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Constructing Exact Solutions for Two Nonlinear Systems

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Abstract Based on the computerized symbolic, a new generalized tanh functions method is used for constructing exact travelling wave solutions of nonlinear partial differential equations (PDES) in a unified way. The main idea of our method is to take full advantage of an auxiliary ordinary differential equation which has more new solutions. At the same time, we present a more general transformation, which is a generalized method for finding more types of travelling wave solutions of nonlinear evolution equations (NLEEs). More new exact travelling wave solutions to two nonlinear systems are explicitly obtained.

Keywords generalized tanh functions method; solitary wave solution; $(2+1)$-dimensional dispersive long-wave system (DLWs); reaction-diffusion equations.

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1. Introduction

In recent yeas, directly serching for exact solutions of nonlinear PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple. A number of methods have been presented, such as inverse scattering method[1], Hirota's bilinear method[2], the truncated Painleve expansion[3], homogeneous balanced method[4], the hyperbolic tangent function series method[5], the sine-cosine method[6], the Jacobi elliptic function expansion method[7] etc. One of the most effectively straightforward method for constructing exact solutions of PDEs is the extended tanh function method[8]. The purpose of this paper is to present a new extended tanh-function method and to solve the $(2+1)$-dimensional dispersive long-wave system (DLWs) and the reaction-diffusion equations.

2. The simple introduction of the algebraic method

The main idea of our method is to use the solutions of an auxiliary ordinary differential equation to replace $\tanh(\xi)$ in tanh-method. The desired auxiliary equation reads:

$$\varphi_{\xi}{}^2 = a\varphi^2 + b\varphi^3 + c\varphi^4, \tag{1}$$

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where a, b, c are real parameters. Eq.(1) admits the following solutions:

$$\varphi_1 = \frac{2a \sec h(\sqrt{a}\xi)}{\sqrt{4ac-b^2}\tanh(\sqrt{a}\xi) - b\sec h(\sqrt{a}\xi)}, \qquad a>0 \quad \text{and} \quad 4ac-b^2>0,$$

$$\varphi_2 = \frac{2a \sec h\,(\sqrt{a}\xi)}{2\,a\tanh\,(\sqrt{a}\xi) - b\sec h\,(\sqrt{a}\xi) + 2\,a}, \qquad a>0,$$

$$\varphi_3 = -\frac{2a \sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right) + b\sec\left(\sqrt{-a}\xi\right)}, \quad a<0 \quad \text{and} \quad b^2-4ac>0,$$

$$\varphi_4 = \frac{2a \sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac} - b\sec\left(\sqrt{-a}\xi\right)}, \qquad a<0 \quad \text{and} \quad b^2-4ac>0,$$

$$\varphi_5 = -\frac{2a \sec\left(\sqrt{-a}\xi\right)}{b\tan\left(\sqrt{-a}\xi\right) - b\sec\left(\sqrt{-a}\xi\right) + 2\sqrt{-\frac{c}{a}}}, \quad a<0 \quad \text{and} \quad c>0.$$

For given partial differential equations (PDEs) with $u(x,y,t)$ and $v(x,y,t)$ in three independent variables x, y, t

$$\begin{cases} H(u,v,u_t,v_t,u_x,v_x,u_y,v_y,u_{xx},v_{xx},...)=0, \\ F(u,v,u_t,v_t,u_x,v_x,u_y,v_y,u_{xx},v_{xx},...)=0. \end{cases} \tag{2}$$

By using travelling transform $u(x,y,t)=U(\xi)$, $v(x,y,t)=v(\xi)$ and $\xi = x+ly+\lambda t$, where l, λ are constants to be determined later, the PDEs.(2) reduces into ordinary differential equations (ODEs)

$$\begin{cases} H_1(U,V,U',V',U'',V'',U''',V'''...)=0, \\ F_1(U,V,U',V',U'',V'',U''',V'''...)=0. \end{cases} \tag{3}$$

We seek its solutions in the form

$$\begin{cases} U = a_0 + \sum\limits_{i=1}^{n_1}(a_i\varphi^i + b_i\dfrac{(\varphi')^i}{\varphi^i}), \\ V = A_0 + \sum\limits_{i=1}^{n_2}(A_i\varphi^i + B_i\dfrac{(\varphi')^i}{\varphi^i}), \end{cases} \tag{4}$$

where $\varphi(\xi)$ satisfies Eq.(1), and $a_0, A_0, a_i, b_i, A_i, B_i$ are constants to be determined later.

The values of n_1,n_2 are obtained by balancing the highest nonlinear terms and the highest-order partial differential terms in Eqs.(3) or Eqs.(2). Substituting Eqs.(4) along with Eq.(1) into Eqs.(3), collecting coefficients of polynomials of φ^i and $\varphi^i(\sqrt{a\varphi^2+b\varphi^3+c\varphi^4})^j$ (i=0,1,2,. . . , j=0,1), and setting each coefficient to be zero, we get a system of algebraic equations. Using Maple to solve the above algebraic equations, one can easily find the exact solutions.

In the following we illustrate the method by considering the reaction-diffusion system and the $(2+1)$-dimensional dispersive long-wave system.

3. The reaction-diffusion system

The reaction-diffusion system reads:

$$\begin{cases} u_t - du_{xx} - u^2v + pu = 0, \\ v_t - dv_{xx} + u^2v - pu = 0. \end{cases} \tag{5}$$

We make the travelling transformation $u = U(\xi), v = V(\xi)$ and $\xi = x + \lambda t$. Then Eqs.(5) reduces to

$$\begin{cases} \lambda U' - dU'' - U^2V + pU = 0, \\ \lambda V' - dV'' + U^2V - pU = 0. \end{cases} \tag{6}$$

Balancing the U_{xx} with U^2V and V_{xx} with U^2V, we can set the leading order $n_1 = 1, n_2 = 1$. Therefore, we may choose the following ansätz

$$\begin{cases} U = a_0 + a_1\varphi(\xi) + b_1\dfrac{\varphi_\xi}{\varphi}, \\ V = A_0 + A_1\varphi(\xi) + B_1\dfrac{\varphi_\xi}{\varphi}, \end{cases} \tag{7}$$

where $\xi = x + \lambda t$, and $a_0, a_1, b_1, A_0, A_1, B_1, \lambda$ are arbitrary constants to be determined later. Substituting Eqs.(7) into Eqs.(6) along with Eq.(1) and setting the coefficients of φ^i and

$$\varphi^i\left(\sqrt{a\varphi^2 + b\varphi^3 + c\varphi^4}\right)^j, \quad i = 0, 1, 2, \ldots, j = 0, 1$$

to zero, we get the following set of algebraic equations namely:

$$-2\,pa_0 + 4\,a_0b_1B_1a + 2\,{a_0}^2A_0 + 2\,{b_1}^2A_0a = 0,$$
$$4\,a_1b_1B_1c + 2\,{a_1}^2A_1 - 4\,dA_1c + 2\,{b_1}^2A_1c = 0,$$
$$2\,{a_0}^2B_1a - 2\,pb_1a + 4\,a_0b_1A_0a + 2\,{b_1}^2B_1a^2 = 0,$$
$$4\,ca_1b_1A_1 + 2\,c^2{b_1}^2B_1 - 4\,c^2dB_1 + 2\,c{a_1}^2B_1 = 0,$$
$$-2\,c^2{b_1}^2B_1 - 2\,c{a_1}^2B_1 - 4\,ca_1b_1A_1 - 4\,c^2db_1 = 0,$$
$$-2\,{a_0}^2A_0 - 2\,{b_1}^2A_0a - 4\,a_0b_1B_1a + 2\,pa_0 = 0,$$
$$-4\,a_1b_1B_1c - 4\,da_1c - 2\,{b_1}^2A_1c - 2\,{a_1}^2A_1 = 0,$$
$$2\,pb_1a - 4\,a_0b_1A_0a - 2\,{b_1}^2B_1a^2 - 2\,{a_0}^2B_1a = 0,$$
$$4\,a_0a_1B_1a + 4\,{b_1}^2B_1ab - dB_1ab - 2\,pb_1b + 2\,\lambda\,A_1a + 4\,a_0b_1A_1a +$$
$$2\,{a_0}^2B_1b + 4\,a_0b_1A_0b + 4\,a_1b_1A_0a = 0,$$
$$4\,a_0b_1A_1b + 2\,{b_1}^2B_1b^2 + 4\,a_0a_1B_1b + 2\,{a_1}^2B_1a - dB_1b^2 - 4\,cdB_1a +$$
$$4ca_0b_1A_0 - 2\,cpb_1 + 2\,\lambda\,A_1b + 4\,c{b_1}^2B_1a + 4\,a_1b_1A_0b + 2\,c{a_0}^2B_1 + 4\,a_1b_1A_1a = 0,$$
$$2c\lambda\,A_1 - 5\,cdB_1b + 4\,ca_0a_1B_1 + 4\,ca_1b_1A_0 + 4\,c{b_1}^2B_1b + 4\,a_1b_1A_1b +$$
$$2\,{a_1}^2B_1b + 4\,ca_0b_1A_1 = 0,$$
$$4\,a_0b_1B_1b - 2\,pa_1 + 2\,{a_0}^2A_1 + 2\,{b_1}^2A_0b - 2\,dA_1a + \lambda\,B_1b +$$
$$4\,a_0a_1A_0 + 4\,a_1b_1B_1a + 2\,{b_1}^2A_1a = 0,$$
$$2\,{a_1}^2A_0 + 4\,a_0b_1B_1c - 3\,dA_1b + 4\,a_1b_1B_1b + 2\,\lambda\,B_1c + 4\,a_0a_1A_1 + 2\,{b_1}^2A_1b + 2\,{b_1}^2A_0c = 0,$$
$$-2\,{b_1}^2A_1a - 2\,da_1a - 4\,a_0a_1A_0 + 2\,pa_1 + \lambda\,b_1b - 2\,{b_1}^2A_0b -$$
$$4\,a_1b_1B_1a - 2\,{a_0}^2A_1 - 4\,a_0b_1B_1b = 0,$$

$$-4a_0b_1B_1c+2\lambda b_1c-3da_1b-4a_1b_1B_1b-4a_0a_1A_1-2{b_1}^2A_0c-2{a_1}^2A_0-2{b_1}^2A_1b=0,$$
$$-4ca_0b_1A_0-4a_0a_1B_1b-4c{b_1}^2B_1a-4a_1b_1A_1a-4a_0b_1A_1b-2{b_1}^2B_1b^2+2\lambda a_1b-$$
$$4a_1b_1A_0b-2c{a_0}^2B_1-4cdb_1a-db_1b^2-2{a_1}^2B_1a+2cpb_1=0,$$
$$-4a_0a_1B_1a-2{a_0}^2B_1b-4a_0b_1A_0b-4a_0b_1A_1a+2\lambda a_1a-db_1ab-$$
$$4a_1b_1A_0a-4{b_1}^2B_1ab+2pb_1b=0,$$
$$-4ca_0b_1A_1-4ca_0a_1B_1-5cdb_1b-4ca_1b_1A_0+2c\lambda a_1-$$
$$4c{b_1}^2B_1b-4a_1b_1A_1b-2{a_1}^2B_1b=0.$$

With the aid of Maple or Mathematica, we find the following results

Case 1

$$a_1=\frac{\sqrt{2cd}}{2},\ A_1=-\frac{\sqrt{2cd}}{2},\ A_0=\frac{\sqrt{2da}\,(da+p)}{2\,da},\ a_0=\frac{\sqrt{2da}}{2},\ \lambda=-\frac{(da-p)\sqrt{a}}{a},$$
$$B_1=-\frac{\sqrt{2d}}{2},\ b_1=\frac{\sqrt{2d}}{2};$$

Case 2

$$A_1=-\frac{\sqrt{2cd}}{2},\ A_0=\frac{\sqrt{2p+da}}{2},\ \lambda=2\sqrt{2dp+d^2a},\ a_1=\frac{\sqrt{2cd}}{2},\ b_1=-\frac{\sqrt{2d}}{2},$$
$$B_1=\frac{\sqrt{2d}}{2},\ a_0=\frac{\sqrt{4\,p+2\,da}}{2};$$

Therefore, we can get the following solitary wave solutions and periodic solutions to Eqs (9):

$$u_1=\frac{\sqrt{2da}}{2}+\frac{\sqrt{2cd}a\sec h\left(\sqrt{a}\xi\right)}{\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)}+\frac{4\,a^2cd-adb^2}{-\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)+b\sec h\left(\sqrt{a}\xi\right)},$$
$$v_1=\frac{\sqrt{2da}\,(da+p)}{2\,da}-\frac{\sqrt{2cd}a\sec h\left(\sqrt{a}\xi\right)}{\left(\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)\right)}+\frac{8\,a^2cd-2adb^2}{2\sqrt{4\,ac-b^2}\tan h\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)},$$

where $a>0, d>0, c>0$ and $4ac-b^2>0$.

$$u_2=\frac{\sqrt{2da}}{2}+\frac{\sqrt{2cd}a\sec h\left(\sqrt{a}\xi\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a}-\frac{\sqrt{2d}a^{\frac{3}{2}}\left(\tanh\left(\sqrt{a}\xi\right)+1\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a},$$
$$v_2=\frac{\sqrt{2da}\,(da+p)}{2\,da}-\frac{\sqrt{2cd}a\sec h\left(\sqrt{a}\xi\right)}{\left(2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a\right)}+$$
$$\frac{\sqrt{2}\sqrt{d}a^{3/2}\left(\tanh\left(\sqrt{a}\xi\right)+1\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a},$$

where $a>0, d>0$ and $c>0$.

$$u_3=\frac{\sqrt{2da}}{2}-\frac{\sqrt{2cd}a\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)}-\frac{8a^2cd-2adb^2}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)},$$
$$v_3=\frac{\sqrt{2da}\,(da+p)}{2\,da}+\frac{\sqrt{2cd}a\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)}+$$
$$\frac{8a^2cd-2adb^2}{2\left(\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)},$$

where $a<0,d<0,c<0$ and $4ac-b^2<0$.

$$u_4=\frac{\sqrt{2da}}{2}+\frac{\sqrt{2cda}\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)}+\frac{8\,a^2cd-2adb^2\tan\left(\sqrt{-a}\xi\right)}{2\,\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)},$$

$$v_4=\frac{\sqrt{2da}\,(da+p)}{2\,da}-\frac{\sqrt{2cda}\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)}+\frac{8\,a^2cd-2adb^2\tan\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)},$$

where $a<0,d<0,c<0$ and $4ac-b^2<0$. In solutions $u_1\sim u_4, v_1\sim v_4$, $\xi=x-\frac{(da-p)\sqrt{a}}{a}t$.

$$u_5=\frac{\sqrt{4\,p+2\,da}}{2}+\frac{\sqrt{2cda}sech\left(\sqrt{a}\xi\right)}{\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)}+$$
$$\frac{8\,a^2cd-adb^2}{2\,\left(\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)\right)},$$

$$v_5=\frac{\sqrt{2p+da}}{2}-\frac{\sqrt{2}(cd)^{\frac{3}{2}}a\sec h\left(\sqrt{a}\xi\right)}{cd\left(\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)\right)}-$$
$$\frac{8\,a^2cd-adb^2}{2\sqrt{4\,ac-b^2}\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)},$$

where $a>0,d>0,c>0$,$4ac-b^2>0$ and $2p+da>0$.

$$u_6=\frac{\sqrt{4\,p+2\,da}}{2}+\frac{\sqrt{2cd}a\sec h\left(\sqrt{a}\xi\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a}+\frac{\sqrt{2d}a^{\frac{3}{2}}\left(\tanh\left(\sqrt{a}\xi\right)+1\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a},$$

$$v_6=\frac{\sqrt{2p+da}}{2}-\frac{\sqrt{2}cda\sec h\left(\sqrt{a}\xi\right)}{\sqrt{cd}\left(2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a\right)}-\frac{\sqrt{2}\sqrt{d}a^{3/2}\left(\tanh\left(\sqrt{a}\xi\right)+1\right)}{2\,a\tanh\left(\sqrt{a}\xi\right)-b\sec h\left(\sqrt{a}\xi\right)+2\,a},$$

where $a>0,cd>0,4ac-b^2>0$ and $2p+da>0$.

$$u_7=\frac{\sqrt{4\,p+2\,da}}{2}-\frac{\sqrt{2cda}\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)}+$$
$$\frac{8a^2cd-2adb^2}{\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)},$$

$$v_7=\frac{\sqrt{2p+da}}{2}+\frac{\sqrt{2}cda\sec\left(\sqrt{-a}\xi\right)}{\sqrt{cd}\left(b^2-4\,ac\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)}-$$
$$\frac{8a^2cd-2adb^2}{2\,\left(\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)},$$

where $a<0,cd>0,4ac-b^2<0$ and $2p+da>0$.

$$u_8=\frac{\sqrt{4\,p+2\,da}}{2}+\frac{\sqrt{2cda}\sec\left(\sqrt{-a}\xi\right)}{\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)}-\frac{8\,a^2cd-2adb^2\tan\left(\sqrt{-a}\xi\right)}{2\,\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)},$$

$$v_8=\frac{\sqrt{2p+da}}{2}-\frac{\sqrt{2}cda\sec\left(\sqrt{-a}\xi\right)}{\sqrt{cd}\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)}-\frac{8\,a^2cd-2adb^2\tan\left(\sqrt{-a}\xi\right)}{\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)},$$

where $a<0,cd>0,4ac-b^2<0$ and $2p+da>0$, In solutions $u_5\sim u_8, v_5\sim v_8$, $\xi=x+2\sqrt{2dp+d^2a}t$.

4. Exact solutions of $(2+1)$-dimensional dispersive long-wave system (DLWs)[9]

$$\begin{cases} u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \\ v_t + u_x + u_x v + u v_x + u_{xxy} = 0. \end{cases} \tag{8}$$

We assume that Eqs.(8) have the solutions in the form

$$u(x,y,t) = U(\xi),\ v(x,y,t) = V(\xi),\ \xi = x + ly + \lambda t, \tag{9}$$

where l and λ are constants to be determined later. Substituting Eq.(9) into Eqs.(8) yields

$$\begin{cases} \lambda l U_{\xi\xi} + V_{\xi\xi} + l(U_\xi)^2 + lUU_{\xi\xi} = 0, \\ \lambda V_\xi + U_\xi V + UV_\xi + U_\xi + lU_{\xi\xi\xi} = 0. \end{cases} \tag{10}$$

Through balancing $U_{\xi\xi}$ with $(U_\xi)^2$ and UV_ξ with $U_{\xi\xi\xi}$ in Eqs.(10), we get $n_1 = 1, n_2 = 2$. So we assume that

$$\begin{cases} u = f + h\varphi + p\dfrac{\varphi_\xi}{\varphi}, \\ v = F + H\varphi + G\varphi^2 + P\dfrac{\varphi_\xi}{\varphi} + Q\dfrac{{\varphi_\xi}^2}{\varphi^2}, \end{cases} \tag{11}$$

where $\varphi(\xi)$ satisfies Eq.(1) and f, h, p, F, H, G, P, Q are constants to be determined later. Substituting Eqs.(11) into Eqs.(10) along with Eq.(1), collecting coefficients of φ^i and $\varphi^i(\sqrt{a\varphi^2 + b\varphi^3 + c\varphi^4})^j$ $(i = 0, 1, 2, \ldots, j = 0, 1)$, and setting it to be zero, we get a set of over-determined algebraic equations with respect to f, h, p, F, H, G, P, Q and l, λ. Using Maple to solve the above system of algebraic equations, we find

Case 1

$$F = -\frac{4\,Qca + 4\,acl - b^2 l + 4\,c}{4c},\ G = -2\,lc - Qc,\ P = \frac{aQ}{c},\ p = \frac{4l}{c},\ Q = Q,\ \lambda = \lambda,\ l = l,$$

$$H = -\frac{b\,(Qc + lc)}{c},\ f = \frac{-2\,\lambda\,c - \sqrt{c}b}{2c},\ h = -2\,\sqrt{c}.$$

Therefore, we can obtain solitary solutions and periodic solutions of Eqs.(8) as follows

$$u_1 = \frac{-2\,\lambda\,c - \sqrt{c}b}{2c} - \frac{4\sqrt{c}a\mathrm{sec}\,h\,(\sqrt{a}\xi)}{\sqrt{4\,ac - b^2}\tanh(\sqrt{a}\xi) - b\mathrm{sec}\,h\,(\sqrt{a}\xi)} + \frac{4\sqrt{4\,a^2c - ab^2}l}{c\left(-\sqrt{4\,ac - b^2}\tanh(\sqrt{a}\xi) + b\mathrm{sec}\,h\,(\xi)\right)}$$

$$v_1 = -\frac{4\,c + 4\,aQc + 4\,acl - b^2 l}{4c} - 2\,\frac{b\,(lc + Qc)\,asech\,(\sqrt{a}\xi)}{c\left(\sqrt{4\,ac - b^2}\tanh(\sqrt{a}\xi) - bsech\,(\sqrt{a}\xi)\right)} + 4\,\frac{(-Qc - 2\,lc)\,a^2\,(\mathrm{sec}\,h\,(\sqrt{a}\xi))^2}{\left(\sqrt{4\,ac - b^2}\tanh(\sqrt{a}\xi) - bsech\,(\sqrt{a}\xi)\right)^2} - \frac{a\left(-4\,ac + b^2\right)Q}{\left(-\sqrt{4\,ac - b^2}\tan(\sqrt{a}\xi) + b\sec(\sqrt{a}\xi)\right)^2} - \frac{a^{3/2}\sqrt{4\,ac - b^2}Q}{\sqrt{4\,ac - b^2}\tanh(\sqrt{a}\xi) - b\mathrm{sec}\,h\,(\sqrt{a}\xi)},$$

where $a > 0, 4ac - b^2 > 0$ and $c > 0$.

$$u_2 = \frac{-2\,\lambda\,c - \sqrt{c}b}{2c} - \frac{4\sqrt{c}a\mathrm{sec}\,h\,(\sqrt{a}\xi)}{2\,a\tanh(\sqrt{a}\xi) - b\mathrm{sec}\,h\,(\sqrt{a}\xi) + 2\,a} + \frac{8a^{3/2}l\,(\tanh(\sqrt{a}\xi) + 1)}{c\,(-2\,a\tanh(\sqrt{a}\xi) + b\mathrm{sec}\,h\,(\sqrt{a}\xi) - 2\,a)}$$

$$v_2 = -\frac{4\,c+4\,aQc+4\,acl-b^2l}{4c} - \frac{2b\,(lc+Qc)\,a\mathrm{sec}\,h\left(\sqrt{a}\xi\right)}{c\,(2\,a\tanh\left(\sqrt{a}\xi\right)-b\mathrm{sec}\,h\left(\sqrt{a}\xi\right)+2\,a)} - \frac{4\,(Qc+2\,lc)\,a^2\mathrm{sec}\,h^2\left(\sqrt{a}\xi\right)}{\left(2\,a\tanh\left(\sqrt{a}\xi\right)-b\mathrm{sec}\,h\left(\sqrt{a}\xi\right)+2\,a\right)^2} + \frac{4a^3\left(\tanh\left(\sqrt{a}\xi\right)+1\right)^2Q}{\left(-2\,a\tanh\left(\sqrt{a}\xi\right)+b\tanh\left(\sqrt{a}\xi\right)-2\,a\right)^2} - \frac{2Qa^{5/2}\left(\tanh\left(\sqrt{a}\xi\right)+1\right)}{c\,(2\,a\tanh\left(\sqrt{a}\xi\right)-b\mathrm{sec}\,h\left(\sqrt{a}\xi\right)+2\,a)},$$

where $a>0$ and $c>0$.

$$u_3 = \frac{-2\,\lambda\,c-\sqrt{c}b}{2c} + \frac{4\sqrt{c}a\sec\left(\sqrt{-a}\xi\right)}{\sqrt{-4\,ac+b^2}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)} - \frac{4\sqrt{4\,a^2c-ab^2}l}{c\left(\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)},$$

$$v_3 = -\frac{4\,c+4\,aQc+4\,acl-b^2l}{4c} + 2\,\frac{b\,(lc+Qc)\,a\sec\left(\sqrt{-a}\xi\right)}{c\left(\sqrt{-4\,ac+b^2}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)} + \frac{4\,(-Qc-2\,lc)\,a^2\sec^2\left(\sqrt{-a}\xi\right)}{\left(\sqrt{-4\,ac+b^2}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)^2} + \frac{\left(-4\,ac+b^2\right)aQ}{(4\,ac-2\,b^2)\left(\sec\left(\sqrt{-a}\xi\right)\right)^2-2\sqrt{-4\,ac+b^2}\tan\left(\sqrt{-a}\xi\right)\sec\left(\sqrt{-a}\xi\right)b+b^2-4\,ac} - \frac{aQ\sqrt{4a^c-ab^2}}{c\left(\sqrt{b^2-4\,ac}\tan\left(\sqrt{-a}\xi\right)+b\sec\left(\sqrt{-a}\xi\right)\right)},$$

where $a<0, 4ac-b^2<0$ and $c>0$.

$$u_4 = \frac{-2\,\lambda\,c-\sqrt{c}b}{2c} - 4\,\frac{\sqrt{c}a\sec\left(\sqrt{-a}\xi\right)}{\sqrt{-4\,ac+b^2}-b\sec\left(\sqrt{-a}\xi\right)} + \frac{4l\sqrt{4\,a^2c-b^2}\tan\left(\sqrt{-a}\xi\right)}{\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)c},$$

$$v_4 = -\frac{4\,c+4\,aQc+4\,acl-b^2l}{4c} - 2\,\frac{b\,(lc+Qc)\,a\sec\left(\sqrt{-a}\xi\right)}{c\left(\sqrt{-4\,ac+b^2}-b\sec\left(\sqrt{-a}\xi\right)\right)} - \frac{4\,(Qc+2\,lc)\,a^2\sec^2\left(\sqrt{-a}\xi\right)}{\left(\sqrt{-4\,ac+b^2}-b\sec\left(\sqrt{-a}\xi\right)\right)^2} - \frac{\left(-4\,ac+b^2\right)a\left(1-\sec^2\left(\sqrt{-a}\xi\right)\right)Q}{4\,ac-b^2+2\,b\sqrt{-4\,ac-b^2}\sec\left(\sqrt{-a}\xi\right)+b^2\left(\sec\left(\sqrt{-a}\xi\right)\right)^2} + \frac{aQ\sqrt{4a^c-b^2}\tan\left(\sqrt{-a}\xi\right)}{c\left(\sqrt{b^2-4\,ac}-b\sec\left(\sqrt{-a}\xi\right)\right)},$$

where $a<0$ and $c>0$.

$$u_5 = \frac{-2\,\lambda\,c-\sqrt{c}b}{2c} + \frac{4\sqrt{c}a\sec\left(\sqrt{-a}\xi\right)}{\left(b\tan\left(\sqrt{-a}\xi\right)-b\sec\left(\sqrt{-a}\xi\right)+2\sqrt{-\frac{c}{a}}\right)} + \frac{4\left(2\tan\left(\sqrt{-a}\xi\right)\sqrt{-\frac{c}{a}}-b\right)\sqrt{-a}l}{c\left(b\tan\left(\sqrt{-a}\xi\right)-b\sec\left(\sqrt{-a}\xi\right)+2\sqrt{-\frac{c}{a}}\right)},$$

$$v_5 = -\frac{4\,c+4\,aQc+4\,acl-b^2l}{4c} + \frac{2\,b\,(lc+Qc)\,a\sec\left(\sqrt{-a}\xi\right)}{c\left(b\tan\left(\sqrt{-a}\xi\right)-b\sec\left(\sqrt{-a}\xi\right)+2\sqrt{-\frac{c}{a}}\right)} +$$

$$\frac{4\,(-Qc-2\,lc)\,a^2\left(\sec\left(\sqrt{-a}\xi\right)\right)^2}{\left(b\tan\left(\sqrt{-a}\xi\right)-b\sec\left(\sqrt{-a}\xi\right)+2\,\sqrt{-\frac{c}{a}}\right)^2}+$$

$$\frac{\left(-2\,\tan\left(\sqrt{-a}\xi\right)\sqrt{-\frac{c}{a}}+b\right)^2 a^2Q}{(N\sec(\sqrt{-a}\xi)+2\,ab^2\tan(\sqrt{-a}\xi)\sec(\sqrt{-a}\xi)-4\,b\tan(\sqrt{-a}\xi)\sqrt{-\frac{c}{a}}a+4\,c+b^2a)}-$$

$$\frac{\sqrt{-a}aQ\left(2\,\tan\left(\sqrt{-a}\xi\right)\sqrt{-\frac{c}{a}}-b\right)}{c\left(b\tan\left(\sqrt{-a}\xi\right)-b\sec\left(\sqrt{-a}\xi\right)+2\,\sqrt{-\frac{c}{a}}\right)},\quad N=4ba\sqrt{-\frac{c}{a}}-2b^2a,$$

where $a<0$ and $c>0$. In solutions $u_1\sim u_5, v_1\sim v_5$, $\xi=x+ly+\lambda t$.

5. Conclusions

In summary, by giving more types of solutions of auxiliary equations, we obtain more types of exact solutions of the reaction-diffusion system and $(2+1)$-dimensional dispersive long-wave system. These solutions contain solitary waves solutions and periodic solutions. The method can be applied to other nonlinear evolution equations. These new solutions cannot be obtained by the tanh-method and its generalizations. This indicates that our method is a new extension of the tanh-function method which can be used to find the exact travelling wave solutions to other NLEEs.

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