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Asymptotic Behavior for Random Walks in Time-Random Environment on Z^1

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Abstract In this paper, we give a general model of random walks in time-random environment in any countable space. Moreover, when the environment is independently identically distributed, a recurrence-transience criterion and the law of large numbers are derived in the nearest-neighbor case on Z^1. At last, under regularity conditions, we prove that the RWIRE $\{X_n\}$ on Z^1 satisfies a central limit theorem, which is similar to the corresponding results in the case of classical random walks.

Keywords Random walks in time-random environment; recurrence-transience criteria; strong law of large numbers; central limit theorem.

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1. Introduction

The general model of random walks in space-random environment including a recurrence-transience criterion had been given in Ref. [6]. Moreover, when the environment is i.i.d., the strong law of large numbers was given in Refs. [1] and [4]. However, the limit theorem had also been investigated in Refs. [2] and [5] where the environment is stationary and ergodic. In this paper, we study some asymptotic behavior for random walks in time-random environment (RWIRE) in the nearest-neighbor case on Z^1 as the environment is i.i.d..

We begin with a general setup, that will be specialized later to the cases of interest to us. Now we let $N = \{0, 1, 2, \ldots\}$. For each $i \in N$, let $M_i(\chi)$ denote the collection of probability measures on χ with support V, $V \subset \chi$, where χ is countable. Formally, an element of $M_i(\chi)$, called a transition law at time i, is a measurable function $\omega_i : \chi \to [0, 1]$ satisfying:

(a) $\omega_i(x) \geq 0, \forall x \in V$;

(b) $\omega_i(x) = 0, \forall x \notin V$;

(c) $\sum_{x \in V} \omega_i(x) = 1$.

We equip $M_i(\chi)$ with the weak topology on probability measures which makes it become a Polish space. Furthermore, it induces a Polish structure on $\Omega = \prod_{i \in N} M_i(\chi)$. Let $\mathcal{F}$ denote

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the Borel σ-algebra on Ω. Given a probability measure P on $(\Omega, \mathcal{F})$. A random environment is an element ω of Ω distributed according to P. One defines naturally the shift T on Ω by $(T\omega)_i = \omega_{i+1}, i \in N$.

For each $\omega \in \Omega$, we define the random walks in the environment ω as the space-homogeneous Markov chains $X = \{X_n, n \geq 0\}$ taking value in χ with transition probabilities

$$P_\omega(X_{n+1} = y | X_n = x) = \omega_n(y - x).$$

Fix an environment $\omega \in \Omega$, $X = \{X_n, n \geq 0\}$ is a time-nonhomogeneous Markov chain. We use P_ω^x to denote the law induced on $(\chi^N, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra generated by cylinder functions and $P_\omega^x(X_0 = x) = 1$.

In the sequel, we refer to $P_\omega^x(\cdot)$ as the quenched law of the random walks $\{X_n, n \geq 0\}$. Note that for each $x \in \chi$ and $G \in \mathcal{B}$, the map $\omega \mapsto P_\omega^x(G)$ is $\mathcal{F}$-measurable.

Hence $P^x := P \otimes P_\omega^x$ on $(\Omega \times \chi^N, \mathcal{F} \times \mathcal{B})$ is the probability measure defined by

$$P^x(F \times G) = \int_F P_\omega^x(G) P(\mathrm{d}\omega), \quad F \in \mathcal{F}, G \in \mathcal{B}.$$

Example Let $\chi = Z^1$ and $V = \{-1, 0, 1\}$. Then according to above definition the RWIRE is called the nearest-neighbor RWIRE on Z^1.

2. Recurrence-transience criteria

In this section, suppose $\{X_n, n \geq 0\}$ is the nearest-neighbor RWIRE on Z^1, starting at 0. We let $\omega_n^+ := \omega_n(1), \omega_n^- := \omega_n(-1)$ and $\omega_n^0 := \omega_n(0)$.

Assumption (I) (a) P is i.i.d.; (b) $P\{(\omega_0^+ + \omega_0^-) > 0\} = 1$.

Theorem 2.1 *Assume Assumption (I). Then*

(1) $E_P\omega_0^- < E_P\omega_0^+ \Longrightarrow \lim_{n\to\infty} X_n = +\infty$,

(2) $E_P\omega_0^- > E_P\omega_0^+ \Longrightarrow \lim_{n\to\infty} X_n = -\infty$,

(3) $E_P\omega_0^- = E_P\omega_0^+ \Longrightarrow -\infty = \lim_{n\to\infty} \inf X_n < \lim_{n\to\infty} \sup X_n = +\infty$

hold P^0-a.s., where E_P is the expectation operator w.r.t. P.

Proof (1) Fix an environment ω with $\frac{E_P\omega_0^-}{E_P\omega_0^+} < \infty$. For each $x \in Z^1$, $l \leq x \leq s$, define

$$H_{l,s,\omega}^x = P_\omega^x(\{X_n\} \text{ hits } l \text{ before hitting } s).$$

From the assumption $\frac{E_P\omega_0^-}{E_P\omega_0^+} < \infty$, for each x, it follows that $H_{l,s,\omega}^x$ is well defined as $P_\omega^x(\{X_n\}$ never hits $[l,s]^c) = 0$. The Markov property of P_ω^x implies

$$\begin{cases} H_{l,s,\omega}^x = \omega_0^+ H_{l,s,T\omega}^{x+1} + \omega_0^0 H_{l,s,T\omega}^x + \omega_0^- H_{l,s,T\omega}^{x-1}, & x \in (l,s); \\ H_{l,s,T^k\omega}^l = 1, & k \geq 1; \\ H_{l,s,T^k\omega}^s = 0, & k \geq 1. \end{cases} \tag{2.1}$$

Since P is i.i.d., and P_ω^x is space-homogeneous for all $\omega \in \Omega$, it follows by taking expectation on

(2.1) w.r.t. P that

$$\begin{cases} (E_P\omega_0^+ + E_P\omega_0^-)E_P H^x_{l,s,\omega} = E_P\omega_0^+ E_P H^{x+1}_{l,s,\omega} + E_P\omega_0^- E_P H^{x-1}_{l,s,\omega}, & x \in (l,s), \\ E_P H^0_{0,s-l,\omega} = 1, \\ E_P H^0_{l-s,0,\omega} = 0. \end{cases} \tag{2.2}$$

Solving (2.2), we obtain:

(i) If $E_P\omega_0 \times E_P\omega_0^- > 0$, then

$$E_P H^x_{l,s,\omega} = \frac{\sum_{j=x+1}^{s-1}(\frac{E_P\omega_0^-}{E_P\omega_0^+})^j}{\sum_{j=l+1}^{s-1}(\frac{E_P\omega_0^-}{E_P\omega_0^+})^j}, \quad E_P H^0_{l,s,\omega} = \frac{\sum_{j=1}^{s-1}(\frac{E_P\omega_0^-}{E_P\omega_0^+})^j}{\sum_{j=l+1}^{s-1}(\frac{E_P\omega_0^-}{E_P\omega_0^+})^j}.$$

There are three cases:

(a) When $E_P\omega_0^- < E_P\omega_0^+$, we have $\lim_{l\to-\infty}\lim_{s\to+\infty} E_P H^0_{l,s,\omega} = 0$, but $0 \le H^1_{0,s,\omega} \le 1$, so $P(\lim_{l\to-\infty}\lim_{s\to+\infty} H^0_{l,s,\omega} = 0) = 1$. We also have $\lim_{s\to+\infty} E_P H^0_{-1,s,\omega} < 1$, hence $P(\lim_{s\to+\infty} H^0_{-1,s,\omega} < 1) > 0$, which implies $\lim_{n\to\infty} X_n = +\infty$ under P^0-a.s..

(b) When $E_P\omega_0^- > E_P\omega_0^+$, similarly, we may get $\lim_{n\to\infty} X_n = -\infty$ under P^0-a.s..

(c) If $E_P\omega_0^- = E_P\omega_0^+$, for any fixed l, $\lim_{s\to+\infty} E_P H^0_{l,s,\omega} = 1$, hence $P(\lim_{s\to+\infty} H^0_{l,s,\omega} = 1) = 1$. Moreover, for any fixed s, $\lim_{l\to-\infty} E_P H^0_{l,s,\omega} = 0$, hence $P(\lim_{l\to-\infty} H^0_{l,s,\omega} = 0) = 1$, so $E_P\omega_0^- = E_P\omega_0^+ \Longrightarrow -\infty = \lim_{n\to\infty}\inf X_n < \lim_{n\to\infty}\sup X_n = +\infty$ under P^0-a.s..

(ii) If $E_P\omega_0^- \times E_P\omega_0^+ = 0$, there are two cases by assumption (I)(b):

(a) When $E_P\omega_0^- = 0, E_P\omega_0^+ > 0$, then by (2.2) and P_ω is space-homogeneous for all ω

$$E_P(H^0_{l,s,\omega}) = E_P(H^0_{l-1,s-1,\omega}) = \cdots = E_P(H^0_{l-s,0,\omega}) = 0.$$

(b) When $E_P\omega_0^- > 0, E_P\omega_0^+ = 0$, similarly, we have

$$E_P(H^0_{l,s,\omega}) = E_P(H^0_{l+1,s+1,\omega}) = \cdots = E_P(H^0_{0,s-l,\omega}) = 1.$$

So the case (a) (or (b)) of (ii) can be included in the case of (a) (or (b)) of (i). We complete the proof by (i) and (ii). □

3. Strong law of large numbers

We introduce hitting times which will serve us later. Let $T_0 = 0$ and

$$T_n = \inf\{k : X_k = n\}, n \ge 1; \inf\varphi = +\infty.$$

Set $\tau_0 = 0$ and $\tau_n = T_n - T_{n-1}, n \ge 1$. Similarly, set $T_{-n} = \inf\{k : X_k = -n\}, n \ge 1$ and $\tau_{-n} = T_{-n} - T_{-n+1}, n \ge 1$, with the convention that $\tau_{\pm n} = +\infty$ if $T_{\pm n} = \infty$.

Theorem 3.1 *Assume Assumption (I). Then under P^0-a.s.*

(1) $E_P\omega_0^+ > E_P\omega_0^- \Longrightarrow \lim_{n\to\infty}\frac{T_n}{n} = (E_P\omega_0^+ - E_P\omega_0^-)^{-1}, \lim_{n\to\infty}\frac{X_n}{n} = (E_P\omega_0^+ - E_P\omega_0^-)$;

(2) $E_P\omega_0^+ < E_P\omega_0^- \Longrightarrow \lim_{n\to\infty}\frac{T_{-n}}{n} = (E_P\omega_0^- - E_P\omega_0^+)^{-1}, \lim_{n\to\infty}\frac{X_n}{n} = (E_P\omega_0^+ - E_P\omega_0^-)$;

(3) $E_P\omega_0^+ = E_P\omega_0^- \Longrightarrow \lim_{n\to\infty}\frac{T_n}{n} = \infty = \lim_{n\to\infty}\frac{T_{-n}}{n}, \lim_{n\to\infty}\frac{X_n}{n} = 0$.

For the proof of Theorem 3.1, we need the following two lemmas.

Lemma 3.1 *Assume Assumption (I). Then* $\{\tau_n, n \geq 1\}$ *is i.i.d. and* $\{\tau_{-n}, n \geq 1\}$ *is i.i.d. under the law* P^0*-a.s..*

Proof When $E_P\omega_0^+ \geq E_P\omega_0^-$, we have $P^0(\lim_{n\to\infty}\sup X_n = +\infty) = 1$, by the definition of τ_n, $P^0(\tau_n < \infty) = 1$ for all $n \geq 1$. To prove $\{\tau_n\}$ is i.i.d., it suffices to show for any positive integer k and m,

$$P^0(\tau_2 = k|\tau_1 = m) = P^0(\tau_2 = k) = P^0(\tau_1 = k).$$

For any fixed $\omega \in \Omega$, by the Markov property and the space-homogeneity of P_ω^0, we have

$$P_\omega^0(\tau_2 = k) = \sum_{m=1}^{\infty} P_\omega^0(\tau_2 = k|\tau_1 = m)P_\omega^0(\tau_1 = m) = \sum_{m=1}^{\infty} P_{T^m\omega}^0(\tau_1 = k)P_\omega^0(\tau_1 = m). \tag{3.1}$$

Since P is i.i.d., it follows by taking expectation on (3.1) w.r.t. P that

$$P^0(\tau_2 = k) = P \otimes P_\omega^0(\tau_1 = k) \sum_{m=1}^{\infty} P \otimes P_\omega^0(\tau_1 = m) = P^0(\tau_1 = k). \tag{3.2}$$

On the other hand

$$\begin{aligned}&P_\omega^0(\tau_2 = k|\tau_1 = m) = P_\omega^0(X_{m+k} = 2, X_{m+s} \neq 2, s = 1, 2, \ldots, k-1| \\ &X_m = 1, X_n \neq 1, n = 1, 2, \ldots, m-1) = P_{T^m\omega}^0(\tau_1 = k).\end{aligned} \tag{3.3}$$

Hence by taking expectation on (3.3) w.r.t. P, we also have

$$P^0(\tau_2 = k|\tau_1 = m) = P^0(\tau_1 = k). \tag{3.4}$$

So $\{\tau_n, n \geq 1\}$ is i.i.d. under P^0-a.s. by (3.2) and (3.4) when $E_P\omega_0^+ \leq E_P\omega_0^-$. Similarly, we may show $\{\tau_{-n}, n \geq 1\}$ is i.i.d. under P^0-a.s..

Lemma 3.2 *Assume Assumption (I). Then*

(1) $E_{P^0}(\tau_1) = \begin{cases} (E_P\omega_0^+ - E_P\omega_0^-)^{-1}, & E_P\omega_0^+ > E_P\omega_0^-, \\ +\infty, & E_P\omega_0^+ = E_P\omega_0^-; \end{cases}$

(2) $E_{P^0}(\tau_{-1}) = \begin{cases} (E_P\omega_0^- - E_P\omega_0^+)^{-1}, & E_P\omega_0^- > E_P\omega_0^+, \\ +\infty, & E_P\omega_0^+ = E_P\omega_0^-. \end{cases}$

Proof We prove only (1) since the proof of (2) is similar. Decompose, with $X_0 = 0$

$$\tau_1 = \chi_{\{X_1=1\}} + \chi_{\{X_1=0\}}(1 + \tau_1') + \chi_{\{X_1=-1\}}(1 + \tau_0'' + \tau_1''). \tag{3.5}$$

Here τ_1' is the first hitting time of 1 after time 1 (possible infinite), $1 + \tau_0''$ is the first hitting time of 0 after time 1, and $1 + \tau_0'' + \tau_1''$ is the first hitting time of 1 after time $1 + \tau_0''$.

Consider first the case $E_{P^0}\tau_1 < \infty$. Then $E_\omega^0\tau_1 < \infty$ under P-a.s.. Taking expectation on (3.5), one gets

$$\begin{aligned}E_\omega^0\tau_1 =& P_\omega^0(X_1 = 1) + P_\omega^0(X_1 = 0)(1 + E_{T\omega}^0\tau_1) + \\ & P_\omega^0(X_1 = -1)(1 + E_{T\omega}^0\tau_1 + \sum_{m=1}^{\infty} P_{T\omega}^0(\tau_1 = m)E_{T^{m+1}\omega}^0\tau_1)\end{aligned}$$

$$=1+(1-\omega_0^+)E^0_{T\omega}\tau_1+\omega_0^-\sum_{m=1}^{\infty}P^0_{T\omega}(\tau_1=m)E^0_{T^{m+1}\omega}\tau_1. \tag{3.6}$$

Taking expectation on (3.6) w.r.t. P, we get that

$$E_{P^0}\tau_1=1+(1-E_P\omega_0^+)E_{P^0}\tau_1+E_P\omega_0^-E_{P^0}\tau_1. \tag{3.7}$$

Hence when $E_P\omega_0^+>E_P\omega_0^-$, we have $E_{P^0}\tau_1=(E_P\omega_0^+-E_P\omega_0^-)^{-1}<\infty$.

Note next that if $E_P\omega_0^+\geq E_P\omega_0^-$, we have by Theorem 2.1 that $E_P(\tau_1\chi_{\tau_1<\infty})=E_P\tau_1$. Hence $E_{P^0}\tau_1=\infty$ implies $E_p\omega_0^+=E_p\omega_0^-$; on the other hand, if $E_p\omega_0^+=E_p\omega_0^-$, by (3.7) we also have $E_{P^0}\tau_1=\infty$. We finish the proof of Lemma 3.2.

Proof of Theorem 3.1 An application of Lemmas 3.1 and 3.2 yields that in case (1)

$$\frac{T_n}{n}=\frac{\sum_{i=1}^n\tau_i}{n}\longrightarrow E_{P^0}\tau_1<\infty. \tag{3.8}$$

Similarly, we use $-n$ instead of n, we also get that in case (2)

$$\frac{T_{-n}}{n}=\frac{\sum_{i=1}^n\tau_{-i}}{n}\longrightarrow E_{P^0}\tau_{-1}<\infty. \tag{3.9}$$

However when $E_p\omega_0^+=E_p\omega_0^-$, $P^0(-\infty=\lim_{n\to\infty}\inf X_n<\lim_{n\to\infty}\sup X_n=+\infty)=1$, for n large enough, $T_n=+\infty=T_{-n}$, so in case (3),

$$P^0(\lim_{n\to\infty}\frac{T_n}{n}=+\infty=\lim_{n\to\infty}\frac{T_{-n}}{n})=1.$$

Now we prove the second limit of the case (1). Let K_n be the unique (random) integers such that $T_{K_n}\leq n<T_{K_n+1}$. Note that $X_n\leq K_n+1$, while $X_n\geq K_n-(n-T_{K_n})$. Hence

$$\frac{K_n}{n}-(1-\frac{T_{K_n}}{n})\leq\frac{X_n}{n}<\frac{K_n+1}{n}. \tag{3.10}$$

Since $P^0(\lim_{n\to\infty}X_n=+\infty)=1$, from the definition of K_n, $P^0(\lim_{n\to\infty}\frac{T_{K_n}}{n}=1)=1$. But the definition K_n also implies

$$P^0(\lim_{n\to\infty}\frac{K_n}{n}=\lim_{n\to\infty}\frac{n}{T_n})=1.$$

Thus it follows from (3.8) and (3.10) that

$$P^0(\lim_{n\to\infty}\frac{X_n}{n}=\lim_{n\to\infty}\frac{n}{T_n}=E_P\omega_0^+-E_P\omega_0^-)=1.$$

Similarly, we may prove the second limit of (2) and (3). □

4. The central limit Theorem

In this section, we study the limiting distribution of the RWIRW $\{X_n\}$, we use following notations:

$$\mu=(E_{P^0}\tau_1)^{-1},\sigma^2=\frac{E_{P^0}(\tau_1)^2-(E_{P^0}\tau_1)^2}{(E_{P^0}\tau_1)^3},D_{P^0}\tau_1=E_{P^0}(\tau_1)^2-(E_{P^0}\tau_1)^2;$$

$$\mu_{-1}=(E_{P^0}\tau_{-1})^{-1},\sigma_{-1}^2=\frac{E_{P^0}(\tau_{-1})^2-(E_{P^0}\tau_{-1})^2}{(E_{P^0}\tau_{-1})^3},D_{P^0}\tau_{-1}=E_{P^0}(\tau_{-1})^2-(E_{P^0}\tau_{-1})^2.$$

Theorem 4.1 *Assume Assumption (I), under* P^0-a.s.,

(1) If $E_P\omega_0^+ > E_P\omega_0^-$ and $D_{P^0}\tau_1 < \infty$, *then*

$$\frac{X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1);\ \frac{T_n - n\mu^{-1}}{\sqrt{nD_{P^0}\tau_1}} \longrightarrow N(0,1),\ n \to \infty.$$

(2) If $E_P\omega_0^+ = E_P\omega_0^-$, *then*

$$\frac{X_n}{\sqrt{2nE_P\omega_0^+}} \longrightarrow N(0,1),\ n \to \infty.$$

(3) If $E_P\omega_0^+ < E_P\omega_0^-$ and $D_{P^0}\tau_{-1} < \infty$, *then*

$$\frac{X_n - n\mu_{-1}}{\sigma_{-1}\sqrt{n}} \longrightarrow N(0,1);\ \frac{T_n - n\mu_{-1}^{-1}}{\sqrt{nD_{P^0}\tau_{-1}}} \longrightarrow N(0,1),\ n \to \infty.$$

Proof (1) For any positive integers n, L and M, by the definition of T_n, we have

$$\begin{aligned}\{T_L \geq n\} &\subset \{X_n \leq L\} \subset (\{T_{L+M} \geq n\} \cup \{(\inf_{s\geq T_{L+M}} X_s \leq L) \cap (T_{L+M} < n)\}) \\ &\subset (\{T_{L+M} \geq n\} \cup \{\inf_{s\geq T_{L+M}} X_s - (L+M) \leq -M\}).\end{aligned} \tag{4.1}$$

Since P is i.i.d., it follows by the Markov property and the space-homogeneity of P_ω^0 that

$$P^0(\inf_{s\geq T_{L+M}} X_s - (L+M) \leq -M) = P^0(\inf_{s\geq 0} X_s \leq -M). \tag{4.2}$$

By Theorem 2.1 we have

$$\lim_{M\to\infty} P^0(\inf_{s\geq 0} X_s \leq -M) = 0. \tag{4.3}$$

Hence for any positive integers l and for any $\delta > 0$, there exists M large enough such that

$$P^0(T_L \geq n) \leq P^0(X_n \leq L) \leq P^0(T_{L+M} \geq n) + \delta. \tag{4.4}$$

Now for any given real x, take

$$L = L(n,x) = n\mu + x\sigma\sqrt{n} + o(\sqrt{n}), \tag{4.5}$$

we have $L(n,x) \longrightarrow +\infty$ *as* $n \to \infty$ and

$$P^0(X_n \leq L(n,x)) \approx P^0(\frac{X_n - n\mu}{\sigma\sqrt{n}} \leq x). \tag{4.6}$$

On the other hand

$$P^0(T_{L(n,x)} \geq n) = P^0(\frac{T_{L(n,x)} - L(n,x)\mu^{-1}}{\sqrt{L(n,x)D_{P^0}\tau_1}} \geq \frac{n - L(n,x)\mu^{-1}}{\sqrt{L(n,x)D_{P^0}\tau_1}}).$$

By the definition of $L(n,x)$, we have

$$\lim_{n\to\infty} \frac{n - L(n,x)\mu^{-1}}{\sqrt{L(n,x)D_{P^0}\tau_1}} = -x.$$

By the Central Limit Theorem of i.i.d. random variable sequence and the given condition we obtain

$$\lim_{n\to\infty} P^0(T_{L(n,x)} \geq n) = \lim_{n\to\infty} P^0(T_{L(n,x)+M} \geq n) = 1 - \Phi(-x) = \Phi(x),$$

where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(\frac{-t^2}{2})\mathrm{d}t$. So by (4.4) and (4.6), the case (1) is proved. Similarly, we can prove the case (3).

(2) When $E_P\omega_0^+=E_P\omega_0^-$, suppose $X_n=\sum_{i=1}^{n}Y_i, Y_i\in\{-1,0,1\}$, where

$$E_\omega(Y_i=1)=\omega_i^+, E_\omega(Y_i=-1)=\omega_i^-, E_\omega(Y_i=0)=\omega_i^0.$$

Since P is i.i.d., by the given condition we have

$$E_{P^0}X_n=E_P[\sum_{i=1}^{n}(\omega_i^+-\omega_i^-)]=\sum_{i=1}^{n}(E_P\omega_i^+-E_P\omega_i^-)=0,$$

$$\begin{aligned}E_{P^0}X_n^2&=E_P(E_\omega^0(\sum_{i=1}^{n}\sum_{j=1}^{n}Y_iY_j))=E_P(E_\omega^0(\sum_{i=1}^{n}Y_i^2+2\sum_{1\le i<j\le n}Y_iY_j))\\&=E_P(\sum_{i=1}^{n}(\omega_i^++\omega_i^-)+2\sum_{1\le i<j\le n}(\omega_i^+-\omega_i^-)(\omega_j^+-\omega_j^-))=2\sum_{i=1}^{n}E_P\omega_i^+=2nE_P\omega_0^+.\end{aligned}$$

Suppose that the characteristic function of $\frac{X_n}{\sqrt{2nE_P\omega_0^+}}$ is $\varphi_n(t)$. Then

$$\begin{aligned}\varphi_n(t)&=E_{P^0}\exp(it\frac{X_n}{\sqrt{2nE_P\omega_0^+}})=E_P(E_\omega^0\prod_{j=1}^{n}\exp(it\frac{Y_j}{\sqrt{2nE_P\omega_0^+}}))\\&=E_P[\prod_{j=1}^{n}(\exp(\frac{it}{\sqrt{2nE_P\omega_0^+}})\omega_j^++\exp(\frac{-it}{\sqrt{2nE_P\omega_0^+}})\omega_j^-+\omega_j^0)]\\&=\prod_{j=1}^{n}[\exp(\frac{it}{\sqrt{2nE_P\omega_0^+}})E_P\omega_j^++\exp(\frac{-it}{\sqrt{2nE_P\omega_0^+}})E_P\omega_j^-+E_P\omega_j^0]\\&=[E_P\omega_0^+(\exp(\frac{it}{\sqrt{2nE_P\omega_0^+}})+\exp(\frac{-it}{\sqrt{2nE_P\omega_0^+}}))+E_P\omega_0^0]^n\\&=[E_P\omega_0^+(1+\frac{it}{\sqrt{2nE_P\omega_0^+}}-\frac{1}{2}\cdot\frac{t^2}{2nE_P\omega_0^+}+o(\frac{1}{n})+1-\\&\qquad\frac{it}{\sqrt{2nE_P\omega_0^+}}-\frac{1}{2}\cdot\frac{t^2}{2nE_P\omega_0^+}+o(\frac{1}{n}))+E_P\omega_0^0]^n\\&=[E_P\omega_0^+(2-\frac{t^2}{2nE_P\omega_0^+}+o(\frac{1}{n}))+E_P\omega_0^0]^n\\&=[1-\frac{t^2}{2n}+o(\frac{1}{n})]^n\longrightarrow\exp(\frac{-t^2}{2}),\ n\longrightarrow\infty.\end{aligned}\tag{4.7}$$

It follows by the Continuous Theorem and (4.7) that $\frac{X_n}{\sqrt{2nE_P\omega_0^+}}\rightarrow N(0,1)$ *as* $n\rightarrow\infty$.

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