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On Fuzzy Interior Ideals in Semigroups

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Abstract The concept of quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set is a generalization of the quasi-coincidence of a fuzzy point in a fuzzy set. With this new concept, the interval valued $(\in, \in \vee q)$-fuzzy interior ideal in semigroups is introduced. In fact, this kind of new fuzzy interior ideals is a generalization of fuzzy interior ideals in semigroups. In this paper, this kind of fuzzy interior ideals and related properties will be investigated. Moreover, the concept of a fuzzy subgroup with threshold is extended to the concept of an interval valued fuzzy interior ideal with threshold in semigroups.

Keywords fuzzy interior ideals; interval valued $(\in, \in \vee q)$-fuzzy interior ideals; interval valued fuzzy interior ideals with thresholds.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh[16] and has been applied to many branches in mathematics. The notion of fuzzy subgroups was initiated by Rosenfeld[14] after fuzzifying the algebraic structure of groups. On the other hand, Zadeh[17] in 1975 introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of the numbers instead of the numbers. Moreover, Biswas[6] defined the interval valued fuzzy subgroups of the same nature which are of the fuzzy subgroups of Rosenfeld. Thus, a new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced. This kind of fuzzy subgroups was first introduced by Bhakat and Das[4], and they used a combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu[13]. The $(\in, \in \vee q)$-fuzzy subgroup now becomes an important concept and is a useful generalization of the fuzzy subgroup of Rosenfeld. This concept has been further studied by many authors[1−5]. Also, a generalization of Rosenfeld's fuzzy group and Bhakat and Das's fuzzy subgroup was given in Ref. [15].

In the literature, it is well known that the l-semigroups are related with the classical relevant logic, non-classical logic and multi-modal arrow logic, in particular, the complete l-semigroups

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appear in a natural manner in the theory of formal languages and programming, in the theory of fuzzy sets, and in the theory automata. The motivation to study the finite semigroups appeared in 1950s and as a consequence, it promoted the study of linguistic models of computation and reasoning. The concept of fuzzy interior ideals in a semigroup was introduced by Hong[7] and he obtained some related properties of such ideals. Recently, Y.B.Jun and others[8] introduced the concept of (α, β)-fuzzy interior ideals in a semigroup and they obtained some important and useful properties. Other important results were obtained in Refs. [9–10, 12, 18].

In this paper, the quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set is a generalized to the quasi-coincidence of a fuzzy point in a fuzzy set. By using this idea, the notion of an interval valued $(\in, \in \vee q)$-fuzzy interior ideal in semigroups is defined, and consequently, some related properties are investigated. Moreover, we also extend the concept of fuzzy subgroups with thresholds to the concept of interval valued fuzzy interior ideals with thresholds in semigroups. For terminologies and notions not mentioned in this paper, we refer to Ref. [11].

2. Preliminaries

Let S be a semigroup. A subsemigroup of S is a non-empty subset G of S such that $G \times G \subseteq G$. A subsemigroup G of a semigroup S is called an interior ideal of S if $SGS \subseteq S$.

Definition 2.1[10] *A fuzzy left ideal F of a semigroup S is called a fuzzy interior ideal if it satisfies:*

(i) $F(xy) \geq \min\{F(x), F(y)\}$, *for all* $x, y \in S$;

(ii) $F(xay) \geq F(a)$, *for all* $a, x, y \in S$.

For any fuzzy set F of S and $t \in (0, 1]$, the set $U(F; t) = \{x \in S \mid F(x) \geq t\}$ is called a level subset of S.

Theorem 2.2[7] *A fuzzy set F in a semigroup S is a fuzzy interior ideal of S if and only if each non-empty level subset $U(F; t)(\neq \emptyset)$ is an interior ideal of S.*

A fuzzy set F of a semigroup S of the form

$$F(y) = \begin{cases} t(\neq 0), & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support x and value t. We now denote this fuzzy point by $U(x; t)$ and we say that a fuzzy point $U(x; t)$ belongs to (resp. is quasi-coincident with) a fuzzy set F, written as $U(x; t) \in F$ (resp. $U(x; t)qF$), if $F(x) \geq t$ (resp. $F(x) + t > 1$). If $U(x; t) \in F$ or (resp. and) $U(x; t)qF$, then we write $U(x; t) \in \vee q$(resp. $\in \wedge q$) F. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold. Using the notion of " belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, we obtain the concept of (α, β)-fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$[4]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the $(\in, \in \vee q)$-fuzzy subgroup. The detailed study of the $(\in, \in \vee q)$-fuzzy subgroup has been recently considered by Bhakat[1].

By an interval number $\tilde{a}$ we mean[17] an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ is identified with the number $a \in [0,1]$.

For the interval numbers $\widetilde{a}_i = [a_i^-, a_i^+] \in D[0,1]$, $i \in I$, we define

$$\text{rmax}\{\widetilde{a}_i, \widetilde{b}_i\} = [\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)],$$

$$\text{rmin}\{\widetilde{a}_i, \widetilde{b}_i\} = [\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)],$$

$$\text{rinf}\widetilde{a}_i = [\bigwedge_{i\in I} a_i^-, \bigwedge_{i\in I} a_i^+], \quad \text{rsup}\widetilde{a}_i = [\bigvee_{i\in I} a_i^-, \bigvee_{i\in I} a_i^+]$$

and put

(1) $\widetilde{a}_1 \leq \widetilde{a}_2 \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$;

(2) $\widetilde{a}_1 = \widetilde{a}_2 \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$;

(3) $\widetilde{a}_1 < \widetilde{a}_2 \Longleftrightarrow \widetilde{a}_1 \leq \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$;

(4) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

It is clear that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $0 = [0,0]$ as the least element and $1 = [1,1]$ as the greatest element.

By an interval valued fuzzy set F on X, we mean[17] the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Put $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$. Then, we see that $F = \{(x, \widetilde{\mu_F}(x)) \mid x \in X\}$, where $\widetilde{\mu_F} : X \to D[0,1]$.

3. Interval valued $(\in, \in \vee q)$-fuzzy interior ideals

On the basis of Refs. [1–5], we can extend the concept of quasi-coincidence of fuzzy point in a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set as follows.

An interval valued fuzzy set F of a semigroup S of the form

$$\widetilde{\mu_F}(y) = \begin{cases} \widetilde{t}(\neq [0,0]), & \text{if } y = x, \\ [0,0], & \text{if } y \neq x \end{cases}$$

is called a fuzzy interval value with support x and interval value $\widetilde{t}$, denoted by $U(x;\widetilde{t})$. Now, we call a fuzzy interval value $U(x;\widetilde{t})$ belonging to (resp. quasi-coincident with) an interval valued fuzzy set F, written as $U(x;\widetilde{t}) \in F$ (resp. $U(x;\widetilde{t})qF$) if $\widetilde{\mu_F}(x) \geq \widetilde{t}$ (resp. $\widetilde{\mu_F}(x) + \widetilde{t} > [1,1]$). If $U(x;\widetilde{t}) \in F$ or (resp. and) $U(x;\widetilde{t})qF$, then we write $U(x;\widetilde{t}) \in \vee q$(resp. $\in \wedge q$) F. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not necessarily hold.

In what follows, let S be a semigroup unless it is otherwise specified. Also we emphasize that $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$ must satisfy the following properties:

$$[\mu_F^-(x), \mu_F^+(x)] < [0.5, 0.5] \quad \text{or} \quad [0.5, 0.5] \leq [\mu_F^-(x), \mu_F^+(x)],$$

for all $x \in S$.

We first extend the concept of fuzzy interior ideals to the concept of interval valued fuzzy interior ideals of S as follows:

Definition 3.1 *An interval valued fuzzy set F of S is said to be an interval valued fuzzy interior ideal of S if for all $x, y \in S$, the following conditions hold:*

(F1) $\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$;

(FF1) $\widetilde{\mu_F}(xay) \geq \widetilde{\mu_F}(a)$.

Let F be an interval valued fuzzy set. Then, for every $t \in (0,1]$, the set $U(F;\widetilde{t}) = \{x \in S | \widetilde{\mu_F}(x) \geq \widetilde{t}\}$ is called the level subset of F.

Now, we characterize interval valued fuzzy interior ideals by using their level interior ideals.

Theorem 3.2 *An interval valued fuzzy set F of S is an interval valued fuzzy interior ideal of S if and only if for any $[0,0] < \widetilde{t} \leq [1,1]$, $U(F;\widetilde{t})(\neq \emptyset)$ is an interior ideal of S.*

Proof The proof is similar to that of Theorem 2.2[7].

Further, we define the following concept:

Definition 3.3 *An interval valued fuzzy set F of S is said to be an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S if for all $t, r \in (0,1]$ and $x, y \in S$, the following conditions hold:*

(F2) $U(x;\widetilde{t}) \in F$ *and* $U(y;\widetilde{r}) \in F$ *imply* $U(xy; \text{rmin}\{\widetilde{t}, \widetilde{r}\}) \in \vee qF$;

(FF2) $U(a;\widetilde{r}) \in F$ *implies* $U(xay;\widetilde{r}) \in \vee qF$.

Note that if F is an interval valued fuzzy interior ideal of S according to Definition 3.1, then F becomes an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S by Definition 3.3. However, the converse statement is in general not true.

Let $\text{Fuz}(S)$ be the set $S \times [0,1]$ and define a binary operation "$\cdot$" on $\text{Fuz}(S)$ as follows:

$(x, m) \cdot (y, n) = (xy, \min\{m, n\})$, for all $(x, m), (y, n) \in \text{Fuz}(S)$. Then, $(\text{Fuz}(S), \cdot)$ is a semigroup[8].

The following is an example of an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of a semigroup.

Example 3.4 Let $G = \{e, a, b, c\}$ be a Klein' 4-group with the following Cayley table:

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Then it is clear that $(\text{Fuz}(G), \cdot)$ is a semigroup but is not a group[8]. Put $x[0,1] = x \times [0,1]$ for every $x \in G$. Then we know that

$$\text{Fuz}(G) = e[0,1] \cup a[0,1] \cup b[0,1] \cup c[0,1].$$

Define an interval valued fuzzy set F in Fuz(G) as follows:

$$\widetilde{\mu_F}(x) = \begin{cases} [0.6, 0.7], & \text{if } x \in e[0,1], \\ [0.8, 0.9], & \text{if } x \in a[0,1], \\ [0.4, 0.5], & \text{if } x \in b[0,1] \cup c[0,1]. \end{cases}$$

Then F is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of Fuz(G).

Theorem 3.5 *The conditions of (F2) and (FF2) in Definition 3.3, are equivalent to the following conditions, respectively:*

(F3) $\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\}$, *for all* $x, y \in S$*;*

(FF3) $\widetilde{\mu_F}(xay) \geq \text{rmin}\{\widetilde{\mu_F}(a), [0.5, 0.5]\}$, *for all* $x, y, a \in S$.

Proof (F2)$\Longrightarrow$ (F3). Suppose that $x, y \in S$. Then we consider the following two cases:

(a) $\text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} < [0.5, 0.5]$;

(b) $\text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geq [0.5, 0.5]$.

Case (a) Assume that $\widetilde{\mu_F}(xy) < \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\}$. Then, it implies that $\widetilde{\mu_F}(xy) < \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$. Choose t such that $\widetilde{\mu_F}(xy) < \widetilde{t} < \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$. Then $U(x; \widetilde{t}) \in F$ and $U(y; \widetilde{t}) \in F$, but $U(xy; \widetilde{t}) \overline{\in \vee q} F$, which contradicts to (F2).

Case (b) Assume that $\widetilde{\mu_F}(xy) < [0.5, 0.5]$. Then $U(x; [0.5, 0.5]) \in F$ and $U(y; [0.5, 0.5]) \in F$, but $U(xy; [0.5, 0.5]) \overline{\in \vee q} F$, a contradiction. Hence, (F3) holds.

(FF2)$\Longrightarrow$ (FF3). Suppose that $x, y, a \in S$. We consider the following two cases:

(a) $\widetilde{\mu_F}(a) < [0.5, 0.5]$;

(b) $\widetilde{\mu_F}(a) \geq [0.5, 0.5]$.

Case (a) If $\widetilde{\mu_F}(xay) < \widetilde{\mu_F}(a)$, then we take $[0, 0] < \widetilde{t} < [0.5, 0.5]$ such that $\widetilde{\mu_F}(xay) < \widetilde{t} < \widetilde{\mu_F}(a)$. Hence $U(a; \widetilde{s}) \in F$, but $U(xay; \widetilde{t}) \overline{\in \vee q} F$. This clearly contradicts to (FF2). Hence $\widetilde{\mu_F}(xy) \geq \widetilde{\mu_F}(y) = \text{rmin}\{\widetilde{\mu_F}(y), [0.5, 0.5]\}$.

Case (b) Let $\widetilde{\mu_F}(y) \geq [0.5, 0.5]$. If $\widetilde{\mu_F}(xay) < \text{rmin}\{\widetilde{\mu_F}(y), [0.5, 0.5]\}$, then $U(a; [0.5, 0.5]) \in F$, but $U(xy; [0.5, 0.5]) \overline{\in \vee q} F$. This contradicts to (FF2). Hence, (FF3) holds.

(F3)$\Longrightarrow$(F2). Let $U(x; \widetilde{t}) \in F$ and $U(y; \widetilde{r}) \in F$. Then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(y) \geq \widetilde{r}$. Now, we have

$$\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\} \geq \text{rmin}\{\widetilde{t}, \widetilde{r}, [0.5, 0.5]\}.$$

If $\text{rmin}\{\widetilde{t}, \widetilde{r}\} > [0.5, 0.5]$, then $\widetilde{\mu_F}(xy) \geq [0.5, 0.5]$, which implies that $\widetilde{\mu_F}(xy) + \text{rmin}\{\widetilde{t}, \widetilde{r}\} > [1, 1]$. If $\text{rmin}\{\widetilde{t}, \widetilde{r}\} \leq [0.5, 0.5]$, then $\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{t}, \widetilde{r}\}$. Therefore $U(xy; \text{rmin}\{\widetilde{t}, \widetilde{r}\}) \in \vee q F$.

(FF3)$\Longrightarrow$(FF2). Suppose that $U(a; \widetilde{t}) \in F$. Then $\widetilde{\mu_F}(a) \geq \widetilde{t}$. For every $x, y \in S$, we have $\widetilde{\mu_F}(xay) \geq \text{rmin}\{\widetilde{\mu_F}(a), [0.5, 0.5]\} \geq \text{rmin}\{\widetilde{t}, [0.5, 0.5]\}$, which implies $\widetilde{\mu_F}(xay) \geq \widetilde{t}$ or $\widetilde{\mu_F}(xay) \geq [0.5, 0.5]$ according as $\widetilde{t} \leq [0.5, 0.5]$ or $\widetilde{t} > [0.5, 0.5]$. Therefore, $U(xay; \widetilde{t}) \in \vee q F$.

By Definition 3.3 and Theorem 3.5, we immediately obtain the following corollary:

Corollary 3.6 *An interval valued fuzzy set* F *of* S *is an interval valued* $(\in, \in \vee q)$*-fuzzy interior* *ideal of S if and only if the conditions (F3) and (FF3) in Theorem 3.5 hold.*

Theorem 3.7 *For any subset A of S, the characteristic function χ_A of A is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if A is an interior ideal of S.*

Proof Assume that A is an interior ideal of S. Then it can be easily verified that χ_A is an interval valued fuzzy interior ideal of S in the sense of Definition 3.1 and so it is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S.

Conversely, assume that χ_A is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S. Then for every $x, y \in A$, we have $\widetilde{\chi_A}(xy) \geq \text{rmin}\{\widetilde{\chi_A}(x), \widetilde{\chi_A}(y), [0.5, 0.5]\} = [0.5, 0.5]$, and so $xy \in A$. Also, for every $x, y \in S$ and $a \in A$, $\widetilde{\chi_A}(xay) \geq \text{rmin}\{\widetilde{\chi_A}(a), [0.5, 0.5]\} = [0.5, 0.5]$. Thus, this implies that $xay \in A$, and hence A is an interior ideal of S.

Now, we characterize the interval valued $(\in, \in \vee q)$-fuzzy interior ideals by using their level interior ideals.

Theorem 3.8 *Let F be an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S. Then for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, $U(F; \widetilde{t})$ is an empty set or an interior ideal of S. Conversely, if F is an interval valued fuzzy set of S such that $U(F; \widetilde{t})(\neq \emptyset)$ is an interior ideal of S for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, then F is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S.*

Proof Let F be an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S and $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. Let $x, y \in U(F; \widetilde{t})$. Then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(y) \geq \widetilde{t}$. Thus, we have

$$\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\} \geq \text{rmin}\{\widetilde{t}, [0.5, 0.5]\} = \widetilde{t}.$$

This implies that $xy \in U(F; \widetilde{t})$. Now, for every $a \in U(F; \widetilde{t})$ and $x, y \in S$, we have

$$\widetilde{\mu_F}(xay) \geq \text{rmin}\{\widetilde{\mu_F}(a), [0.5, 0.5]\} \geq \text{rmin}\{\widetilde{t}, [0.5, 0.5]\} = \widetilde{t}$$

which implies $xay \in U(F; \widetilde{t})$. Hence $U(F; \widetilde{t})$ is an interior ideal of S.

Conversely, let F be an interval valued fuzzy set of S such that $U(F; \widetilde{t})(\neq \emptyset)$ is an interior ideal of S, for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. For every $x, y \in S$, we now write

$$\widetilde{\mu_F}(x) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\} = \widetilde{t_0}, \quad \widetilde{\mu_F}(y) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\} = \widetilde{t_0},$$

then $x, y \in U(F; \widetilde{t_0})$, and so $xy \in U(F; \widetilde{t_0})$. Hence $\widetilde{\mu_F}(xy) \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\}$.

Also, we have $\widetilde{\mu_F}(a) \geq \text{rmin}\{\widetilde{\mu_F}(a), [0.5, 0.5]\} = \widetilde{s_0}$. Hence $a \in U(F; \widetilde{s_0})$. For every $x, y \in S$, we have $xay \in U(F; \widetilde{s_0})$. This leads to $\widetilde{\mu_F}(xay) \geq \widetilde{s_0} = \text{rmin}\{\widetilde{\mu_F}(a), [0.5, 0.5]\}$. Therefore, F is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S.

Naturally, a corresponding result can be deduced if $U(F; \widetilde{t})$ is an interior ideal of S, for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.

Theorem 3.9 *Let F be an interval valued fuzzy set of S. Then $U(F; \widetilde{t})(\neq \emptyset)$ is an interior ideal of S for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$ if and only if, for all $x, y, a \in S$,*

(F4) $\text{rmax}\{\widetilde{\mu_F}(xy), [0.5, 0.5]\} \geq \text{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$;

(FF4) $\text{rmax}\{\widetilde{\mu_F}(xay), [0.5, 0.5]\} \geq \widetilde{\mu_F}(a)$.

Proof Assume that $U(F;\widetilde{t})(\neq \emptyset)$ is an interior ideal of S. Suppose that $\text{rmax}\{\widetilde{\mu_F}(xy),[0.5,0.5]\} < \text{rmin}\{\widetilde{\mu_F}(x),\widetilde{\mu_F}(y)\} = \widetilde{t}$, for some $x,y \in S$. Then $[0.5,0.5] < \widetilde{t} \leq [1,1]$, $\widetilde{\mu_F}(xy) < \widetilde{t}$, and $x,y \in U(F;\widetilde{t})$. Since $x,y \in U(F;\widetilde{t})$ and $U(F;\widetilde{t})$ is an interior ideal, we have $xy \in U(F;\widetilde{t})$ or $\widetilde{\mu_F}(xy) \geq \widetilde{t}$, which contradicts to $\widetilde{\mu_F}(xy) < \widetilde{t}$. Hence, (F4) holds.

If there exist $x,y,a \in S$ such that $\text{rmax}\{\widetilde{\mu_F}(xay),[0.5,0.5]\} < \widetilde{\mu_F}(a) = \widetilde{t}$, then $[0.5,0.5] < \widetilde{t} < [1,1]$, $\widetilde{\mu_F}(xay) < \widetilde{t}$ and $a \in U(F;\widetilde{t})$. Since $a \in U(F;\widetilde{t})$, we have $xay \in U(F;\widetilde{t})$ or $\widetilde{\mu_F}(xay) \geq t$, which is a contradiction. Hence (FF4) holds.

Conversely, suppose that the conditions (F4) and (FF4) hold. Then, we need to show that $U(F;\widetilde{t})$ is an interior ideal of S. Assume that $[0.5,0.5] < \widetilde{t} < [1,1], x,y \in U(F;\widetilde{t})$ and $a \in S$. Then

$$[0.5,0.5] < \widetilde{t} \leq \text{rmin}\{\widetilde{\mu_F}(x),\widetilde{\mu_F}(y)\} \leq \text{rmax}\{\widetilde{\mu_F}(xy),[0.5,0.5]\} < \widetilde{\mu_F}(xy),$$

$$[0.5,0.5] < \widetilde{t} < \widetilde{\mu_F}(a) \leq \text{rmax}\{\widetilde{\mu_F}(xay),[0.5,0.5]\} < \widetilde{\mu_F}(xay),$$

and hence $xy \in U(F;\widetilde{t})$ and $xay \in U(F;\widetilde{t})$. Thus, $U(F;\widetilde{t})$ is an interior ideal of S.

Let F be an interval valued fuzzy set of a semigroup S and $J = \{t | t \in (0,1]$ and $U(F;\widetilde{t})$ is an empty set or an interior ideal of $S\}$. In particular, if $J = (0,1]$, then F is an ordinary interval valued fuzzy interior ideal of S (Theorem 3.2); if $J = (0,0.5)$, then F is an interval valued $(\in, \in \vee q)$-fuzzy interior ideal of S (Theorem 3.8).

In Ref. [15], Yuan et al gave the definition of a fuzzy subgroup with thresholds. This is in fact a generalization of Rosenfeld's fuzzy subgroup, and Bhkat and Das's fuzzy subgroup. Based on Ref. [15], a fuzzy subgroup with thresholds can be extended to the fuzzy interior ideals with thresholds in the following way:

Definition 3.10 *Let $\alpha,\beta \in [0,1]$ and $\widetilde{\alpha} < \widetilde{\beta}$. Then an interval valued fuzzy set F of S is called an interval valued fuzzy interior ideal with thresholds $(\widetilde{\alpha},\widetilde{\beta})$ of S if for all $x,y,a \in S$, the following conditions hold:*

(F5) $\text{rmax}\{\widetilde{\mu_F}(xy),\widetilde{\alpha}\} \geq \text{rmin}\{\widetilde{\mu_F}(x),\widetilde{\mu_F}(y),\widetilde{\beta}\}$,

(FF5) $\text{rmax}\{\widetilde{\mu_F}(xay),\widetilde{\alpha}\} \geq \text{rmin}\{\widetilde{\mu_F}(a),\widetilde{\beta}\}$.

Now, we characterize the interval valued fuzzy interior ideals with thresholds by using their level interior ideals.

Theorem 3.11 *An interval valued fuzzy set F of S is an interval valued fuzzy interior ideal with thresholds $(\widetilde{\alpha},\widetilde{\beta})$ of S if and only if $U(F;\widetilde{t})(\neq \emptyset)$ is an interior ideal of S, for all $\widetilde{\alpha} < \widetilde{t} \leq \widetilde{\beta}$.*

Proof The proof is similar to those of Theorems 3.8 and 3.9. We hence omit the details.

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