Journal of Mathematical Research & Exposition
Feb., 2008, Vol. 28, No. 1, pp. 169–176
DOI:10.3770/j.issn:1000-341X.2008.01.022
Http://jmre.dlut.edu.cn

Approximating Fixed Points of Pseudocontractive Mapping in Banach Spaces

YAO Yong-hong, CHEN Ru-dong
(Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China)
(E-mail: yuyanrong@tjpu.edu.cn)

Abstract Let K be a nonempty closed convex subset of a real p-uniformly convex Banach space E and T be a Lipschitz pseudocontractive self-mapping of K with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from $x_1 \in K$ by $x_{n+1} = a_n x_n + b_n T y_n + c_n u_n$, $y_n = a'_n x_n + b'_n T x_n + c'_n v_n$ for all integers $n \geq 1$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Moreover, if T is completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

Keywords pseudocontractive mappings; p-uniformly convex Banach spaces; Ishikawa iteration process with errors.

Document code A
MR(2000) Subject Classification 47H05; 47H10; 47H17
Chinese Library Classification O177.91

1. Introduction

Let K be a nonempty subset of a real Banach space E with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Recall that a mapping $T : K \to K$ is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2$$

for all $x, y \in K$. A mapping $T : K \to K$ is called Lipschitzian if there exists a constant $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for each $x, y \in K$. If $L = 1$, then T is called nonexpansive.

Apart from being an important generalization of nonexpansive mappings, interest in pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear accretive operators, where a mapping U with domain $D(U)$ and range $R(U)$ in E is called accretive if the inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\|$$

holds for every $x, y \in D(U)$ and for all $s > 0$. It is well known[1] that if T is accretive, then the solutions of the equation $Tx = 0$ correspond to the equilibrium points of some evolution systems.

Received date: 2005-09-05; **Accepted date**: 2006-04-12

Consequently, considerable research efforts, especially within the past 20 years or so, have been devoted to iterative methods for approximating fixed points of T when T is pseudocontractive (see, for example, Refs. [2–6] and the references therein).

Let $T : K \to K$ be a nonexpansive self-mapping on a convex subset K of a normed linear space E. Let $S_\lambda := \lambda I + (1-\lambda)T$, $\lambda \in (0,1)$, where I denotes the identity mapping of K. Then for fixed $x_0 \in K$, $\{S_\lambda^n(x_0)\}$ is defined by $S_\lambda^n(x_0) := \lambda x_n + (1-\lambda)Tx_n$, where $x_n := S_\lambda^{n-1}(x_0)$. In 1955, Krasnoseleskii[7] proved that if E is uniformly convex and K is compact, then for any $x_0 \in K$, the iterative sequence $\{S_{\frac{1}{2}}^n(x_0)\}$ converges to a fixed point of T, where $S_{\frac{1}{2}} := \frac{1}{2}(I+T)$. Schaefer[8] observed that the same result holds for any S_λ with $\lambda \in (0,1)$, and Edelstein[9] proved that strict convexity of E suffices. The important and natural question of whether strict convexity can be removed remained open for many years. In 1976, this question was resolved in the affirmative in the following theorem.

Theorem I[10] *Let K be a nonempty subset of a Banach space E and let $T : K \to E$ be a nonexpansive mapping. For $x_0 \in K$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1-c_n)x_n + c_nTx_n, \tag{1}$$

where the real sequence $\{c_n\}$ satisfies the following conditions: (a) $\sum_{n=0}^\infty c_n = \infty$; (b) $0 \le c_n \le 1$ for all positive integers n; and (c) $x_n \in K$ for all positive integers n. If $\{x_n\}$ is bounded, then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

The iteration method of Theorem I is now referred to as the Mann iteration method in the light of Ref. [3] and has been studied extensively by various authors. One consequence of this theorem is that if K is closed and T is completely continuous, then T has a fixed point and the sequence $\{x_n\}$ defined by (1) converges strongly to a fixed point of T (see, for example, Theorem 1 of Ref. [10]). Any sequence satisfying the conclusion of Theorem I, i.e., $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, is called an approximate fixed point sequence for T.

The importance of approximate fixed point sequences is that once a sequence has been constructed and proved to be an appropriate fixed point sequence for a nonexpansive mapping T, convergence of that sequence to a fixed point of T is then achieved under some mild compactness-type assumptions either on T or on its domain.

Our concern now is the following: Is it possible to extend Theorem I to the case where T is a Lipschitz pseudocontractive mapping? In this connection, Chidume and Mutangadura[11] have recently given an example of a Lipschitz pseudocontractive self-mapping of a compact convex subset of a Hilbert space with a unique fixed point to which no Mann sequence converges. Consequently, for this class of mappings, the Mann sequence cannot give the conclusion of Theorem I.

In 1974, Ishikawa[12] introduced an iteration process which, in some sense, is more general than that of Mann and which converges to a fixed point of Lipschitzian pseudocontractive self-mapping T of K. He proved the following theorem.

Theorem IS[12] *Suppose K is a compact convex subset of a Hilbert space H and $T : K \to K$*

is a Lipschitzian pseudocontractive mapping. For $x_0 \in K$, define the sequence $\{x_n\}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \ n \geq 0, \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

(i) $0 \leq \alpha_n \leq \beta_n < 1$; *(ii)* $\lim_{n\to\infty} \beta_n = 0$; *(iii)* $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point of T.

The iteration method of Theorem IS, now referred to as the Ishikawa iteration method, has been studied extensively by various authors. However, it is still an open question whether or not this method can be employed to approximate fixed points of Lipschitz pseudocontractive mappings in space more general than Hilbert spaces[5,13,14].

It is our purpose in this paper to give affirmative answer to the above question. Let K be a nonempty closed convex subset of a real p-uniformly convex Banach space and T be a Lipschitz pseudocontractive self-mapping of K with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from $x_1 \in K$ by $x_{n+1} = a_n x_n + b_n T y_n + c_n u_n$, $y_n = a'_n x_n + b'_n T x_n + c'_n v_n$ for all integers $n \geq 1$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Moreover, if T be completely continuous, then $\{x_n\}$ converges strongly to a fixed point of T.

2. Preliminaries

Let E be a Banach space, the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \epsilon\}.$$

A Banach space E is called uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. For $p > 1$, the (generalized) duality mapping $J_p : E \to 2^{E^*}$ is defined as $J_p(x) := \{f \in E : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}$. In particular, $J = J_2$ is the normalized duality mapping on E. It is known that $J_p(x) = \|x\|^{p-2}J(x), x \neq 0$. A Banach space E is called p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, $\forall\ 0 < \epsilon \leq 2$. It is known[15] that L_p is

$$\begin{aligned} &\text{2-uniformly convex}, \quad \text{if } \ 1 < p \leq 2, \\ &p\text{-uniformly convex}, \quad \text{if } \ p \geq 2. \end{aligned}$$

Lemma 2.1[5] *Let $p > 1$ be a given real number. Then the following statements about a Banach space E are equivalent:*

(i) E is p-uniformly convex;

(ii) There is a constant $c_p > 0$ such that for every $x, y \in E$, $j_p(x) \in J_p(x)$, the following inequality holds:

$$\|x+y\|^p \geq \|x\|^p + p\langle y, j_p(x)\rangle + c_p\|y\|^p. \tag{3}$$

Remark 2.1 Replacing x by $(x+y)$ and y by $(-y)$ in Inequality (3) and using the Cauchy-Schwarz inequality, we can obtain

$$\|x+y\|^p \leq \|x\|^p + p\|y\| \cdot \|x+y\|^{p-1}.$$

Lemma 2.2[15] *Let $p > 1$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $d > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)d\|x-y\|^p \tag{4}$$

for all $\lambda \in [0,1]$ and $x, y \in E$, where $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Lemma 2.3[16] *Let $\{\rho_n\}, \{\sigma_n\}$ be two nonnegative sequences and for all integers $n \ge N_0$ (for some fixed N_0), $\rho_{n+1} \le \rho_n + \sigma_n$.*

(a) *If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \rho_n$ exists;*

(b) *If $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\{\rho_n\}$ has a sequence converging to zero, then $\lim_{n\to\infty} \rho_n = 0$.*

3. Main results

In the sequel, c_p, d will denote the constants appearing in Inequalities (3) and (4), respectively. For the rest of this paper, we shall assume that E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$. For L_p spaces with $1 < p \le 2$, the following inequalities hold[15,p1131–1132]:

$$\|x+y\|^2 \ge \|x\|^2 + 2\langle y, J(x)\rangle + c_p\|y\|^2,$$

and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - W_2(\lambda)(p-1)\|x-y\|^2$$

for $\lambda \in [0,1]$ and $\forall\ x, y \in E$, where $c_p = [1 + t_p^{(p-1)}][(1+t_p)^{-(p-1)}]$ and for $0 < t_p < 1$, t_p is the unique solution of the equation $g(t) = (p-2)t^{(p-1)} + (p-1)t^{(p-2)} - 1 = 0$. We observe that the function $h : [0,1] \to [0,\infty)$ defined by $h(x) = \frac{1+x^{p-1}}{(1+x)^{p-1}}$ is increasing on

$$[0,1](h'(x) = \frac{(1+x)^{p-2}(p-1)(x^{p-2}-1)}{(1+x)^{2p-2}} \ge 0),$$

hence for $L_p (1 < p \le 2)$ we have $c_p \ge 1$ and $d = p-1$. Therefore, the conditions $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$ are satisfied.

Lemma 3.1 *Let E be a real p-uniformly convex Banach space, $\emptyset \ne K \subset E$ be convex and bounded, $T : K \to K$ be a pseudocontractive mapping. Then, for each $x, y \in K$ and for each integer $n \ge 1$, the following inequality holds:*

$$c_p\|Tx - Ty\|^p \le (p-1)\|x-y\|^p + \|(I-T)x - (I-T)y\|^p.$$

Proof Replacing x by $\frac{1}{2}(x-y)$ and y by $-\frac{1}{2}(Tx - Ty)$ in Inequality (3), we can get

$$\begin{aligned}\|x-y-(Tx-Ty)\|^p \ge& \|x-y\|^p - p2^{p-1}\langle Tx-Ty, j_p(\frac{1}{2}(x-y))\rangle + \\ & c_p\|Tx-Ty\|^P \\ \ge& \|x-y\|^p - p\|x-y\|^p + c_p\|Tx-Ty\|^p.\end{aligned}$$

Since

$$j_p(\frac{1}{2}(x-y)) \in J_p(\frac{1}{2}(x-y)) = 2^{-(p-1)}\|x-y\|^{(p-2)}J(x-y),$$

we have

$$c_p\|Tx - Ty\|^p \leq (p-1)\|x-y\|^p + \|x - y - (Tx - Ty)\|^p. \tag{5}$$

The proof is completed. $\square$

Remark 3.1 We observe that the function $f : [0,\infty) \to (-\infty, +\infty)$ defined by $f(x) = L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1}$ is strictly increasing on $(0,\infty)$. Hence, it has at most one zero on $(0,\infty)$, provided that $f(0) = (p-1)c_p^{-1} - dp2^{-(p-2)} < 0$. In this case, since $f(1) = L^p + (p-1)c_p^{-1} > 0$, it follows that the zero $t_p \in (0,1)$.

Lemma 3.2 *Let E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \leq 1 + c_p$. Let K be a nonempty bounded convex subset of E, $T : K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}$, $\{a_n^{'}\}, \{b_n^{'}\}$, and $\{c_n^{'}\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1, \forall\ n \geq 1$;

(ii) $\sum_{n=0}^{\infty} c_n < \infty,\ \sum_{n=0}^{\infty} c_n^{'} < \infty$;

(iii) $\epsilon \leq 1 - dc_p(1-\alpha_n)2^{-(p-2)} \leq \beta_n \leq b$ for all integers $n \geq 1$, some $\epsilon > 0$ and $b \in (0, t_p)$, where $\alpha_n = b_n + c_n, \beta_n = b_n^{'} + c_n^{'}$ and t_p is the unique solution of the equation:

$$L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1} = 0 \tag{6}$$

on $(0,\infty)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \quad y_n = a_n' x_n + b_n' T x_n + c_n^{'} v_n, \quad n \geq 1,$$

where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof Let $x^* \in F(T)$. Using Inequality (4) and the boundedness of K, for some constant $M \geq 0$, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^p =& \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*) - c_n(Ty_n - u_n)\|^p \\ \leq& (1-\alpha_n)\|x_n - x^*\|^p + \alpha_n\|Ty_n - x^*\|^p - \\ & W_p(\alpha_n)d\|x_n - Ty_n\|^p + Mc_n.\end{aligned} \tag{7}$$

Notice that

$$c_p\|Tx_n - x^*\|^p \leq (p-1)\|x_n - x^*\|^p + \|x_n - Tx_n\|^p, \tag{8}$$

and

$$c_p\|Ty_n - x^*\|^p \leq (p-1)\|y_n - x^*\|^p + \|y_n - Ty_n\|^p. \tag{9}$$

Moreover, for some constants $M_1 \geq 0, M_2 \geq 0$, we have

$$\begin{aligned}\|y_n - x^*\|^p =& \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*) - c_n'(Tx_n - v_n)\|^p \\ \leq& (1-\beta_n)\|x_n - x^*\|^p + \beta_n\|Tx_n - x^*\|^p - \\ & W_p(\beta_n)d\|x_n - Tx_n\|^p + M_1 c_n',\end{aligned} \tag{10}$$

and

$$\|y_n - Ty_n\|^p = \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n) - c_n'(Tx_n - v_n)\|^p$$

$$\leq (1-\beta_n)\|x_n - Ty_n\|^p + \beta_n\|Tx_n - Ty_n\|^p - W_p(\beta_n)d\|x_n - Tx_n\|^p + M_2c'_n. \tag{11}$$

Substituting (8) into (10), one gets

$$\begin{aligned}\|y_n - x^*\|^p \leq &(1-\beta_n)\|x_n - x^*\|^p + \beta_n c_p^{-1}\{(p-1)\|x_n - x^*\|^p + \\ &\|x_n - Tx_n\|^p\} - W_p(\beta_n)d\|x_n - Tx_n\|^p + M_1c'_n \\ =&[1+\beta_n c_p^{-1}(p-1-c_p)]\|x_n - x^*\|^p + \\ &[\beta_n c_p^{-1} - W_p(\beta_n)d]\|x_n - Tx_n\|^p + M_1c'_n.\end{aligned} \tag{12}$$

Set $t_n = \beta_n c_p^{-1}(p-1-c_p)$, $r_n = \beta_n c_p^{-1} - W_p(\beta_n)d$. Then

$$\|y_n - x^*\|^p \leq (1+t_n)\|x_n - x^*\|^p + r_n\|x_n - Tx_n\|^p + M_1c'_n. \tag{13}$$

Substituting (13) and (11) into (9) yields

$$\begin{aligned}c_p\|Ty_n - x^*\|^p \leq &(p-1)(1+t_n)\|x_n - x^*\|^p + (p-1)r_n\|x_n - Tx_n\|^p + \\ &(1-\beta_n)\|x_n - Ty_n\|^p + \beta_n\|Tx_n - Ty_n\|^p - \\ &W_p(\beta_n)d\|x_n - Tx_n\|^p + [(p-1)M_1 + M_2]c'_n.\end{aligned}$$

Substituting this inequality into (7) now gives

$$\begin{aligned}\|x_{n+1} - x^*\|^p \leq &\{1+\alpha_n[(p-1)c_p^{-1}(1+t_n) - 1]\}\|x_n - x^*\|^p - \\ &[W_p(\alpha_n)d - c_p^{-1}\alpha_n(1-\beta_n)]\|x_n - Ty_n\|^p - \\ &c_p^{-1}\alpha_n[W_p(\beta_n)d - (p-1)r_n]\|x_n - Tx_n\|^p + \\ &\alpha_n\beta_n c_p^{-1}\|Tx_n - Ty_n\|^p + M_3(c_n + c'_n)\end{aligned} \tag{14}$$

for some $M_3 > 0$. Observe that $c_p^{-1}(p-1)(1+t_n) - 1 = c_p^{-2}(p-1-c_p)[(p-1)\beta_n + c_p]$ and that by condition (iii), $W_p(\alpha_n)d - c_p^{-1}\alpha_n(1-\beta_n) \geq 0$ since $W_p(\alpha_n) \geq \alpha_n(1-\alpha_n)2^{-(p-2)}$. Therefore

$$\begin{aligned}\|x_{n+1} - x^*\|^p \leq &\{1+\alpha_n c_p^{-2}(p-1-c_p)[(p-1)\beta_n + c_p]\}\|x_n - x^*\|^p - \\ &\alpha_n c_p^{-1}[W_p(\beta_n)d - (p-1)r_n]\|x_n - Tx_n\|^p + \\ &\alpha_n\beta_n c_p^{-1}\|Tx_n - Ty_n\|^p + M_3(c_n + c'_n).\end{aligned}$$

Since T is Lipschitzian, we have, for some constant $M_4 > 0$, that

$$\begin{aligned}\|Tx_n - Ty_n\|^p &\leq L^p\|x_n - y_n\|^p = L^p\|\beta_n(x_n - Tx_n) + c'_n(Tx_n - v_n)\|^p \\ &\leq L^p\beta_n^p\|x_n - Tx_n\|^p + M_4c'_n.\end{aligned}$$

Hence by the assumption $p \leq 1 + c_p$,

$$\begin{aligned}\|x_{n+1} - x^*\|^p \leq &\|x_n - x^*\|^p - \alpha_n\beta_n c_p^{-1}[dp(1-\beta_n)2^{-(p-2)} - (p-1)c_p^{-1} - \\ &\beta_n^pL^p]\|x_n - Tx_n\|^p + M_5(c_n + c'_n)\end{aligned} \tag{15}$$

for some constant $M_5 \geq 0$. Since $b \in (0, t_p)$, it follows that

$$\delta = dp(1-b)2^{-(p-2)} - (p-1)c_p^{-1} - b^pL^p > 0.$$

We can choose some ϵ such that $\epsilon' = 1 - (1-\epsilon)2^{-(p-2)}c_p d > 0$. Then condition (iii) implies $\alpha_n \geq \epsilon' > 0$. Furthermore, Inequality (15) now yields the following estimates

$$\|x_{n+1} - x^*\|^p \leq \|x_n - x^*\|^p - \epsilon\epsilon' c_p^{-1}\delta\|x_n - Tx_n\|^p + M_5(c_n + c_n'). \tag{16}$$

Since $\sum_{n=0}^{\infty}(c_n + c_n') < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - x^*\|^p$ exists. Let $\lim_{n\to\infty}\|x_n - x^*\|^p = r$. Inequality (16) also yields

$$0 < \epsilon\epsilon^{'} c_p^{-1}\delta\|x_n - Tx_n\|^p \leq \|x_n - x^*\|^p - \|x_{n+1} - x^*\|^p + M_5(c_n + c_n') \to 0.$$

Hence, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The proof is completed. □

Theorem 3.1 *Let E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \leq 1 + c_p$. Let K be a nonempty closed convex and bounded subset of E, $T: K \to K$ be a completely continuous Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Let $\{a_n\}, \{b_n\}, \{c_n\}$, $\{a_n'\}, \{b_n'\}$, and $\{c_n'\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1, \forall\ n \geq 1$;

(ii) $\sum_{n=0}^{\infty} c_n < \infty$, $\sum_{n=0}^{\infty} c_n' < \infty$;

(iii) $\epsilon \leq 1 - dc_p(1-\alpha_n)2^{-(p-2)} \leq \beta_n \leq b$ *for all integers $n \geq 1$, some $\epsilon > 0$ and $b \in (0, t_p)$, where $\alpha_n = b_n + c_n, \beta_n = b_n' + c_n'$ and t_p is the unique solution of the equation:*

$$L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1} = 0 \tag{17}$$

on $(0,\infty)$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n; \quad y_n = a_n' x_n + b_n' T x_n + c_n' v_n, \quad n \geq 1,$$

where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof By Lemma 3.2, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since T is completely continuous, there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y^*$. This implies, by Lemma 3.2, that

$$x_{n_i} \to y^*. \tag{18}$$

By the continuity of T and Lemma 3.2, we obtain $Ty^* = y^*$, i.e., y^* is a fixed point of T. Replacing the x^* by y^* in Inequality (16), we obtain that

$$\|x_{n+1} - y^*\|^p \leq \|x_n - y^*\|^p - \epsilon\epsilon' c_p^{-1}\delta\|x_n - Tx_n\|^p + M_5(c_n + c_n'). \tag{19}$$

From (18) we know that $\{\|x_n - y^*\|\}$ has a sequence converging to zero. In view of the conditions $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} c_n' < \infty$, from Inequality (19) and Lemma 2.3, we can conclude that $\|x_n - x^*\| \to \infty$ as $n \to \infty$, i.e., $\{x_n\}$ converges to a fixed point of T. The proof is completed. □

References

[1] DEIMLING K. *ZDeimling, Klaus Zeros of accretive operators* [J]. Manuscripta Math., 1974, **13**: 365–374.

[2] CHIDUME C E, MOORE C. *The solution by iteration of nonlinear equations in uniformly smooth Banach spaces* [J]. J. Math. Anal. Appl., 1997, **215**(1): 132–146.

[3] MANN W R. *Mean value methods in iteration* [J]. Proc. Amer. Math. Soc., 1953, **4**: 506–510.

[4] OSILIKE M O. *Iterative solution of nonlinear equations of the ϕ-strongly accretive type* [J]. J. Math. Anal. Appl., 1996, **200**(2): 259–271.

[5] LIU Qi-hou. *The convergence theorems of the sequence of Ishikawa iterates for hemicontractive mappings* [J]. J. Math. Anal. Appl., 1990, **148**(1): 55–62.

[6] REICH S. *Iterative Methods for Accretive Sets in "Nonlinear Equations in Abstract Space"* [M]. Academic Press, New York, 1978, 317–326.

[7] KRASNOSELSKIL M A. *Two observations about the method of successive approximations* [J]. Uspehi Math, Nauk, 1955, **10**: 123–127.

[8] SCHAEFER H. *Über die methode sukzessiver approximationen* [J]. Jber. Deutsch. Math. Verein., 1957, **59**(1): 131–140.

[9] EDELSTEIN. *A remark on a theorem of Krasnoselskii* [J]. Amer. Math. Monthly, 1966, **13**: 507–510.

[10] ISHIKAWA S. *Fixed points and iteration of a nonexpansive mapping in a Banach space* [J]. Proc. Amer. Math. Soc., 1976, **59**(1): 65–71.

[11] CHIDUME C E, MUTANGADURA S A. *An example of the Mann iteration method for Lipschitz pseudocontractions* [J]. Proc. Amer. Math. Soc., 2001, **129**(8): 2359–2363.

[12] ISHIKAWA S. *Fixed points by a new iteration method* [J]. Proc. Amer. Math. Soc., 1974, **44**: 147–150.

[13] CHIDUME C E, MOORE C. *Fixed point iteration for pseudocontractive maps* [J]. Proc. Amer. Math. Soc., 1999, **127**(4): 1163–1170.

[14] LIU Qi-hou. *On Naimpally and Singh's open questions* [J]. J. Math. Anal. Appl., 1987, **124**(1): 157–164.

[15] XU Hong-kun. *Inequalities in Banach spaces with applications* [J]. Nonlinear Anal., 1991, **16**(12): 1127–1138.

[16] TAN K K, XU Hong-kun. *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process* [J]. J. Math. Anal. Appl., 1993, **178**(2): 301–308.