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On a Linear Combination Operator of Neumann-Bessel Series

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Abstract In this paper we construct a new operator $H_{n,r}^{(N,B)}(f;z)$ by means of the partial sums $S_n^{(N,B)}(f;z)$ of Neumann-Bessel series. The operator converges uniformly to any fixed continuous function $f(z)$ on the unit circle $\mid z \mid= 1$ and has the best approximation order for $f(z)$ on $\mid z \mid= 1$.

Keywords Neumann-Bessel series; kernel function; best approximation order; uniform convergence.

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1. Introduction

Let $J_n(z)$ be Bessel functions and let $Q_n(z)$ be Neumann polynomials:

$$J_n(z) = (\frac{z}{2})^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(\frac{z}{2})^{2k}, \quad n = 0, 1, 2, \ldots \tag{1}$$

$$Q_0(z) = \frac{1}{z},$$

$$Q_n(z) = \frac{1}{4}(\frac{2}{z})^{n+1} \sum_{k=0}^{[\frac{n}{2}]} \frac{n(n-k-1)!}{k!}(\frac{z}{2})^{2k}, \quad n = 1, 2, \ldots \tag{2}$$

where $[a]$ is the integer part of number a.

Let Γ be the unit circle $\mid z \mid= 1$ and let $f(z)$ be L-integrabel on Γ, write

$$A_n = \frac{\varepsilon_n}{i\pi} \oint_{\Gamma} f(\zeta)Q_n(\zeta)\mathrm{d}\zeta, \quad B_n = \frac{\varepsilon_n}{i\pi} \oint_{\Gamma} f(\zeta)J_n(\zeta)\mathrm{d}\zeta, \tag{3}$$

$$\varepsilon_0 = \frac{1}{2}, \quad \varepsilon_n = 1, \quad n = 1, 2, \ldots, i = \sqrt{-1}.$$

The series $\sum_{k=0}^{\infty}(A_k J_k(z)+B_k Q_k(z))$, $z \in \Gamma$, is called the Neumann-Bessel series. Let $S_n^{(N,B)}(f;z)$ denote the n-th partial sums of Neumann-Bessel series, i.e.,

$$S_n^{(N,B)}(f;z) = \sum_{k=0}^{n}(A_k J_k(z) + B_k Q_k(z)) = \frac{1}{2\pi i} \oint_{\Gamma} f(\zeta)K_n^{(N,B)}(z,\zeta)\mathrm{d}\zeta, \quad z \in \Gamma, \tag{4}$$

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where

$$K_n^{(N,B)}(z,\zeta) = Q_0(\zeta)J_0(z) + Q_0(z)J_0(\zeta) + 2\sum_{k=0}^{n}(Q_k(\zeta)J_k(z) + Q_k(z)J_k(\zeta)) \tag{5}$$

are called the kernel functions.

Since $S_n^{(N,B)}(f;z)$ cannot converge uniformly to each continuous function $f(z)$ on the unit circle $\mid z \mid= 1$. Mu[3] considered Fejér sums of $S_n^{(N,B)}(f;z)$ as follows:

$$\sigma_n^{(N,B)}(f;z) = \frac{1}{n}\sum_{k=0}^{n-1} S_k^{(N,B)}(f;z),$$

and obtained an asymptotic formula as follows

Theorem *Let $f(z)$ be a function of bounded variation on Γ and let Z_0 be a point in Γ. If the two one-side derivatives $f'_+(z_0)$ and $f'_-(z_0)$ of $f(z)$ at Z_0 exist, then we have*

$$\sigma_n^{(N,B)}(f;z_0) - f(z_0) = \frac{2iz_0 \ln n}{n\pi}(f'_+(z_0) - f'_-(z_0)) + O(\frac{\ln n}{n}), \quad n \to \infty, \tag{6}$$

where

$$f'_+(z_0) = \lim_{z\to z_0, z\in\Gamma} \frac{f(z) - f(z_0)}{z - z_0} \quad (clockwise),$$
$$f'_-(z_0) = \lim_{z\to z_0, z\in\Gamma} \frac{f(z) - f(z_0)}{z - z_0} \quad (count\ clockwise).$$

Fomula (6) shows that the convergence order of $\sigma_n^{(N,B)}(f;z)$ does not reach the order of the best approximation for any fixed continuous function $f(z)$ on Γ. In the paper we use $S_n^{(N,B)}(f;z)$ to construct a new operator $H_{n,r}^{(N,B)}(f;z)$ which converges to any fixed continuous function $f(z)$ on Γ uniformly and has the best approximation order for $f(z)$ on Γ. $H_{n,r}^{(N,B)}(f;z)$ is determined as follows. Let r be an arbitrary odd natural number and let $h = \frac{\pi}{n+1}$. Then $H_{n,r}^{(N,B)}(f;z)$ is defined by

$$H_{n,r}^{(N,B)}(f;z) = S_n^{(N,B)}(f;z) - (-\frac{1}{4})^{\frac{r+1}{2}}\sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}S_n^{(N,B)}(f;ze^{i(k-\frac{r+1}{2})h}). \tag{7}$$

We have the following result concerning $H_{n,r}^{(N,B)}(f;z)$.

Theorem 1 *Let $f(z)$ be a continuous function on Γ. Then*

$$\mid H_{n,r}^{(N,B)}(f;z) - f(z) \mid= O(\frac{1}{n} + \omega(f,\frac{1}{n})), \quad z \in \Gamma,$$

where "O" is independent of n and $\omega(f,\delta)$ is the modulus of continuity of $f(z)$ on Γ.

For any fixed continuous function $f(z)$ on Γ, Theorem 1 shows that the convergence order of $H_{n,r}^{(N,B)}(f;z)$ reaches the order of the best approximation for the function $f(z)$. In addition, by Theorem 1, the following convergence theorem holds.

Theorem 2 *Let $f(z)$ be a continuous function on Γ. Then $\lim_{n\to\infty} H_{n,r}^{(N,B)}(f;z) = f(z)$, $z \in \Gamma$, is valid uniformly on Γ.*

2. Some formulas

Since $J_0(z)+2\sum_{j=1}^{\infty}J_{2j}(z)=1,\ J_n(z)=O(\frac{1}{2^n n!})$, we have

$$S_n^{(N,B)}(1;z)=1-2\sum_{j=[\frac{n}{2}]+1}^{\infty}J_{2j}(z)=1+O(\frac{1}{2^n n!}).$$

Therefore,

$$H_{n,r}^{(N,B)}(1;z)=1+O(\frac{1}{2^n n!}). \tag{8}$$

Let $z=e^{i\theta}$ and $\zeta=e^{is}$. We have

$$K_n^{(N,B)}(z;\zeta)=e^{-\frac{i(s+\theta)}{2}}\frac{\sin(n+1)(s-\theta)}{\sin(\frac{s-\theta}{2})}+(e^{is}+e^{i\theta})\frac{\cos(n+1)(s-\theta)}{2n}+O(\frac{1}{n^2}), \tag{9}$$

where "O" is independent of n. Denote $t=s-\theta$. Then it follows from (9) that

$$\begin{aligned}
&K_n^{(N,B)}(ze^{ih},\zeta)+2K_n^{(N,B)}(z,\zeta)+K_n^{(N,B)}(ze^{-ih},\zeta)\\
&=(e^{\frac{-i(s+\theta+h)}{2}}-e^{\frac{-i(s+\theta)}{2}})\frac{\sin(n+1)(t-h)}{\sin(\frac{t-h}{2})}+\\
&\quad e^{\frac{-i(s+\theta)}{2}}(\frac{\sin(n+1)(t-h)}{\sin(\frac{t-h}{2})}+2\frac{\sin(n+1)t}{\sin(\frac{t}{2})}+\frac{\sin(n+1)(t+h)}{\sin(\frac{t+h}{2})})+\\
&\quad (e^{\frac{-i(s+\theta-h)}{2}}-e^{\frac{-i(s+\theta)}{2}})\frac{\sin(n+1)(t+h)}{\sin(\frac{t+h}{2})}+(e^{i(\theta+h)}-e^{i\theta})\frac{\cos(n+1)(t-h)}{2n}+\\
&\quad (e^{is}+e^{i\theta})\frac{\cos(n+1)(t+h)+2\cos(n+1)t+\cos(n+1)(t-h)}{2n}+\\
&\quad (e^{i(\theta-h)}-e^{i\theta})\frac{\cos(n+1)(t+h)}{2n}+O(\frac{1}{n^2})\\
&=\sum_{j=1}^{7}A_j.
\end{aligned} \tag{10}$$

Since $h=\frac{\pi}{n+1}$, we have $A_5=0$. In view of the trigonometric equations, one can obtain

$$\begin{aligned}
&\frac{\sin(n+1)(t-h)}{\sin(\frac{t-h}{2})}+2\frac{\sin(n+1)t}{\sin(\frac{t}{2})}+\frac{\sin(n+1)(t+h)}{\sin(\frac{t+h}{2})}\\
&=-2\sin(n+1)t\sin\frac{h}{4}\frac{\cos t\sin\frac{h}{4}+\sin\frac{3}{4}h}{\sin\frac{t-h}{2}\sin\frac{t}{2}\sin\frac{t+h}{2}}.
\end{aligned} \tag{11}$$

Let $f(z)$ be a continuous function on the unit circle $\mid z\mid=1$. The modulus of continuity of $f(z)$ is given by

$$\omega(f,\delta)=\sup_{|s-\theta|\le\delta}\mid f(e^{is})-f(e^{i\theta})\mid. \tag{12}$$

Note that it is obvious that

$$\omega(f,a\delta)=(a+1)\omega(f,\delta),\quad a>0 \tag{13}$$

and

$$H_{n,1}^{(N,B)}(f;z)=\frac{1}{4}\{S_n^{(N,B)}(f;ze^{ih})+2S_n^{(N,B)}(f;z)+S_n^{(N,B)}(f;ze^{-ih})\}=R_n^{(N,B)}(f;z).$$

For $r = 3, 5, \ldots$, the following equation holds:

$$H_{n,r}^{(N,B)}(f;z) = R_n^{(N,B)}(f;z) - \frac{1}{4}\{H_{n,r-2}^{(N,B)}(f;ze^{ih}) - 2H_{n,r-2}^{(N,B)}(f;z) + H_{n,r-2}^{(N,B)}(f;ze^{-ih})\}. \ (*)$$

In fact, For $r = 1$ and $r = 3$, respectively, formula $(*)$ holds obviously. Now assume formula $(*)$ is valid for $r = v > 3$. Then, for $r = v + 2$, we have

$$\begin{aligned}
&S_n^{(N,B)}(f;z) - (-\frac{1}{4})^{\frac{v+3}{2}} \sum_{k=0}^{v+3} (-1)^k \binom{v+3}{k} S_n^{(N,B)}(f;ze^{i(k-\frac{v+3}{2})h}) \\
&= S_n^{(N,B)}(f;z) + \frac{1}{4}(-\frac{1}{4})^{\frac{v+1}{2}} \sum_{k=0}^{2}(-1)^k \binom{2}{k} \times \sum_{j=0}^{v+1}(-1)^j \binom{v+1}{j} S_n^{(N,B)}(f;ze^{i(k+j-\frac{v+3}{2})h}) \\
&= S_n^{(N,B)}(f;z) + \frac{1}{4}\sum_{k=0}^{2}(-1)^k \binom{2}{k} \times S_n^{(N,B)}(f;ze^{i(k-1)h}) - \frac{1}{4}\sum_{k=0}^{2}(-1)^k \binom{2}{k} \times \\
&\quad \{S_n^{(N,B)}(f;ze^{i(k-1)h}) - (-\frac{1}{4})^{\frac{v+1}{2}} \sum_{j=0}^{v+1}(-1)^j \binom{v+1}{j} S_n^{(N,B)}(f;ze^{i((k-1)h+(j-\frac{v+1}{2})h)})\} \\
&= R_n^{(N,B)}(f;z) - \frac{1}{4}\{H_{n,v}^{(N,B)}(f;ze^{ih}) - 2H_{n,v}^{(N,B)}(f;z) + H_{n,v}^{(N,B)}(f;ze^{-ih})\} \\
&= H_{n,v+2}^{(N,B)}(f;z).
\end{aligned}$$

Which implies that formula $(*)$ holds for $r = v + 2$. Therefore, by the mathematical inductive method, formula $(*)$ holds for all $r = 1, 3, 5, \ldots$.

3. Proof of Theorem 1

By (5) and (9), we have

$$H_{n,1}^{(N,B)}(f;z) = \frac{1}{8\pi i}\oint_\Gamma f(\zeta)\{K_n^{(N,B)}(ze^{ih};\zeta) + 2K_n^{(N,B)}(z;\zeta) + K_n^{(N,B)}(ze^{-ih};\zeta)\}\mathrm{d}\zeta.$$

Let $z = e^{i\theta}, \zeta = e^{is}, t = s - \theta$, and $h = \frac{\pi}{n+1}$. Then, in view of Eqs.(8) and (10), we have

$$\begin{aligned}
&H_{n,1}^{(N,B)}(f;z) - f(z) \\
&= \frac{1}{8\pi i}\oint_\Gamma [f(\zeta) - f(z)]\{K_n^{(N,B)}(ze^{ih};\zeta) + 2K_n^{(N,B)}(z;\zeta) + K_n^{(N,B)}(ze^{-ih};\zeta)\}\mathrm{d}\zeta + \\
&\quad O(\frac{1}{2^n n!}) \\
&= \frac{1}{8\pi}\sum_{j=1}^{7}\int_{\theta-\pi}^{\theta+\pi}(f(e^{is}) - f(e^{i\theta}))e^{is}A_j\mathrm{d}s + O(\frac{1}{2^n n!}) \\
&= \frac{1}{8\pi}\sum_{j=1}^{7}\int_{-\pi}^{\pi}(f(e^{is}) - f(e^{i\theta}))e^{is}A_j\mathrm{d}s + O(\frac{1}{2^n n!}) \\
&= \sum_{j=1}^{7} B_j + O(\frac{1}{2^n n!}).
\end{aligned} \tag{14}$$

Now, since $A_5 = 0, B_5 = 0$. By Euler formula, we have

$$| e^{\frac{-i(s+\theta\pm h)}{2}} - e^{\frac{-i(s+\theta)}{2}} |= O(h), \quad | e^{i(\theta\pm h)} - e^{i\theta} |= O(h). \tag{15}$$

From (12) and (13), it follows that $| f(e^{is} - f(e^{i\theta})) | \leq (n \mid t \mid +1)\omega(f, \frac{1}{n})$. Obviously,

$$| \frac{\sin(n+1)v}{\sin\frac{v}{2}} | \leq 2(n+1),$$
$$\frac{2}{\pi} \leq \sin v \leq v, \quad 0 \leq v \leq \frac{\pi}{2}. \tag{16}$$

Thus,

$$\begin{aligned} | B_1 | &= O(\omega(f, \frac{1}{n}))\int_{-\pi}^{\pi}(n \mid t \mid +1) \mid A_1 \mid \mathrm{d}t \\ &= O(\omega(f, \frac{1}{n}))\int_{-\pi}^{\pi}(\frac{nh \mid t-h \mid}{\mid \sin\frac{t-h}{2} \mid} + h(n+1))\mathrm{d}t \\ &= O(\omega(f, \frac{1}{n})). \end{aligned}$$

Similarly, we can prove that

$$| B_3 |= O(\omega(f, \frac{1}{n})), \quad | B_j |= O(\omega(f, \frac{1}{n})) \quad j = 4, 6, 7.$$

Now we estimate B_2. By Eqs.(14) we have

$$\begin{aligned} | B_2 | &= O(\omega(f, \frac{1}{n}))\int_{0}^{\pi}(nt+1) \mid A_2 \mid \mathrm{d}t \\ &= O(\omega(f, \frac{1}{n}))\{\int_{0}^{2h}(nt+1) \mid A_2 \mid \mathrm{d}t + \int_{2h}^{\pi}(nt+1) \mid A_2 \mid \mathrm{d}t\}. \end{aligned}$$

Moreover, from Eqs.(15) we have

$$\int_{0}^{2h}(nt+1) \mid A_2 \mid \mathrm{d}t = O(1).$$

And from Eqs.(16) and (11) we have

$$\int_{2h}^{\pi}(nt+1) \mid A_2 \mid \mathrm{d}t = O(\int_{2h}^{\pi}\frac{h}{(t-h)(t+h)}\mathrm{d}t) = O(\ln\frac{t-h}{t+h}|_{2h}^{\pi}) = O(1).$$

Therefore, $| B_2 |= O(\omega(f, \frac{1}{n}))$. Combining the above expressions of $B_j, j = 1, 2, \ldots, 7$, we have

$$| H_{n,1}^{(N,B)}(f; z) - f(z) |= O(\frac{1}{n} + \omega(f, \frac{1}{n})) \quad z \in \Gamma.$$

Therefore, the estimation in Theorem 1 is valid for $r = 1$. It is obviously also valid for an arbitrary odd natural number r by $(*)$ and the mathematical inductive method. This completes the proof of Theorem 1. □

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