

# A New Fixed Point Theorem in Noncompact Hyperconvex Metric Spaces and Its Application to Saddle Point Problems

WEN Kai-ting

(Department of Mathematics, Bijie University, Guizhou 551700, China)

(E-mail: wenkaiting\_2004@sina.com.cn)

**Abstract** In this paper, a new fixed point theorem is established in noncompact hyperconvex metric spaces. As applications, a continuous selection and its fixed point theorem, an existence theorem for maximal elements, a Ky Fan minimax inequality and an existence theorem for saddle points are obtained.

**Keywords** hyperconvex metric space; sub-admissible set; fixed point; continuous selection; maximal element; minimax inequality; saddle point.

**Document code** A

**MR(2000) Subject Classification** 47H04; 52A99; 49J35

**Chinese Library Classification** O177.91

The notion of hyperconvex metric spaces was introduced by Aronszajn and Panitchakdi<sup>[1]</sup> in 1956, and then, Sine<sup>[2]</sup> and Soardi<sup>[3]</sup> proved independently that the fixed point property for the nonexpansive mapping holds in bounded hyperconvex metric spaces. Since then hyperconvex metric spaces have been widely studied and many interesting fixed point theorems have been established. Kirk et al.<sup>[4–7]</sup> established the fixed point theory for nonexpansive mappings in hyperconvex metric spaces. Khamsi et al.<sup>[8]</sup> studied fixed points of commuting nonexpansive maps in hyperconvex metric spaces. Kirk<sup>[9]</sup> obtained fixed point theorems for continuous mappings in compact hyperconvex metric space. Khamsi<sup>[10]</sup> established fixed point theorems, KKM and Ky Fan theorems in hyperconvex metric spaces. Recently, Yuan<sup>[11]</sup> studied the characterization for a mapping with finitely metrically open values being a generalized metric KKM mapping in hyperconvex metric spaces, and obtained fixed point theorems, Ky Fan type matching theorems for both closed and open covers in hyperconvex metric spaces. Khamsi et al.<sup>[12]</sup> studied fixed points and selection theorems in hyperconvex metric spaces. Park<sup>[13]</sup> obtained a Ky Fan matching theorem for open covers, a coincidence theorem and fixed point theorems for hyperconvex metric spaces. Kirk et al.<sup>[14]</sup> established the characterization of the KKM principle in hyperconvex metric spaces, and as applications of their results, the hyperconvex version of Fan's minimax principle, Fan's best approximation theorem for mappings, Nash equilibrium, Browder-Fan fixed point theorem and some other results were given. Zhang<sup>[15]</sup> gave a fixed point theorem, a coincidence theorem and some other results in hyperconvex metric spaces. Chen and Shen<sup>[16]</sup> yielded

---

**Received date:** 2006-04-28; **Accepted date:** 2007-01-16

**Foundation item:** the Science Research Foundation of Bijie University (No. 20062002).

a continuous selection theorem, a coincidence theorem and fixed point theorems in hyperconvex metric spaces. In Ref. [17], we established a Browder fixed point theorem in noncompact admissible subsets of noncompact hyperconvex metric spaces, which was used to derive two Ky Fan coincidence theorems.

In this paper, we first establish a new fixed point theorem in noncompact hyperconvex metric spaces. As applications, we obtain a continuous selection and its fixed point theorem, an existence theorem for maximal elements, a Ky Fan minimax inequality and an existence theorem for saddle points. Our results improve and generalize corresponding results in Refs. [11, 13–16].

## 1. Preliminaries

Let  $A$  be a nonempty subset of a topological space  $X$ . We denote by  $\mathcal{F}(A)$  and  $2^A$  the family of all nonempty finite subsets of  $A$  and the family of all subsets of  $A$ , respectively, and by  $\text{int}_X A$  the interior of  $A$  in  $X$ . Let  $\{G_i\}_{i \in I}$  be a family of subsets of the topological space  $X$ .  $\{G_i\}_{i \in I}$  is said to be transfer open (resp., transfer closed) if for each  $i \in I$ ,  $x \in G_i$  (resp.,  $x \notin G_i$ ), there exists an  $i' \in I$  such that  $x \in \text{int}_X G_{i'}$  (resp.,  $x \notin \text{cl}_X G_{i'}$ ). Let  $X$  and  $Y$  be two topological spaces. A set-valued mapping (in short, mapping)  $T : X \rightarrow 2^Y$  is said to be transfer open valued (resp., transfer closed valued) if  $\{T(x)\}_{x \in X}$  is transfer open (resp., transfer closed).  $T$  is said to have the local intersection property if for each  $x \in X$  such that  $T(x) \neq \emptyset$ , there exists an open neighborhood  $\mathcal{N}(x)$  of  $x$  such that  $\bigcap_{z \in \mathcal{N}(x)} T(z) \neq \emptyset$ .

Following Sine et al.<sup>[2–17]</sup>, a metric space  $(M, d)$  is called a hyperconvex metric space (in short, h.c.m.s.) if for any collection of points  $\{x_i\}_{i \in I} \subset M$  and for any collection  $\{r_i\}_{i \in I}$  of nonnegative real numbers for which  $d(x_i, x_j) \leq r_i + r_j$ , it is the case that  $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ . Where  $B(x, r)$  denotes the closed ball centered at  $x \in M$  with radius  $r \geq 0$ . For any nonempty bounded subset  $A$  of  $M$ , its closed ball hull  $\text{co}(A)$  is defined by

$$\text{co}(A) := \bigcap \{B \subset M : B \text{ is a closed ball containing } A\}.$$

The subset  $A$  is called admissible if  $A = \text{co}(A)$ . The family of all admissible subsets of  $M$  is defined by  $\mathcal{A}(M) := \{A \subset M : A = \text{co}(A)\}$ . Let  $X \in \mathcal{A}(M)$  be an admissible subset of an h.c.m.s.  $M$ . Then  $X$  is a closed hyperconvex metric subspace of  $M$  by Definition 1.2 (2) of Kirk et al.<sup>[14]</sup>.

Following Zhang<sup>[15]</sup>, a subset  $A$  of a metric space  $M$  is called sub-admissible if for each  $F \in \mathcal{F}(A)$ ,  $\text{co}(F) \subset A$ . We define the family of all sub-admissible subsets of  $M$  by

$$\mathcal{B}(M) := \{A \subset M : \text{co}(F) \subset A \text{ for each } F \in \mathcal{F}(A)\}.$$

Obviously, each admissible subset of  $M$  is sub-admissible, but the inverse is not true.

Following Kirk et al.<sup>[14]</sup>, let  $X$  be a nonempty subset of a metric space  $M$ . A mapping  $G : X \rightarrow 2^M \setminus \{\emptyset\}$  is said to be a metric KKM (MKKM) mapping if for each  $F \in \mathcal{F}(X)$ ,  $\text{co}(F) \subset \bigcup_{x \in F} G(x)$ . Let  $X$  be a nonempty subset of an h.c.m.s.  $M$ . A function  $f : X \rightarrow \bar{R} := [-\infty, +\infty]$  is said to be hyper quasi-convex (resp., hyper quasi-concave) if  $\{x \in X : f(x) \leq \lambda\} \in \mathcal{A}(M)$  (resp.,  $\{x \in X : f(x) \geq \lambda\} \in \mathcal{A}(M)$ ) for each  $\lambda \in R$ . Where the inequality  $\leq$  (resp.,  $\geq$ ) can be

replaced equivalently by the strict inequality  $<$  (resp.,  $>$ ).

Following Tian<sup>[18]</sup>, let  $X$  and  $Y$  be two topological spaces and  $\gamma \in R$  a real number. A function  $\varphi(x, y): X \times Y \rightarrow \bar{R}$  is said to be  $\gamma$ -transfer lower semicontinuous (in short,  $\gamma$ -t.l.s.c.) in  $x$  if the mapping  $F: Y \rightarrow 2^X$  defined by  $F(y) := \{x \in X : \varphi(x, y) \leq \gamma\}$  for each  $y \in Y$  is transfer closed valued.  $\varphi(x, y)$  is said to be  $\gamma$ -transfer upper semicontinuous (in short,  $\gamma$ -t.u.s.c.) in  $x$  if  $-\varphi(x, y)$  is  $\gamma$ -t.l.s.c. in  $x$ . Since a closed valued (resp., open valued) mapping is always transfer closed valued (resp., transfer open valued), then a function  $\varphi(x, y)$  is  $\gamma$ -t.l.s.c. (resp.,  $\gamma$ -t.u.s.c.) in  $x$  if  $x \mapsto \varphi(x, y)$  is lower semicontinuous (resp., upper semicontinuous) for each fixed  $y \in Y$ .

Now, we introduce the following definition and lemma.

**Definition 1.1** Let  $X$  be a nonempty subset of a metric space  $M$  and  $\lambda \in R$  a real number. A function  $f: X \rightarrow \bar{R}$  is said to be weakly hyper  $\lambda$ -quasi-convex (resp. weakly hyper  $\lambda$ -quasi-concave) if  $\{x \in X : f(x) \leq \lambda\} = \emptyset$  or  $\in \mathcal{B}(M)$  (resp.,  $\{x \in X : f(x) \geq \lambda\} = \emptyset$  or  $\in \mathcal{B}(M)$ ). Where the inequality  $\leq$  (resp.,  $\geq$ ) can be replaced equivalently by the strict inequality  $<$  (resp.,  $>$ ).

**Remark 1.1** Clearly, a function  $f: X \rightarrow \bar{R}$  is weakly hyper  $\lambda$ -quasi-convex (resp., weakly hyper  $\lambda$ -quasi-concave) if  $f$  is hyper quasi-convex (resp., hyper quasi-concave), but the inverse is not true.

The following result is a special case of Lemma 1.1 of Ding<sup>[19]</sup>.

**Lemma 1.1** Let  $X$  and  $Y$  be topological spaces and  $G: X \rightarrow 2^Y$  be a mapping with nonempty values. Then the following conditions are equivalent:

- (a)  $G$  has the local intersection property;
- (b) For each  $y \in Y$ , there exists an open subset  $O_y$  of  $X$  (which may be empty) such that  $O_y \subset G^{-1}(y)$  and  $X = \bigcup_{y \in Y} O_y$ ;
- (c) There exists a mapping  $F: X \rightarrow 2^Y$  such that  $F(x) \subset G(x)$  for each  $x \in X$  and  $X = \bigcup_{y \in Y} F^{-1}(y)$ ;
- (d)  $X = \bigcup_{y \in Y} \text{int}_X G^{-1}(y)$ ;
- (e)  $G^{-1}$  is transfer open valued.

## 2. Main results

In this section, we first establish the following fixed point theorem in noncompact h.c.m.s..

**Theorem 2.1** Let  $X$  be an h.c.m.s. and  $T: X \rightarrow 2^X \setminus \{\emptyset\}$  a mapping such that

- (i) For each  $x \in X$ ,  $T(x) \in \mathcal{B}(X)$ ;
- (ii)  $T$  satisfies one of the conditions (a)  $\sim$  (e) in Lemma 1.1;
- (iii) There exists an  $x_0 \in X$  such that  $T(x_0)$  is compact.

Then  $T$  has a fixed point in  $X$ .

**Proof** Since  $X$  is a metric space,  $X$  is paracompact. By (1), (2) and Corollary 1 of Chen and Shen<sup>[16]</sup>,  $T$  has a continuous selection  $f: X \rightarrow X$  such that  $f(x) \in T(x)$  for each  $x \in X$ . Define

a mapping  $G : X \rightarrow 2^X$  by

$$G(x) := \{y \in X : d(y, f(y)) \leq d(x, f(y))\}$$

for each  $x \in X$ . Then  $x \in G(x)$ , and hence  $G(x) \neq \emptyset$  for each  $x \in X$ . By the continuity of  $f$ ,  $G$  is closed valued.

We claim that  $G$  is an MKKM mapping. Otherwise, there exist  $\{x_1, \dots, x_n\} \in \mathcal{F}(X)$  and  $y \in \text{co}\{x_1, \dots, x_n\}$  such that  $y \notin \bigcup_{i=1}^n G(x_i)$ , i.e.,  $d(x_i, f(y)) < d(y, f(y))$  for each  $i \in \{1, \dots, n\}$ . Let

$$\epsilon := \frac{1}{2} \min_{1 \leq i \leq n} \{d(y, f(y)) - d(x_i, f(y))\}.$$

Then  $d(x_i, f(y)) < d(y, f(y)) - \epsilon$ , and hence  $x_i \in B(f(y), d(y, f(y)) - \epsilon)$  for each  $i \in \{1, \dots, n\}$ . Thus

$$y \in \text{co}\{x_1, \dots, x_n\} \subset B(f(y), d(y, f(y)) - \epsilon).$$

Consequently,

$$d(y, f(y)) \leq d(y, f(y)) - \epsilon < d(y, f(y)),$$

which is a contradiction.

Now, note that  $G$  is an MKKM mapping with closed values. By (3) and Theorem 4 of Khamsi<sup>[10]</sup>, there exists a  $y_0 \in \bigcap_{x \in X} G(x)$ , i.e.,  $d(y_0, f(y_0)) = \inf_{x \in X} d(x, f(y_0))$ .

We claim that  $y_0 = f(y_0)$ . Otherwise, by hyperconvexity of  $X$ , for any  $\alpha \in (0, 1)$ ,

$$B(y_0, \alpha d(y_0, f(y_0))) \cap B(f(y_0), (1 - \alpha)d(y_0, f(y_0))) \neq \emptyset.$$

Take  $z \in B(y_0, \alpha d(y_0, f(y_0))) \cap B(f(y_0), (1 - \alpha)d(y_0, f(y_0)))$ . Then  $d(y_0, z) \leq \alpha d(y_0, f(y_0))$  and  $d(y_0, z) \leq (1 - \alpha)d(y_0, f(y_0))$ . Moreover,  $d(y_0, z) = \alpha d(y_0, f(y_0))$  and  $d(f(y_0), z) = (1 - \alpha)d(y_0, f(y_0))$ . Otherwise,

$$\begin{aligned} d(y_0, f(y_0)) &= \alpha d(y_0, f(y_0)) + (1 - \alpha)d(y_0, f(y_0)) \\ &> d(y_0, z) + d(f(y_0), z) \\ &\geq d(y_0, f(y_0)). \end{aligned}$$

This is a contradiction. Thus,

$$\begin{aligned} d(z, f(y_0)) &= (1 - \alpha)d(y_0, f(y_0)) = (1 - \alpha) \inf_{x \in X} d(x, f(y_0)) \\ &\leq (1 - \alpha)d(z, f(y_0)) < d(z, f(y_0)). \end{aligned}$$

This is a contradiction, too.

Finally, since  $f$  is the continuous selection of  $T$ ,  $y_0 = f(y_0) \in T(y_0)$ .

The proof is completed.  $\square$

**Remark 2.1.1** Let  $X$  be a compact h.c.m.s. or a compact admissible subset of an h.c.m.s. and  $T(x) \in \mathcal{A}(X)$  for each  $x \in X$ . Then the condition (i) of Theorem 2.1 is satisfied trivially. Moreover, since  $T(x) \in \mathcal{A}(X)$  is closed and  $X$  is compact,  $T(x)$  is also compact for each  $x \in X$ , and hence the condition (iii) of Theorem 2.1 is satisfied naturally. Suppose  $T^{-1}$  is open valued.

Then  $T^{-1}$  is transfer open valued, and hence, the condition (ii) of Theorem 2.1 is satisfied, certainly. Therefore, Theorem 2.1 improves and generalizes Theorem 3.6 of Yuan<sup>[11]</sup>, Theorem 3 of Park<sup>[13]</sup>, Theorem 3.1 of Kirk et al.<sup>[14]</sup>, Lemma 2.2 of Zhang<sup>[15]</sup>, Corollary 3 of Chen and Shen<sup>[16]</sup> by relaxing the compactness of  $X$  and weakening assumptions of  $T$ .

**Remark 2.1.2** The proof method of Theorem 2.1 is different from that of corresponding theorems in the references cited above.

In the proof of Theorem 2.1, we obtain the following continuous selection and its fixed point theorem.

**Theorem 2.2** Let  $X$  be an h.c.m.s. and  $T : X \rightarrow 2^X \setminus \{\emptyset\}$  a mapping such that

- (i) For each  $x \in X$ ,  $T(x) \in \mathcal{B}(X)$ ;
- (ii)  $T$  satisfies one of the conditions (a)  $\sim$  (e) in Lemma 1.1;
- (iii) There exists an  $x_0 \in X$  such that  $T(x_0)$  is compact.

Then  $T$  has a continuous selection  $f : X \rightarrow X$  and  $f$  has a fixed point in  $X$ .

**Remark 2.2** Theorem 2.2 strengthens the conclusions of Theorem 1 of Khamsi et al.<sup>[12]</sup>, Lemma 2.1 of Zhang<sup>[15]</sup>, Theorem 1 of Chen and Shen<sup>[16]</sup>, Theorems 2.3 and 2.4 of Wu<sup>[20]</sup>.

As an immediate application of Theorem 2.1, we get the following existence theorem for maximal elements.

**Theorem 2.3** Let  $X$  be an h.c.m.s. and  $T : X \rightarrow 2^X$  be a mapping such that

- (i) For each  $x \in X$ ,  $T(x) = \emptyset$  or  $\in \mathcal{B}(X)$ ;
- (ii)  $T$  satisfies one of the conditions (a)  $\sim$  (e) in Lemma 1.1;
- (iii) There exists an  $\bar{x} \in X$  such that  $T(\bar{x})$  is compact;
- (iv) For each  $x \in X$ ,  $x \notin T(x)$ .

Then there exists an  $x_0 \in X$  such that  $T(x_0) = \emptyset$ .

**Remark 2.3** As shown in Remark 2.1.1, Theorem 2.3 improves and generalizes Theorem 3.4 of Kirk et al.<sup>[14]</sup> by relaxing the compactness of  $X$  and weakening assumptions of  $T$ .

In virtue of Theorem 2.3, we have the following Ky Fan minimax inequality.

**Theorem 2.4** Let  $X$  be an h.c.m.s.,  $\gamma \in \mathbb{R}$  be a real number and  $\varphi(x, y) : X \times X \rightarrow \bar{\mathbb{R}}$  be a function satisfying

- (i) There exists an  $\bar{x} \in X$  such that  $\{y \in X : \varphi(\bar{x}, y) > \gamma\}$  is compact;
- (ii) For each fixed  $x \in X$ ,  $y \mapsto \varphi(x, y)$  is weakly hyper  $\gamma$ -quasi-concave;
- (iii)  $\varphi(x, y)$  is  $\gamma$ -t.l.s.c. in  $x$ ;
- (iv) For each  $x \in X$ ,  $\varphi(x, x) \leq \gamma$ .

Then there exists an  $x_0 \in X$  such that  $\sup_{y \in X} \varphi(x_0, y) \leq \gamma$ .

**Proof** Define two mappings  $F, G : X \rightarrow 2^X$  by

$$F(x) := \{y \in X : \varphi(x, y) > \gamma\}$$

and

$$G(x) := X \setminus F^{-1}(x) = \{y \in X : \varphi(y, x) \leq \gamma\}$$

for each  $x \in X$ , respectively. Then by (i), there exists an  $\bar{x} \in X$  such that  $F(\bar{x})$  is compact. By (ii), for each  $x \in X$ ,  $F(x) = \emptyset$  or  $\in \mathcal{B}(X)$ . By (iii),  $G$  is transfer closed valued, and hence  $F^{-1}$  is transfer open valued by Remark 2.2 of Kirk et al.<sup>[14]</sup>. Thus,  $F$  satisfies the condition (e) in Lemma 1.1. By (iv),  $x \notin F(x)$  for each  $x \in X$ . Therefore, in virtue of Theorem 2.3, there exists an  $x_0 \in X$  such that  $F(x_0) = \emptyset$ , i.e.,  $\varphi(x_0, y) \leq \gamma$  for all  $y \in X$ , and hence

$$\sup_{y \in X} \varphi(x_0, y) \leq \gamma.$$

The proof is completed. □

**Remark 2.4** Let  $X \in \mathcal{A}(M)$  be a compact admissible subset of an h.c.m.s.  $M$  and for each fixed  $y \in X$ ,  $x \mapsto \varphi(x, y)$  be lower semicontinuous. Then  $X$  is an h.c.m.s., and conditions (i) and (iii) of Theorem 2.4 are satisfied naturally. Hence Theorem 2.4 improves Theorem 2.8 of Kirk et al.<sup>[14]</sup> in the following ways:

- (a) The compactness of  $X$  is relaxed;
- (b) The condition that  $x \mapsto \varphi(x, y)$  is lower semicontinuous for each fixed  $y \in X$  is replaced by the weaker condition that  $\varphi(x, y)$  is  $\gamma$ -t.l.s.c. in  $x$ .

By using Theorem 2.4, we yield the following existence theorem for saddle points.

**Theorem 2.5** Let  $X$  be an h.c.m.s. and  $\varphi: X \times X \rightarrow \bar{R}$  satisfy

- (i) There exist  $x_1, x_2 \in X$  such that  $\{y \in X : \varphi(x_1, y) > 0\}$  and  $\{y \in X : \varphi(y, x_2) < 0\}$  are compact;
- (ii)  $\varphi(x, y)$  is 0-t.l.s.c. in  $x$ ; for each fixed  $x \in X$ ,  $y \mapsto \varphi(x, y)$  is weakly hyper 0-quasi-concave;
- (iii)  $\varphi(x, y)$  is 0-t.u.s.c. in  $y$ ; for each fixed  $y \in X$ ,  $x \mapsto \varphi(x, y)$  is weakly hyper 0-quasi-convex;
- (iv) For each  $x \in X$ ,  $\varphi(x, x) = 0$ .

Then  $\varphi$  has a saddle point in  $X \times X$ , i.e., there exists an  $(x_0, y_0) \in X \times X$  such that

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) = \varphi(x_0, y_0) = \inf_{x \in X} \sup_{y \in X} \varphi(x, y).$$

**Proof** By conditions (i), (ii) and (iv), in virtue of Theorem 2.4, there exists an  $x_0 \in X$  such that

$$\sup_{y \in X} \varphi(x_0, y) \leq 0. \tag{1}$$

Define  $\phi: X \times X \rightarrow \bar{R}$  by

$$\phi(x, y) = -\varphi(y, x), \quad \text{for each } (x, y) \in X \times X.$$

Then by condition (iii),  $\phi(x, y)$  is 0-t.l.s.c. in  $x$ ; for each fixed  $x \in X$ ,  $y \mapsto \phi(x, y)$  is weakly hyper 0-quasi-concave. By conditions (i), (iv) and Theorem 2.4, there exists a  $y_0 \in X$  such that

$$\sup_{x \in X} \phi(y_0, x) \leq 0,$$

i.e.,

$$\inf_{x \in X} \varphi(x, y_0) \geq 0. \quad (2)$$

By Inequalities (1) and (2), we have

$$\varphi(x_0, y_0) = 0. \quad (3)$$

Moreover, Inequalities (1)–(3) imply

$$\inf_{x \in X} \varphi(x, y) \leq \sup_{y \in X} \varphi(x_0, y) \leq \varphi(x_0, y_0) \leq \inf_{x \in X} \varphi(x, y_0) \leq \sup_{y \in X} \varphi(x, y). \quad (4)$$

In turn Inequality (4) implies

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) \leq \varphi(x_0, y_0) \leq \inf_{x \in X} \sup_{y \in X} \varphi(x, y), \quad (5)$$

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) \geq \inf_{x \in X} \varphi(x, y_0) \geq \varphi(x_0, y_0), \quad (6)$$

$$\inf_{x \in X} \sup_{y \in X} \varphi(x, y) \leq \sup_{y \in X} \varphi(x_0, y) \leq \varphi(x_0, y_0). \quad (7)$$

Therefore,

$$\sup_{y \in X} \inf_{x \in X} \varphi(x, y) = \varphi(x_0, y_0) = \inf_{x \in X} \sup_{y \in X} \varphi(x, y),$$

i.e.,  $(x_0, y_0)$  is a saddle point of  $\varphi$ . The proof is completed.  $\square$

**Remark 2.5** Let  $X \in \mathcal{A}(M)$  be a compact admissible subset of an h.c.m.s.  $M$ , for each fixed  $y \in X$ ,  $x \mapsto \varphi(x, y)$  be lower semicontinuous and hyper 0-quasi-convex, for each fixed  $x \in X$ ,  $y \mapsto \varphi(x, y)$  be upper semicontinuous and hyper 0-quasi-concave. Then  $X$  is an h.c.m.s. and assumptions (i), (ii) and (iii) of Theorem 2.5 are fulfilled trivially. Hence Theorem 2.5 improves Theorem 5.1 of Kirk et al.<sup>[14]</sup> in the following ways:

- (a) The compactness of  $X$  is relaxed;
- (b) The condition that  $x \mapsto \varphi(x, y)$  is lower semicontinuous and hyper 0-quasi-convex for each fixed  $y \in X$  is replaced by the weaker conditions that  $\varphi(x, y)$  is  $\gamma$ -t.l.s.c. in  $x$  and  $x \mapsto \varphi(x, y)$  is weakly hyper 0-quasi-convex.
- (c) The condition that  $y \mapsto \varphi(x, y)$  is upper semicontinuous and hyper 0-quasi-concave for each fixed  $x \in X$  is replaced by the weaker conditions that  $\varphi(x, y)$  is  $\gamma$ -t.u.s.c. in  $y$  and  $y \mapsto \varphi(x, y)$  is weakly hyper 0-quasi-concave.

**Acknowledgment** The author thanks the referee for pointing out many misprints and grammatical mistakes in the first draft and helpful suggestions.

## References

- [1] ARONSZAJN N, PANITCHPAKDI P. *Extension of uniformly continuous transformations and hyperconvex metric spaces* [J]. Pacific J. Math., 1956, **6**: 405–439.
- [2] SINE R C. *On nonlinear contraction semigroups in sup-norm spaces* [J]. Nonlinear Anal., 1979, **3**(6): 885–890.
- [3] SOARDI P M. *Existence of fixed points of nonexpansive mappings in certain Banach lattices* [J]. Proc. Amer. Math. Soc., 1979, **73**(1): 25–29.
- [4] KIRK W A. *Fixed point theory for nonexpansive mappings* [C]. Fixed points and nonexpansive mappings (Cincinnati, Ohio, 1982), 121–140, Contemp. Math., 18, Amer. Math. Soc., Providence, RI, 1983.

- [5] BAILON J B. *Nonexpansive mapping and hyperconvex spaces* [C]. Fixed point theory and its applications (Berkeley, CA, 1986), 11–19, Contemp. Math., 72, Amer. Math. Soc., Providence, RI, 1988.
- [6] GOEBEL K, KIRK W A. *Topics in Metric Fixed Point Theory* [M]. Cambridge University Press, Cambridge, 1990.
- [7] KHAMSI M A, REICH S. *Nonexpansive mappings and semigroups in hyperconvex spaces* [J]. Math. Japon., 1990, **35**(3): 467–471.
- [8] KHAMSI M A, LIN M, SINE R C. *On the fixed points of commuting nonexpansive maps in hyperconvex spaces* [J]. J. Math. Anal. Appl., 1992, **168**(2): 372–380.
- [9] KIRK W A. *Continuous mapping in compact hyperconvex metric space* [J]. Numer. Funct. Anal. Optim., 1996, **17**(5-6): 599–603.
- [10] KHAMSI M A. *KKM and Ky Fan theorems in hyperconvex metric spaces* [J]. J. Math. Anal. Appl., 1996, **204**(1): 298–306.
- [11] YUAN Xian-Zhi. *The characterization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications* [J]. J. Math. Anal. Appl., 1999, **235**(1): 315–325.
- [12] KHAMSI M A, KIRK W A, YANEZ C M. *Fixed point and selection theorems in hyperconvex spaces* [J]. Proc. Amer. Math. Soc., 2000, **128**(11): 3275–3283.
- [13] PARK S. *Fixed point theorems in hyperconvex metric spaces* [J]. Nonlinear Anal., 1999, **37**(4): 467–472.
- [14] KIRK W A, SIMS B, YUAN Xian-Zhi. *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications* [J]. Nonlinear Anal., 2000, **39**(5): 611–627.
- [15] ZHANG H L. *Some nonlinear problems in hyperconvex metric spaces* [J]. J. Appl. Anal., 2003, **9**(2): 225–235.
- [16] CHEN Feng-juan, SHEN Zi-fei. *Continuous selection theorem and coincidence theorem on hyperconvex spaces* [J]. Adv. Math. (China), 2005, **34**(5): 614–618.
- [17] WEN Kai-ting. *A Browder fixed point theorem in noncompact hyperconvex metric spaces and its applications to coincidence problems* [J]. Adv. Math. (China), 2005, **34**(2): 208–212. (in Chinese)
- [18] TIAN Guo-qiang. *Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity* [J]. J. Math. Anal. Appl., 1992, **170**(2): 457–471.
- [19] DING Xie-ping. *Coincidence theorems in topological spaces and their applications* [J]. Appl. Math. Lett., 1999, **12**(7): 99–105.
- [20] WU Xian. *Michael selection problem in hyperconvex metric spaces* [J]. Z. Anal. Anwendungen, 2003, **22**(3): 505–516.