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Orders of the Renner Monoids of Adjoint Type

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Abstract In this paper, we find the orders of the Renner monoids for $\mathcal{J}$-irreducible monoids $\overline{K^*\rho(G)}$, where G is a simple algebraic group over an algebraically closed field K, and $\rho: G \to \mathrm{GL}(V)$ is the irreducible representation associated with the highest root.

Keywords Renner monoid; Weyl group; type map; $\mathcal{J}$-irreducible monoid; order.

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1. Introduction

A linear algebraic monoid is an affine variety defined over an algebraically closed field K together with an associative morphism and an identity. An algebraic monoid is irreducible if it is irreducible as a variety. The unit group of an algebraic monoid is an algebraic group. An irreducible monoid is reductive if its unit group is a reductive group.

Let M be a reductive monoid with unit group G, and let $B \subset G$ be a Borel subgroup, $T \subset B$ be the maximal torus, and $W = N_G(T)/T$ be the Weyl group. Let $\overline{N_G(T)}$ be the Zariski closure of $N_G(T)$ in M. Then $R = \overline{N_G(T)}/T$, called the Renner monoid of M, is an inverse monoid with unit group W. Let $\overline{T}$ be the Zariski closure of T in M and $E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}$ be the set of idempotents in $\overline{T}$. Then we have $R = \langle W, \Lambda\rangle$, where $\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}$ is the cross-section lattice of M.

Definition 1.1[7] *Let M, G, B, $T \subset B$, W be as above. Let Δ be the fundamental root system relative to T and B, and $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of simple reflections that generate the Weyl group. The Putcha's type map $\lambda: \Lambda \to 2^\Delta$ is defined by $\lambda(e) = \{\alpha \in \Delta \mid s_\alpha e = e s_\alpha,\ s_\alpha \in S\}$.*

Definition 1.2[7] *Let M be a reductive monoid with zero. The monoid M is called $\mathcal{J}$-irreducible if $\Lambda \setminus \{0\}$ has a unique minimal idempotent.*

Let G be a simple algebraic group, and let $\rho: G \to \mathrm{GL}(V)$ be an irreducible representation. Then $M = \overline{K^*\rho(G)}$ is a $\mathcal{J}$-irreducible monoid[8, Corollary 8.3.3]. Let $\lambda_*(e) = \cap_{f \le e}\lambda(f)$, $\lambda^*(e) = \cap_{f \ge e}\lambda(f)$, and $W(e) = W_{\lambda(e)}$, the associated parabolic subgroup of W as in Section 7.5 of Ref. [9]. Then we have the following theorem[1], which offers a general formula for the order of a

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Renner monoid of a reductive monoid with zero.

Theorem 1.1 (Theorem 3.2 of Ref. [1]) *Let M be a reductive monoid with zero and type map λ. Let λ^* and λ_* be defined as above. Then the order of the Renner monoid R of M is:*

$$|R| = 1 + \sum_{e\in\Lambda\setminus\{0\}} |WeW| = 1 + \sum_{e\in\Lambda\setminus\{0\}} \frac{|W|^2}{|W_{\lambda^*(e)}| \times |W_{\lambda_*(e)}|^2}.$$

Note that $\lambda^*(e)$ and $\lambda_*(e)$ are subsets of the set S given in Definition 1.1. Let $W_{\lambda^*(e)}$ and $W_{\lambda_*(e)}$ be the parabolic subgroups associated with $\lambda^*(e)$ and $\lambda_*(e)$ of W respectively. We put a brief and alternate proof of the above theorem in the next section, and after that we give a complete list of the orders of the Renner monoids of all $\mathcal{J}$-irreducible monoids for adjoint type in Section 3.

2. An alternate proof of Theorem 1.1

Let $W \times W$ act on R by: $(w_1, w_2)\cdot r = w_1 r w_2^{-1}$. Then the set of $(W\times W)$-orbits of R is isomorphic to Λ as a lattice and $R = \bigsqcup_{e\in\Lambda} WeW$. Clearly, $|R| = \sum_{e\in\Lambda} |WeW|$. This tells us that as long as a formula for the number of elements in each WeW is obtained, where $e \in \Lambda$, then the order of R is done. We give a basic proof here. First of all, we need the following results, which are due to Putcha and Renner[9, Section 7.5.1].

(i) $\lambda^*(e) = \{a \in \Delta \mid s_\alpha e = e s_\alpha \neq e\}$.

(ii) $\lambda_*(e) = \{a \in \Delta \mid s_\alpha e = e s_\alpha = e\}$.

(iii) For $e \in \Lambda$, $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$.

(iv) For $e \in \Lambda$, $W(e) \cong W_{\lambda^*(e)} \times W_{\lambda_*(e)}$ and $w^* w_* = w_* w^*$ for $w^* \in W_{\lambda^*(e)}$ and $w_* \in W_{\lambda_*(e)}$.

Then let e and f be two arbitrary idempotents in $E(R)(= E(\overline{T}))$ and let w be an arbitrary element in the Weyl group W. Firstly, if $we = f$, then $f = we = wee = fe$. Since $we = f$, we get $e = w^{-1}f = w^{-1}ff = ef$. Therefore, $e = f$. Similarly, if $ew = f$ then $e = f$. Secondly, if $we = e$, then $wew^{-1}w = e$, which means $wew^{-1} = e$. Hence, $ew = we = e$. Similarly, if $ew = e$, then $we = e$. Finally, we come to the conclusion that $\{w \in W \mid we = ew = e\} = \{w \in W \mid we = e\} = \{w \in W \mid ew = e\}$. Hence, it follows from Definition 1.1, Definition 1.2 and (ii) that if $e \in \Lambda$, then $W_{\lambda(e)} = \{w \in W \mid we = ew\}$, and $W_{\lambda_*(e)} = \{w \in W \mid we = e\} = \{w \in W \mid ew = e\}$.

For $e \in \Lambda$, let $(W\times W)_e = \{(w_1, w_2) \in W \times W \mid w_1, w_2 \in W\ , w_1 e w_2^{-1} = e\}$ be the isotropic group of e. Now we prove that $(W \times W)_e = \{(w, ww_*) \in W \mid w \in W_{\lambda(e)} \text{ and } w_* \in W_{\lambda_*(e)}\}$. Actually, it is straightforward to check that the set on the right-hand side is contained in the one of the left-hand side. On the other hand, for any $(w_1, w_2) \in (W\times W)_e$, $w_1 e w_2^{-1} = e$ and then $w_1 e w_1^{-1}(w_1 w_2^{-1}) = e$. According to the former argument, we know it means that $w_1 e w_1^{-1} = e$. Hence $w_1 \in W_{\lambda(e)}$. Similarly, $w_2 \in W_{\lambda(e)}$. It follows from $w_1 e w_1^{-1} = e$ and $w_1 e w_1^{-1} w_1 w_2^{-1} = e$ that $e w_1 w_2^{-1} = e$. Thus, $w_1 w_2^{-1} \in W_{\lambda_*(e)}$, and $(w_1, w_2) = (w_1, w_1(w_1^{-1}w_2))$ belongs to the set on the right-hand side. As a natural result, we finally get

(i) $|(W\times W)_e| = |W_{\lambda(e)}| \times |W_{\lambda_*(e)}| = |W_{\lambda^*(e)}| \times |W_{\lambda_*(e)}|^2$,

(ii) $|WeW| = \frac{|W|^2}{|(W\times W)_e|} = \frac{|W|^2}{|W_{\lambda^*(e)}|\times|W_{\lambda_*(e)}|^2}$.

Then Theorem 1.1 follows from (ii).

3. Application to $\mathcal{J}$-irreducible monoids

For completeness, we list all Dynkin diagrams and orders of the Weyl groups of simple groups here for the reference of the next subsections.

A_l:

$\alpha_0 = \lambda_1 + \lambda_l = \alpha_1 + \alpha_2 + \cdots + \alpha_l; \quad |W| = (l+1)!, \ l \geq 1.$

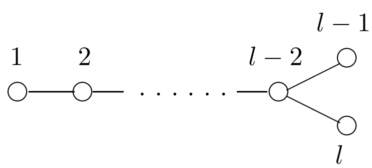
B_l :

$\alpha_0 = \lambda_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l; \quad |W| = (l!)2^l, \ l \geq 2.$

C_l :

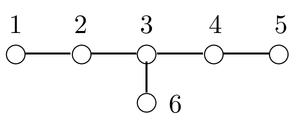
$\alpha_0 = 2\lambda_1 = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l; \quad |W| = (l!)2^l, \ l \geq 3.$

D_l:



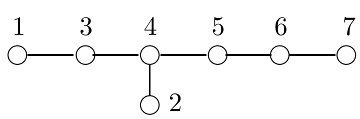
$\alpha_0 = \lambda_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l; \quad |W| = (l!)2^{l-1}, \ l \geq 4.$

E_6:



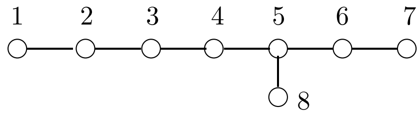
$\alpha_0 = \lambda_6 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6; \quad |W| = 2^7 3^4 5.$

E_7:



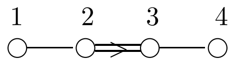
$\alpha_0 = \lambda_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7; \quad |W| = 2^{10} 3^4 5 \times 7.$

E_8:



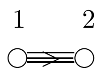
$\alpha_0 = \lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$; $|W| = 2^{14}3^5 5^2 7$.

F_4:



$\alpha_0 = \lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$; $\quad |W| = 2^7 3^2$.

G_2:



$\alpha_0 = \lambda_1 = 2\alpha_1 + 3\alpha_2$; $\quad |W| = 12$.

Note that the number above or beside each node is the index of the associated fundamental root, α_0 is the highest root and λ_i is the i-th fundamental dominant weight relative to the fundamental root system Δ.

3.1. Type map of $\mathcal{J}$-irreducible monoids

Putcha and Renner found the type maps of the $\mathcal{J}$-irreducible monoids in Ref. [7].

Theorem 3.1 (Theorem 4.16 of Ref. [7]) *Let M be a $\mathcal{J}$-irreducible monoid associated with a dominant weight μ and $J_0 = \{\alpha \in \Delta \mid \langle \mu, \alpha \rangle = 0\}$(see [9, p.16] for the bracket $\langle\ ,\ \rangle$). Let e_0 be the unique minimal idempotent in $\Lambda \setminus \{0\}$. Then*

(i) $\lambda^*(\Lambda \setminus \{0\}) = \{X \subset \Delta \mid X$ has no connected component that lies entirely in $J_0\}$.

(ii) $\lambda^*(e) \in \lambda^*(\Lambda \setminus \{0\})$ and $\lambda_*(e) = \{a \in J_0 \setminus \lambda^*(e) \mid s_a s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)\}$ for $e \in \Lambda \setminus \{0\}$. Specially, $\lambda(e_0) = \lambda_*(e_0) = J_0$.

For the remainder of this paper, we assume that $M = \overline{K^*\rho(G)}$ where G is a simple algebraic group over K, and $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation associated with the highest root α_0.

3.2. Orders of Renner monoids for adjoint type

The orders of Renner monoids of $\mathcal{J}$-irreducible monoids associated with the first fundamental dominant weight λ_1 were found by Li, Li and Cao[1]. Since the highest root is just the first fundamental dominant weight in the cases of type E_7, E_8, F_4, G_2, and two times in the case of C_l. For these cases the orders of the Renner monoids are completely the same as those in Ref. [1]. Therefore, we list the orders for all the cases but omit the proof for types C_l, E_7, E_8, F_4 and G_2.

Theorem 3.2 *Let R be the Renner monoids of $\mathcal{J}$-irreducible monoids M associated with the highest root. Then*

(A_l) $|R| = 2(l+1)^2 \sum_{r=0}^{l} \binom{l}{r}^2 r! + \sum_{i=1}^{l-2} \sum_{j=1}^{l-1-i} \binom{l+1}{i+1}^2 (i+1)! \binom{l-i}{j+1}^2 (j+1)! - l^4 - 2l^3 - 3l^2 -$

$4l-1+(l+1)!$.

(B_l) $|R|=\sum_{r=0}^{l}4^r\binom{l}{r}^2(r+1)!-20l^4+40l^3-28l^2+2^{l+1}l\cdot l!+2^l l!$.

(C_l) $|R|=\sum_{r=0}^{l}4^r\binom{l}{r}^2 r!+2^l l!$.

(D_l) $|R|=\sum_{r=0}^{l}4^r\binom{l}{r}^2(r+1)!-20l^4+40l^3-28l^2-2^{2l-1}(l+1)!+2^l l\cdot l!+2^{l-1}l!$.

(E_6) $|R|=113068225=5^2\times 4522729$.

(E_7) $|R|=44520456709=281\times 158435789$.

(E_8) $|R|=332011601568001=4969\times 7187\times 9296867$.

(F_4) $|R|=103105=5\times 17\times 1213$.

(G_2) $|R|=121=11^2$.

Proof The main procedure of the proof for each case is to calculate the orders of two-sided W orbits WeW for $e\in\Lambda\setminus\{0\}$ by using Theorem 1.1.

(a) Type A_l: $J_0=\{\alpha_2,\alpha_3,\dots,\alpha_{l-1}\}$. It follows from Theorem 3.1 that

$$\begin{aligned}\lambda^*(\Lambda\setminus\{0\})=\big\{&\phi,\{\alpha_1\},\{\alpha_1,\alpha_2\},\dots,\{\alpha_1,\alpha_2,\dots,\alpha_{l-1}\},\\&\{\alpha_l\},\{\alpha_{l-1},\alpha_l\},\dots,\{\alpha_2,\alpha_3,\dots,\alpha_l\},\\&\{\alpha_1,\alpha_2,\dots,\alpha_l\},\{\alpha_1,\dots,\alpha_{i-1},\alpha_i,\alpha_{l-j+1},\dots,\alpha_l\}:1\le i\le l-j-1\le l-2\big\}.\end{aligned}$$

If $e=e_0$ is the minimal idempotent in $\Lambda\setminus\{0\}$, we get $\lambda^*(e_0)=\phi,\lambda_*(e_0)=J_0$. Hence

$$|We_0W|=\frac{|W|^2}{|W_{J_0}|^2}=\frac{|W(A_l)|^2}{|W(A_{l-2})|^2}=\frac{((l+1)!)^2}{((l-1)!)^2}=l^2(l+1)^2. \tag{1}$$

If e is any idempotent other than e_0 in $\Lambda\setminus\{0\}$, and when $\lambda^*(e)=\{\alpha_1,\dots,\alpha_r\}$ with $1\le r\le l-1$, then by Theorem 3.1, $\lambda_*(e)=\{\alpha_{r+2},\dots,\alpha_{l-1}\}$ for $1\le r\le l-3$ and $\lambda_*(e)=\phi$ for $r=l-2$ and $l-1$. Hence, for $1\le r\le l-3$, $W_{\lambda^*(e)}\cong W(A_r),W_{\lambda_*(e)}\cong W(A_{l-r-2})$.

For $r=l-2$ and $l-1$, $W_{\lambda^*(e)}\cong W(A_r),W_{\lambda_*(e)}\cong 1$.

So, $|W_{\lambda^*(e)}|=(r+1)!$ and $|W_{\lambda_*(e)}|=(l-r-1)!$ for $1\le r\le l-1$. It follows from Theorem 1.1 that,

$$|WeW|=\frac{((l+1)!)^2}{(r+1)!((l-r-1)!)^2}. \tag{2}$$

Obviously, when $\lambda_*(e)=\{\alpha_{l-r+1},\alpha_{l-r},\dots,\alpha_l\}$ with $1\le r\le l-1$, this case is complete the same as the above case.

When $\lambda^*(e)=\{\alpha_1,\alpha_2,\dots,\alpha_l\}$, e is the identity element in $\Lambda\setminus\{0\}$. Hence,

$$|WeW|=|W|=(l+1)!. \tag{3}$$

When $\lambda^*(e)=\{\alpha_1,\dots,\alpha_{i-1},\alpha_i,\alpha_{l-j+1},\dots,\alpha_l\},1\le i\le l-j-1\le l-2$, we have $\lambda_*(e)=\{\alpha_{i+2},\alpha_{i+3},\dots,\alpha_{l-j-2},\alpha_{l-j-1}\}$ for $l-j-i\ge 3$ and $\lambda_*(e)=\phi$ for $l-j-i=2$ and 1. Hence, for $l-j-i\ge 3$, $W_{\lambda^*(e)}\cong W(A_i)\times W(A_j),W_{\lambda_*(e)}\cong W(A_{l-j-i-2})$ and for $l-j-i=2$ and 1, $W_{\lambda^*(e)}\cong W(A_i)\times W(A_j)$, $W_{\lambda_*(e)}\cong 1$. Thus, $|W_{\lambda^*(e)}|=(i+1)!(j+1)!$ and $|W_{\lambda_*(e)}|=(l-j-i-1)!$, for $1\le i\le l-j-1\le l-2$. Hence,

$$|WeW|=\frac{((l+1)!)^2}{(i+1)!(j+1)!((l-j-i-1)!)^2}. \tag{4}$$

Therefore, from (1)–(4) and Theorem 1.1, we have

$$\begin{aligned}|R| =&1 + l^2(l+1)^2 + 2\sum_{r=1}^{l-1} \frac{((l+1)!)^2}{(r+1)!((l-r-1)!)^2} + (l+1)!+\\ &\sum_{1\le i\le l-j-1\le l-2} \frac{((l+1)!)^2}{(i+1)!(j+1)!((l-j-i-1)!)^2}\\ =&2(l+1)^2\sum_{r=0}^{l}\binom{l}{r}^2 r! + \sum_{i=1}^{l-2}\sum_{j=1}^{l-1-i}\binom{l+1}{i+1}^2(i+1)!\binom{l-i}{j+1}^2(j+1)! -\\ &l^4 - 2l^3 - 3l^2 - 4l - 1 + (l+1)!.\end{aligned}$$

(b) Type B_l: $J_0 = \{\alpha_1, \alpha_3, \ldots, \alpha_l\}$. It follows from Theorem 3.1 that

$$\begin{aligned}\lambda^*(\Lambda \setminus \{0\}) =&\{\phi, \{\alpha_2\}, \{\alpha_2,\alpha_3\}, \ldots, \{\alpha_2,\alpha_3,\ldots,\alpha_l\},\\ &\{\alpha_1,\alpha_2\}, \{\alpha_1,\alpha_2,\alpha_3\}, \ldots, \{\alpha_1,\alpha_2,\ldots,\alpha_l\}\}.\end{aligned}$$

If $e = e_0$ is the minimal idempotent in $\Lambda \setminus \{0\}$, we get

$$|We_0W| = \frac{|W|^2}{|W_{J_0}|^2} = \frac{|W(B_l)|^2}{|W(A_1)\times W(B_{l-2})|^2} = \frac{(2^l l!)^2}{(2!(l-2)!2^{l-2})^2} = 4(l-1)^2l^2. \tag{5}$$

If e is any idempotent other than e_0 in $\Lambda \setminus \{0\}$, and when $\lambda^*(e) = \{\alpha_2, \alpha_3, \ldots, \alpha_{r+1}\}$ with $1 \le r \le l-1$, by Theorem 3.1, $\lambda_*(e) = \{\alpha_{r+3}, \ldots, \alpha_l\}$ for $1 \le r \le l-3$ and $\lambda_*(e) = \phi$ for $r = l-2$ and $l-1$. Therefore, for $1 \le r \le l-3$, $W_{\lambda^*(e)} \cong W(A_r), W_{\lambda_*(e)} \cong W(B_{l-r-2})$, where $B_1 \cong A_1$ when $r = l-3$, for $r = l-2$, $W_{\lambda^*(e)} \cong W(A_{l-2}), W_{\lambda_*(e)} \cong 1$, and for $r = l-1, W_{\lambda^*(e)} \cong W(B_{l-1}), W_{\lambda_*(e)} \cong 1$.

So, $|W_{\lambda^*(e)}| = (r+1)!$ and $|W_{\lambda_*(e)}| = 2^{l-r-2}(l-r-2)!$ for $1 \le r \le l-2$, $|W_{\lambda^*(e)}| = 2^{l-1}(l-1)!$ and $|W_{\lambda_*(e)}| = 1$ for $r = l-1$. It follows from Theorem 1.1 that for $1 \le r \le l-2$,

$$|WeW| = \frac{(2^l l!)^2}{(r+1)!(2^{l-r-2}(l-r-2)!)^2} = 4^{r+2}(r+2)\binom{l}{r+1}^2(r+2)!. \tag{6}$$

and for $r = l-1$,

$$|WeW| = \frac{(2^l l!)^2}{2^{l-1}(l-1)!} = 2^{l+1}l\, l!. \tag{7}$$

It is similar to calculate the remaining cases:

For $\lambda^*(e) = \{\alpha_1, \alpha_2, \ldots, \alpha_{r+1}\}$ with $1 \le r \le l-1$, we have $\lambda_*(e) = \{\alpha_{r+3}, \ldots, \alpha_l\}$ for $1 \le r \le l-3$ and $\lambda_*(e) = \phi$ for $r = l-2$ and $l-1$. It is easy to get that for $1 \le r \le l-2$,

$$|WeW| = \frac{(2^l l!)^2}{(r+2)!(2^{l-r-2}(l-r-2)!)^2} = 4^{r+2}\binom{l}{r+2}^2(r+2)!. \tag{8}$$

and for $r = l-1$, $\lambda^*(e) = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, e is the identity element and so

$$|WeW| = |W| = 2^l l!. \tag{9}$$

Therefore, from (5)–(9) and Theorem 1.1, we have

$$|R| =1 + 4(l-1)^2l^2 + \sum_{r=1}^{l-2} 4^{r+2}(r+2)\binom{l}{r+2}^2(r+2)! + 2^{l+1}l\cdot l!+$$

$$\sum_{r=1}^{l-2} 4^{r+2}\binom{l}{r+2}^2 (r+2)! + 2^l l!$$
$$= \sum_{r=0}^{l} 4^r \binom{l}{r}^2 (r+1)! - 20l^4 + 40l^3 - 28l^2 + 2^{l+1} l \cdot l! + 2^l l!.$$

(d) Type D_l: $J_0 = \{\alpha_1, \alpha_3, \ldots, \alpha_l\}$. It follows from Theorem 3.1 that

$$\begin{aligned}\lambda^*(\Lambda \setminus \{0\}) =& \{\phi, \{\alpha_2\}, \{\alpha_2, \alpha_3\}, \ldots, \{\alpha_2, \alpha_3, \ldots, \alpha_{l-2}\},\\ &\{\alpha_2, \ldots, \alpha_{l-2}, \alpha_{l-1}\}, \{\alpha_2, \ldots, \alpha_{l-2}, \alpha_l\}, \{\alpha_2, \ldots, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\},\\ &\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_2, \alpha_3\}, \ldots, \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{l-2}\},\\ &\{\alpha_1, \ldots, \alpha_{l-2}, \alpha_{l-1}\}, \{\alpha_1, \ldots, \alpha_{l-2}, \alpha_l\}, \{\alpha_1, \ldots, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\}\}.\end{aligned}$$

If $e = e_0$ is the minimal idempotent in $\Lambda \setminus \{0\}$, we get

$$|We_0W| = \frac{|W|^2}{|W_{J_0}|^2} = \frac{|W(D_l)|^2}{|W(A_1) \times W(D_{l-2})|^2} = \frac{(2^{l-1} l!)^2}{(2!\, 2^{l-3}\, (l-2)!)^2} = 4(l-1)^2 l^2. \tag{10}$$

For $1 \le r \le l-5$ and $\lambda^*(e) = \{\alpha_2, \alpha_3, \ldots, \alpha_{r+1}\}$, by Theorem 3.1, we have $\lambda_*(e) = \{\alpha_{r+3}, \ldots, \alpha_l\}$. So, $W_{\lambda^*(e)} \cong W(A_r)$ and $W_{\lambda_*(e)} \cong W(D_{l-r-2})$, where $D_3 \cong A_3$ when $r = l-5$. Thus, $|W_{\lambda^*(e)}| = (r+1)!$ and $|W_{\lambda_*(e)}| = 2^{l-r-3}(l-r-2)!$. Therefore,

$$|WeW| = \frac{(2^{l-1} l!)^2}{(r+1)!(2^{l-r-3}(l-r-2)!)^2} = 4^{r+2}(r+2)\binom{l}{r+2}^2 (r+2)!. \tag{11}$$

For $r = l-4$ and $\lambda^*(e) = \{\alpha_2, \ldots, \alpha_{l-3}\}$, we have $\lambda_*(e) = \{\alpha_{l-1}, \alpha_l\}$. So, $W_{\lambda^*(e)} \cong W(A_{l-4})$ and $W_{\lambda_*(e)} \cong W(A_1) \times W(A_1)$. It follows that $|W_{\lambda^*(e)}| = (l-3)!$ and $|W_{\lambda_*(e)}| = 4$. Therefore,

$$|WeW| = \frac{(2^{l-1} l!)^2}{(l-3)!\, 4^2} = 2^{2l-6}(l-2)(l-1)l \cdot l!. \tag{12}$$

For $r = l-3$ and $\lambda^*(e) = \{\alpha_2, \ldots, \alpha_{l-2}\}$, we have $\lambda_*(e) = \phi$. So, $W_{\lambda^*(e)} \cong W(A_{l-3})$ and $W_{\lambda_*(e)} \cong 1$. We have $|W_{\lambda^*(e)}| = (l-2)!$ and $|W_{\lambda_*(e)}| = 1$. Therefore,

$$|WeW| = \frac{(2^{l-1} l!)^2}{(l-2)!} = 2^{2l-2}(l-1)l \cdot l!. \tag{13}$$

For $r = l-2$, it follows that $\lambda^*(e) = \{\alpha_2, \ldots, \alpha_{l-2}, \alpha_{l-1}\}$ or $\lambda^*(e) = \{\alpha_2, \ldots, \alpha_{l-2}, \alpha_l\}$, we have both $\lambda_*(e) = \phi$, $W_{\lambda^*(e)} \cong W(A_{l-2})$ and $W_{\lambda_*(e)} \cong 1$. Thus, $|W_{\lambda^*(e)}| = (l-1)!$ and $|W_{\lambda_*(e)}| = 1$. Therefore,

$$|WeW| = \frac{(2^{l-1} l!)^2}{(l-1)!} = 2^{2l-2} l \cdot l!. \tag{14}$$

For $r = l-1$, $\lambda^*(e) = \{\alpha_2, \ldots, \alpha_l\}$, we have $\lambda_*(e) = \phi$. So, $W_{\lambda^*(e)} \cong W(D_{l-1})$ and $W_{\lambda_*(e)} \cong 1$. Thus, $|W_{\lambda^*(e)}| = 2^{l-2}(l-1)!$ and $|W_{\lambda_*(e)}| = 1$. Hence,

$$|WeW| = \frac{(2^{l-1} l!)^2}{2^{l-2}(l-1)!} = 2^l l \cdot l!. \tag{15}$$

The argument is similar for the remaining cases. It is easy to find that:

For $1 \le r \le l-5$ and $\lambda^*(e) = \{\alpha_1, \dots, \alpha_{r+1}\}$, by Theorem 3.1, we have $\lambda_*(e) = \{\alpha_{r+3}, \dots, \alpha_l\}$, and

$$|WeW| = \frac{(2^{l-1}l!)^2}{(r+2)!(2^{l-r-3}(l-r-2)!)^2} = 4^{r+2}\binom{l}{r+2}^2 (r+2)!. \tag{16}$$

For $r = l-4$ and $\lambda^*(e) = \{\alpha_1, \dots, \alpha_{l-3}\}$, we have $\lambda_*(e) = \{\alpha_{l-1}, \alpha_l\}$, and

$$|WeW| = \frac{(2^{l-1}l!)^2}{(l-2)!\,4^2} = 2^{2l-6}(l-1)l \cdot l!. \tag{17}$$

For $r = l-3$ and $\lambda^*(e) = \{\alpha_1, \dots, \alpha_{l-2}\}$, we have $\lambda_*(e) = \phi$, and

$$|WeW| = \frac{(2^{l-1}l!)^2}{(l-1)!} = 2^{2l-2}l \cdot l!. \tag{18}$$

For $r = l-2$, it follows that $\lambda^*(e) = \{\alpha_1, \dots, \alpha_{l-2}, \alpha_{l-1}\}$ or $\lambda^*(e) = \{\alpha_1, \dots, \alpha_{l-2}, \alpha_l\}$, and for both of the cases, $\lambda_*(e) = \phi$,

$$|WeW| = \frac{(2^{l-1}l!)^2}{l!} = 2^{2l-2}l!. \tag{19}$$

For $r = l-1$, $\lambda^*(e) = \{\alpha_1, \dots, \alpha_l\}$. Then e is the identity element in $\Lambda \setminus \{0\}$, and hence

$$|WeW| = |W| = 2^{l-1}l!. \tag{20}$$

Therefore, by (10)–(20), and Theorem 1.1, we have

$$\begin{aligned}
|R| =& 1 + 4(l-1)^2l^2 + \sum_{r=1}^{l-5} 4^{r+2}(r+2)\binom{l}{r+2}^2(r+2)! + 2^{2l-6}(l-2)(l-1)l \cdot l! + \\
& 2^{2l-2}(l-1)l \cdot l! + 2 \times 2^{2l-2}l \cdot l! + 2^l l \cdot l! + \sum_{r=1}^{l-5} 4^{r+2}\binom{l}{r+2}^2(r+2)! + \\
& 2^{2l-6}(l-1)l \cdot l! + 2^{2l-2}l \cdot l! + 2 \times 2^{2l-2}l! + 2^{l-1}l! \\
=& \sum_{r=0}^{l} 4^r\binom{l}{r}^2(r+1)! - 20l^4 + 40l^3 - 28l^2 - 2^{2l-1}(l+1)! + 2^l l \cdot l! + 2^{l-1}l!.
\end{aligned}$$

(e_6) Type E_6: $J_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. It follows from Theorem 3.1 that

$$\begin{aligned}
\lambda^*(\Lambda \setminus \{0\}) =& \{\phi, \{\alpha_6\}, \{\alpha_6, \alpha_3\}, \{\alpha_6, \alpha_3, \alpha_2\}, \{\alpha_6, \alpha_3, \alpha_4\}, \\
& \{\alpha_6, \alpha_3, \alpha_2, \alpha_1\}, \{\alpha_6, \alpha_3, \alpha_4, \alpha_5\}, \{\alpha_6, \alpha_3, \alpha_2, \alpha_4\}, \\
& \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_1\}, \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_5\}, \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_1, \alpha_5\}\}.
\end{aligned}$$

If $e = e_0$ is the minimal idempotent in $\Lambda \setminus \{0\}$, we have

$$|We_0W| = \frac{|W|^2}{|W_{J_0}|^2} = \frac{|W(E_6)|^2}{|W(A_5)|^2} = \frac{(2^7 3^4 5)^2}{(6!)^2} = 2^6 3^4. \tag{21}$$

For $\lambda^*(e) = \{\alpha_6\}$, by Theorem 3.1 we have $\lambda_*(e) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$. So, $W_{\lambda^*(e)} \cong W(A_1)$ and $W_{\lambda_*(e)} \cong W(A_2) \times W(A_2)$. Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{2!(3!)^4} = 2^9 3^4 5^2. \tag{22}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3\}$, we have $\lambda_*(e) = \{\alpha_1, \alpha_5\}$. So, $W_{\lambda^*(e)} \cong W(A_2)$ and $W_{\lambda_*(e)} \cong W(A_1) \times W(A_1)$, Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{3!\, 4^2} = 2^9 3^7 5^2. \tag{23}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2\}$, this case is the same as $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_4\}$. By Theorem 3.1, we have $\lambda_*(e) = \{\alpha_5\}$. So, $W_{\lambda^*(e)} \cong W(A_3)$ and $W_{\lambda_*(e)} \cong W(A_1)$, Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{4!(2!)^2} = 2^9 3^7 5^2. \tag{24}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2, \alpha_1\}$, this case is the same as $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_4, \alpha_5\}$. By Theorem 3.1, we have $\lambda_*(e) = \{a_5\}$. So, $W_{\lambda^*(e)} \cong W(A_4)$ and $W_{\lambda_*(e)} \cong W(A_1)$. Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{5!\, 2^2} = 2^9 3^7 5. \tag{25}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2, \alpha_4\}$, by Theorem 3.1, we have $\lambda_*(e) = \phi$. So, $W_{\lambda^*(e)} \cong W(D_4)$ and $W_{\lambda_*(e)} \cong 1$. Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{2^3 4!} = 2^8 3^7 5^2. \tag{26}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_1\}$, this case is the same as $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_5\}$. By Theorem 3.1, we have $\lambda_*(e) = \phi$. So, $W_{\lambda^*(e)} \cong W(D_5)$ and $W_{\lambda_*(e)} \cong 1$. Therefore,

$$|WeW| = \frac{(2^7 3^4 5)^2}{2^4 5!} = 2^7 3^7 5. \tag{27}$$

For $\lambda^*(e) = \{\alpha_6, \alpha_3, \alpha_2, \alpha_4, \alpha_1, \alpha_5\}$, e is the identity element, and hence

$$|WeW| = |W(E_6)| = 2^7 3^4 5. \tag{28}$$

It follows from (21)–(28), and Theorem 1.1 that

$$|R| = 113068225 = 5^2 \times 4522729.$$

The proof for the case E_6 is completed. □

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