and only if ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary and every f-projective right R-module has an $\mathcal{F}$-inj-envelope with the unique mapping property if and only if ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary and every f-projective right R-module is f-injective. Finally, as a generalization of the well known result that R is a right coherent ring if and only if every direct limit of FP-injective right R-modules is FP-injective, we get that R is a right coherent ring if and only if every direct limit of f-injective right R-modules is f-injective.

2. Definition and general results

We start with the following

Definition 2.1 *Let M be a right R-module. M is called an f-projective module if* $\operatorname{Ext}^1(M, N) = 0$ *for any f-injective right R-module N.*

Remark 2.2 Any finitely presented cyclic R-module is f-projective, and it is easily seen that all (left) right R-modules are f-projective if and only if ring R is (left)right Noetherian.

Recall that a pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules is called a cotorsion theory[6] if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$, where $\mathcal{F}^{\perp} = \{C : \operatorname{Ext}^1(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$, and ${}^{\perp}\mathcal{C} = \{F : \operatorname{Ext}^1(F, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\varphi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of M[6] if for any homomorphism f: $M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\varphi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \varphi$, the $\mathcal{C}$-preenvelope φ is called a $\mathcal{C}$-*envelope* of M. Following Ref. [6, Definition 7.1.6], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of M if $\operatorname{coker}(\alpha) \in {}^{\perp}\mathcal{C}$. Dually we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. Special $\mathcal{C}$-preenvelopes (resp., special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp., $\mathcal{C}$-precovers).

A $\mathcal{C}$-envelope $\varphi : M \to F$ is said to have the unique mapping property[5] if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a unique homomorphism $g : F \to F'$ such that $g\varphi = f$.

Proposition 2.3 *Let R be a right self f-injective ring and M a right R-module. Then the following are equivalent:*

(1) M is f-projective;

(2) M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$, where A is f-injective;

(3) For every exact sequence $0 \to K \to F \to M \to 0$, where F is f-injective, $K \to F$ is an f-injective preenvelope of K;

(4) M is a cokernel of an f-injective preenvelope $K \to F$ with F projective.

Proof (1)$\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). For every f-injective right R-module N, there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective, which induces an exact sequence $\mathrm{Hom}(M,E) \to \mathrm{Hom}(M,L) \to \mathrm{Ext}^1(M,N) \to 0$. Since $\mathrm{Hom}(M,E) \to \mathrm{Hom}(M,L) \to 0$ is exact by (2), $\mathrm{Ext}^1(M,N) = 0$. So (1) follows.

(1) $\Rightarrow$ (3) is easy to verify.

(3) $\Rightarrow$ (4). Let $0 \to K \to P \to M \to 0$ be an exact sequence with P projective. Note that P is f-injective by hypothesis, thus $K \to P$ is an f-injective preenvelope.

(4) $\Rightarrow$ (1). By (4), there is an exact sequence $0 \to K \to P \to M \to 0$, where $K \to P$ is an f-injective preenvelope with P projective. It gives rise to the exactness of $\mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to \mathrm{Ext}^1(M,N) \to 0$ for each f-injective right R-module N. Note that $\mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(M,N) = 0$, as desired.

Denote by $\mathcal{F}$-proj ($\mathcal{F}$-inj) the class of all f-projective (f-injective) right R-modules. Then we have

Theorem 2.4 *Let R be a ring. Then ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is a cotorsion theory. Moreover, every right R-module has a special $\mathcal{F}$-inj-preenvelope and every right R-module has a special $\mathcal{F}$-proj-precover.*

Proof Let X be the set of representatives of finitely presented cyclic right R-modules. Thus $\mathcal{F}$-inj$= X^{\perp}$. Since $\mathcal{F}$-proj$=^{\perp} (X^{\perp})$, the results follow from Ref. [6, Definition 7.1.5] and [7, Theorem 10].

Remark 2.5 The statement of Theorem 2.4 is the best possible in the sense that ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is not a perfect cotorsion theory because f-injective envelopes may not exist in general[14, Proposition 4.8]. However, if $\mathcal{F}$-proj is closed under direct limits, then ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is a perfect cotorsion theory by Theorem 2.4 and Ref. [6, Theorem 7.2.6].

Corollary 2.6 *The following are equivalent for a ring R:*

(1) Every right R-module is f-projective;

(2) Every cyclic right R-module is f-projective;

(3) Every f-injective right R-module is injective;

(4) ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary, and every f-injective right R-module is f-projective.

In this case, R is right Noetherian.

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let N be any f-injective right R-module and I any right ideal of R. Then $\mathrm{Ext}^1(R/I,N) = 0$ by (2). Thus N is injective, as desired.

(3) $\Rightarrow$ (1) holds by Theorem 2.4.

(1) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). By Theorem 2.4, for any right R-module M, there is a short exact sequence $0 \to M \to F \to L \to 0$, where F is f-injective and L is f-projective. So (1) follows from (4).

In this case, R is right Noetherian since every f-injective right R-module is injective.

3. Applications

Recall that a ring R is right semihereditary if and only if every finitely generated right ideal of R is projective.

Theorem 3.1 *The following are equivalent for a ring R:*

(1) R is a right semihereditary ring;

(2) Every quotient module of any (f-)injective right R-module is f-injective;

(3) Every (quotient module of any injective) right R-module M has a monic $\mathcal{F}$-inj-cover $\varphi : F \to M$;

(4) $\mathrm{pd}(R/I) \leq 1$ for every right R-module R/I with I finitely generated right ideal of R;

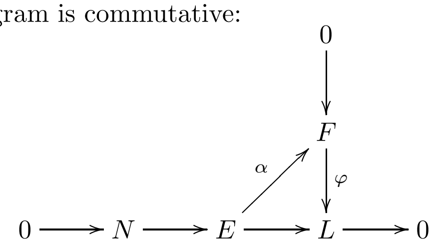
(5) $\mathrm{pd}(M) \leq 1$ for every f-projective right R-module M;

(6) ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary, and every f-projective right R-module has a monic $\mathcal{F}$-inj-cover.

Proof (1) $\Leftrightarrow$ (2) holds by Ref. [8, Theorem 3.2].

(2) $\Rightarrow$ (3). Let M be any right R-module. Write $F = \sum\{N \leq M : N \in \mathcal{F}\text{-inj}\}$ and $G = \oplus\{N \leq M : N \in \mathcal{F}\text{-inj}\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Note that $G \in \mathcal{F}$-inj, so $F \in \mathcal{F}$-inj by (2). Next we prove that the inclusion $i : F \to M$ is an $\mathcal{F}$-inj-cover of M. Let $\psi : F' \to M$ with $F' \in \mathcal{F}$-inj be an arbitrary right R-homomorphism. Note that $\psi(F') \leq F$ by (2). Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is an $\mathcal{F}$-inj-precover of M. In addition, it is clear that the identity map 1_F of F is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (3) follows.

(3) $\Rightarrow$ (2). Let M be any f-injective right R-module and N any submodule of M. We shall show that M/N is f-injective. Indeed, there exists an exact sequence $0 \to N \to E \to L \to 0$ with E injective. Since L has a monic $\mathcal{F}$-inj-cover $\varphi : F \to L$ by (3), there is $\alpha : E \to F$ such that the following exact diagram is commutative:



Thus φ is epic, and hence it is an isomorphism. Therefore L is f-injective. For any finitely presented cyclic right R-module K, we have

$$0 = \mathrm{Ext}^1(K, L) \to \mathrm{Ext}^2(K, N) \to \mathrm{Ext}^2(K, E) = 0.$$

Therefore $\mathrm{Ext}^2(K, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^1(K, M) \to \mathrm{Ext}^1(K, M/N) \to \mathrm{Ext}^2(K, N) = 0.$$

Therefore $\mathrm{Ext}^1(K, M/N) = 0$, as desired.

(2) $\Rightarrow$ (5). Let M be any f-projective right R-module and N any right R-module. There exists an exact sequence $0 \to N \to E \to L \to 0$ with E injective, and hence L is f-injective by (2), so we have the exact sequence

$$0 = \mathrm{Ext}^1(M, L) \to \mathrm{Ext}^2(M, N) \to \mathrm{Ext}^2(M, E) = 0.$$

Therefore $\mathrm{Ext}^2(M, N) = 0$, which implies $pd(M) \leq 1$.

(5) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (6) is clear by the equivalence of (2) and (3).

(6) $\Rightarrow$ (2). Let M be any f-injective right R-module and N any submodule of M. We have to prove that M/N is f-injective. In fact, note that N has a special $\mathcal{F}$-inj-preenvelope, i.e., there exists an exact sequence $0 \to N \to E \to L \to 0$ with $E \in \mathcal{F}$-Inj and $L \in \mathcal{F}$-Proj. The rest of the proof is similar to that of (3) $\Rightarrow$ (2) by noting that $\mathrm{Ext}^2(K, E) = 0$ for any finitely presented cyclic right R-module K, since ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary. This completes the proof.

A right R-module M is called cotorsion[6, Definition 5.3.22] if $\mathrm{Ext}^1(F, M) = 0$ for all flat right R-modules F. It is well known that a ring R is von Neumann regular if and only if every (cyclic) right R-module is flat if and only if every cotorsion right R-module is flat if and only if every cotorsion right R-module is injective. Now we have

Theorem 3.2 *The following are equivalent for a ring R:*

(1) R is a von Neumann regular ring;

(2) Every right R-module is f-injective;

(3) Every cotorsion right R-module is f-injective;

(4) Every f-projective right R-module is projective;

(5) Every f-projective right R-module is flat;

(6) Every finitely presented cyclic right R-module is flat;

(7) Every right R-module has an $\mathcal{F}$-inj-envelope with the unique mapping property;

(8) ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary and every f-projective right R-module has an $\mathcal{F}$-inj-envelope with the unique mapping property;

(9) ($\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary and every f-projective right R-module is f-injective.

Proof (1) $\Leftrightarrow$ (2) holds by Ref. [4, Proposition 1.11], (2)$\Rightarrow$ (3) and (4) through (6) are obvious.

(3) $\Leftrightarrow$ (6) follows from Ref. [10, Proposition 2.10].

(6) $\Rightarrow$ (1). Note that R/I is flat for any finitely generated right ideal I of R by (6). Thus R/I is projective since R/I is finitely presented. It follows that I is a direct summand of R, which implies that R is von Neumann regular.

(2) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) are clear.

(7) ⇒ (2). Let M be a right R-module. There is the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\sigma_M} & \mathrm{FI}(M) & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\ & & & \searrow^{0} & & \searrow^{\sigma_L\gamma} & \downarrow{\scriptstyle \sigma_L} & & \\ & & & & & & \mathrm{FI}(L) & & \end{array}$$

Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (7). Therefore $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence M is f-injective.

(8) ⇒ (9). Let M be an f-projective right R-module. By (8), there is an exact sequence

$$0 \longrightarrow M \xrightarrow{\sigma_M} \mathrm{FI}(M) \xrightarrow{\gamma} L \longrightarrow 0\ ,$$

where L is f-projective by Wakamatsu's Lemma[6, Proposition 7.2.4]. Thus M is f-injective by the proof of (7) ⇒ (2).

(9) ⇒ (2). Let M be any right R-module. Note that M has a special $\mathcal{F}$-proj-precover, i.e., there exists an exact sequence $0 \to K \to L \to M \to 0$ with $K \in \mathcal{F}$-inj and $L \in \mathcal{F}$-proj. Thus $L \in \mathcal{F}$-inj, and $M \in \mathcal{F}$-inj by (9).

By Ref. [13, Theorem 3.2], a ring R is right coherent if and only if every direct limit of FP-injective right R-modules is FP-injective. We generalize this result as follows.

Theorem 3.3 *The following are equivalent for a ring R:*

(1) Every factor module of an f-injective right R-module by a pure submodule is f-injective;

(2) Every direct limit of f-injective right R-modules is f-injective;

(3) R is a right coherent ring.

Proof (1) ⇒ (2). Let $(M_i, f_{ij})_\Lambda$ be a direct system of f-injective right R-modules. Then $\oplus M_i$ is f-injective by Ref. [8, Theorem 2.4], and the canonical epimorphism $\oplus M_i \to \varinjlim M_i$ is pure by Ref. [15, 33.9]. So $\varinjlim M_i$ is f-injective by (1).

(2) ⇔ (3). Let I be any finitely generated right ideal of R and $(M_i, f_{ij})_\Lambda$ a direct system of right R-module with M_i f-injective for all $i \in \Lambda$. Then the exactness of the sequence $0 \to I \to R$ induces the following commutative diagram:

$$\begin{array}{ccccc} \varinjlim \mathrm{Hom}(R, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}(I, M_i) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_I} & & \\ \mathrm{Hom}(R, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}(I, \varinjlim M_i) & \longrightarrow & 0 \end{array}$$

with φ being an isomorphism and φ_I being a monomorphism by Ref. [15, 24.9] (for I is finitely generated), where the first row is exact since $M_i \in \mathcal{F}$-inj for all $i \in \Lambda$. So we have that R is right coherent if and only if I_R is a finitely presented right R-module if and only if φ_I is an isomorphism if and only if the bottom row is exact if and only if $\varinjlim M_i$ is f-injective.

(3) ⇒ (1) is clear.

Remark 3.4 Let m and n be fixed positive integers. Recall that a right R-module M is said to be (m,n)-injective[16] if $\mathrm{Ext}^1(P,M)=0$ for any (m,n)-presented right R-module P; A ring R is called right (m,n)-coherent in case each n-generated submodule of the right R-module R^m is finitely presented.

From the proof of Theorem 3.3, we may obtain the following general result:

R is right (m,n)-coherent if and only if every direct limit of (m,n)-injective right R-modules is (m,n)-injective.

We conclude the paper with the following

Corollary 3.5 *If R is a left (m,n)-coherent ring, then M^+ has an (m,n)-injective cover for any right R-module M.*

Proof By Ref. [11, Corollary 3.3], [6, Corollary 5.2.7] and Remark 3.4. □

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