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# The Kinematic Density for Pairs of Intersecting Lines

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**Abstract** In this paper, we get a kinematic density formula for pairs of intersecting lines which has not yet been gotten in integral geometry by using the moving orthogonal frames method. And we obtain a kinematic formula of the intersection of the pairs of intersecting lines belonging to convex body $K$ by using it.

**Keywords** pairs of intesecting lines; kinematic density; kinematic formula.

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## 1. Introduction

The kinematic density for pairs of intersecting lines in $\mathbb{R}^2$ is the same as the kinematic density for pairs of lines in $\mathbb{R}^2$. Let $G$ and $L$ be two lines which intersect at the point $P$, $\theta$ be the angle between $G$ and $L$, and $\alpha$ be the angle $G$ with the $x$-axis. We can get the formula[1,2]

$$\mathrm{d}G\mathrm{d}L = \sin\theta \mathrm{d}P\mathrm{d}\theta\mathrm{d}\alpha. \tag{1}$$

The kinematic density for pairs of intersecting lines in $\mathbb{R}^3$ is different from the kinematic density for pairs of lines in $\mathbb{R}^3$. The kinematic density for pairs of lines in $\mathbb{R}^3$ is[3]

$$\mathrm{d}G\mathrm{d}L = \sin^2\theta \mathrm{d}P_N\mathrm{d}Q_N\mathrm{d}G_N\mathrm{d}L_N\mathrm{d}N. \tag{2}$$

Where $N$ is the common perpendicular of lines $G$ and $L$, $P{=}N\cap G$, $Q{=}N\cap L$, $\theta$ denotes the angle between $G$ and $L$, $\mathrm{d}P_N$ and $\mathrm{d}Q_N$ are the densities of $P$ and $Q$ on the common perpendicular $N$, respectively, and $\mathrm{d}G_N$ and $\mathrm{d}L_N$ are the densities of $G$ and $L$ for the rotations around $N$, respectively.

The kinematic density for pairs in $\mathbb{R}^n$ is[4,5]

$$\mathrm{d}G\mathrm{d}L = t^{n-3}\sin^2\theta \mathrm{d}P_N\mathrm{d}Q_N\mathrm{d}G_N\mathrm{d}L_N\mathrm{d}N, \tag{3}$$

where $t$ denotes the length of $PQ$, $\theta$ denotes the angle between $G$ and $L$, $\mathrm{d}P_N$ and $\mathrm{d}Q_N$ are the densities of $P$ and $Q$ on the common perpendicular $N$, respectively, and $\mathrm{d}G_N$ and $\mathrm{d}L_N$ are the densities of $G$ and $L$ for the rotations around $N$, respectively.

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Let $K$ be a convex body in $\mathbb{R}^n$, and $\sigma$ the chord intersected by a random line $G$ with $K$. Consider integrals[1]

$$I_m^{(n)}(K) = \int_{G\cap K} \sigma^m \mathrm{d}G,$$

where $m$ is a nonnegative integer and $\mathrm{d}G$ is the density of lines. The integral $I_m^{(n)}(K)$ is called the $m$th chord-power integral of $K$.

In particular,

$$I_1^{(n)}(K) = \frac{1}{2} O_{n-1} V, \tag{4}$$

where $O_{n-1}$ is the surface area of $(n-1)$-dimensional unit sphere and $V$ is the volume of $K$.

When we study geometric probability[5], we may meet these problems which need using the related formulas, say, caculating the kinematic measure of the intersection of the lines $G$ and $L$ belonging to convex body $K$ in $\mathbb{R}^n$. Meanwhile, we notice that these are conditional probability problems.

## 2. Main results

**Lemma** *The kinematic density for pairs of intersecting lines $G$ and $L$ in $\mathbb{R}^n$ is*

$$\mathrm{d}G\mathrm{d}L = \sin^{n-1}\theta \mathrm{d}\theta \omega_1 \omega_2 \cdots \omega_n \omega_{12} \cdots \omega_{1n} \omega_{23} \cdots \omega_{2n}, \tag{5}$$

*where $\theta$ is the angle between the lines $G$ and $L$, and $\omega_i$, $\omega_{ij}$ are 1-forms in $\mathbb{R}^n$,*

$$i = 1, 2, \ldots, n; j = 1, 2, \ldots, n.$$

**Proof** Let $g_1$ be the unit vector parallel to the line $G$ and $l_1$ be the unit vector parallel to the line $L$. Let $\{P, g_1, g_2, g_3, \ldots, g_n\}$ be the moving orthogonal frames, such that the plane $\pi$ which is spanned by the lines $G$ and $L$ will be perpendicular to $\mathrm{span}(P; g_3, g_4, \ldots, g_n)$. And the lines $l_1$ and $l_2$ are also in the plane $\pi$. Denote $g_i = l_i$, $i = 3, 4, \ldots, n$. Then

$$\begin{aligned}
l_1 &= \cos\theta g_1 + \sin\theta g_2, \quad l_2 = -\sin\theta g_1 + \cos\theta g_2, \\
\mathrm{d}l_1 &= \cos\theta \mathrm{d}g_1 + \sin\theta \mathrm{d}g_2 + (\cos\theta g_2 - \sin\theta g_1)\mathrm{d}\theta, \\
\mathrm{d}l_2 &= -\sin\theta \mathrm{d}g_1 + \cos\theta \mathrm{d}g_2 - (\cos\theta g_1 + \sin\theta g_2)\mathrm{d}\theta.
\end{aligned}$$

Because the arbitary line $G$ always intersects the line $L$, we have

$$\begin{aligned}
\mathrm{d}L &= \mathrm{d}Pl_2 \wedge \mathrm{d}l_1 l_2 \wedge \mathrm{d}l_1 g_3 \wedge \cdots \wedge \mathrm{d}l_1 g_n \\
&= \mathrm{d}P(-\sin\theta g_1 + \cos\theta g_2) \\
&\wedge (\cos\theta \mathrm{d}g_1 + \sin\theta \mathrm{d}g_2 + (\cos\theta g_2 - \sin\theta g_1)\mathrm{d}\theta)(-\sin\theta g_1 + \cos\theta g_2) \\
&\wedge (\cos\theta \mathrm{d}g_1 + \sin\theta \mathrm{d}g_2 + (\cos\theta g_2 - \sin\theta g_1)\mathrm{d}\theta) g_3 \\
&\wedge \cdots \\
&\wedge (\cos\theta \mathrm{d}g_1 + \sin\theta \mathrm{d}g_2 + (\cos\theta g_2 - \sin\theta g_1)\mathrm{d}\theta) g_n \\
&= (-\sin\theta \mathrm{d}Pg_1 + \cos\theta \mathrm{d}Pg_2) \wedge (\mathrm{d}\theta + \mathrm{d}g_1 g_2) \wedge (\cos\theta \mathrm{d}g_1 g_3 + \sin\theta \mathrm{d}g_2 g_3) \\
&\wedge \cdots
\end{aligned}$$

$$\wedge (\cos\theta \mathrm{d}g_1g_n + \sin\theta \mathrm{d}g_2g_n).$$

Since[2] $\mathrm{d}G = \mathrm{d}Pg_2 \wedge \cdots \wedge \mathrm{d}Pg_n \wedge \mathrm{d}g_1g_2 \wedge \cdots \wedge \mathrm{d}g_1g_n$, we have

$$\begin{aligned}\mathrm{d}G\mathrm{d}L = {} & \mathrm{d}Pg_2 \wedge \cdots \wedge \mathrm{d}Pg_n \wedge \mathrm{d}g_1g_2 \wedge \cdots \wedge \mathrm{d}g_1g_n \\ & \wedge (-\sin\theta \mathrm{d}Pg_1 + \cos\theta \mathrm{d}Pg_2) \wedge (\mathrm{d}\theta + \mathrm{d}g_1g_2) \wedge (\cos\theta \mathrm{d}g_1g_3 + \sin\theta \mathrm{d}g_2g_3) \\ & \wedge \cdots \\ & \wedge (\cos\theta \mathrm{d}g_1g_n + \sin\theta \mathrm{d}g_2g_n) \\ & = \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}Pg_1 \wedge \mathrm{d}Pg_2 \cdots \wedge \mathrm{d}Pg_n \wedge \mathrm{d}g_1g_2 \wedge \cdots \wedge \mathrm{d}g_1g_n \wedge \mathrm{d}g_2g_3 \\ & \wedge \cdots \wedge \mathrm{d}g_2g_n \\ & = \sin^{n-1}\theta \mathrm{d}\theta \omega_1\omega_2 \cdots \omega_n\omega_{12} \cdots \omega_{1n}\omega_{23} \cdots \omega_{2n}.\end{aligned}$$

**Theorem 1** *The kinematic density for pairs of intersecting lines $G$ and $L$ in $\mathbb{R}^n$ is*

$$\mathrm{d}G\mathrm{d}L = \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}P \mathrm{d}u_{n-1} \mathrm{d}u_{n-2}, \tag{6}$$

*where $P$ is the intersection of the lines $G$ and $L$, $\mathrm{d}u_{n-1}$ denotes $(n-1)$-dimensional volume element of unit sphere $U_{n-1}$, and $\mathrm{d}u_{n-2}$ denotes $(n-2)$-dimensional volume element of unit sphere $U_{n-2}$ in $\mathbb{R}^n$.*

**Remark 1** The formula (6) of kinematic density for pairs of intersecting lines is different from the formula (3) about kinematic density for pairs of intersecting lines.

**Remark 2** For $n = 3$, we have[4]

$$\mathrm{d}G\mathrm{d}L = \sin^2\theta \mathrm{d}\theta \mathrm{d}P \mathrm{d}u_1 \mathrm{d}u_2,$$

which is different from the formula (2).

**Remark 3** But, for $n = 2$, we have

$$\mathrm{d}G\mathrm{d}L = \sin\theta \mathrm{d}\theta \mathrm{d}P \mathrm{d}u_1,$$

which is the same as the formula (1).

**Proof** Since[2]

$$\mathrm{d}P = \omega_1\omega_2 \cdots \omega_n, \quad \mathrm{d}u_{n-1} = \omega_{12} \cdots \omega_{1n}, \quad \mathrm{d}u_{n-2} = \omega_{23} \cdots \omega_{2n},$$

by formula (5), we get

$$\begin{aligned}\mathrm{d}G\mathrm{d}L &= \sin^{n-1}\theta \mathrm{d}\theta \omega_1\omega_2 \cdots \omega_n\omega_{12} \cdots \omega_{1n}\omega_{23} \cdots \omega_{2n} \\ &= \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}P \mathrm{d}u_{n-1} \mathrm{d}u_{n-2}.\end{aligned}$$

**Theorem 2** *The kinematic density for pairs of intersecting lines $G$ and $L$ in $\mathbb{R}^n$ is*

$$\mathrm{d}G\mathrm{d}L = \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}\Sigma_G \mathrm{d}P_G \mathrm{d}G, \tag{7}$$

*where $\Sigma$ is the plane spanned by the lines $G$ and $L$, $\mathrm{d}\Sigma_G$ is the kinematic density of $\Sigma$ for the rotations around $G$, and $\mathrm{d}P_G$ is the kinematic density of $P$ on $G$.*

**Proof** Since[2]

$$\mathrm{d}P_G = \omega_1, \quad \mathrm{d}G = \omega_2 \cdots \omega_n \omega_{12} \cdots \omega_{1n}, \quad \mathrm{d}\Sigma_G = \omega_{23} \cdots \omega_{2n},$$

it follows from formula (5) that

$$\begin{aligned}\mathrm{d}G\mathrm{d}L &= \sin^{n-1}\theta \mathrm{d}\theta \omega_1 \cdots \omega_n \omega_{12} \cdots \omega_{1n} \omega_{23} \cdots \omega_{2n} \\ &= \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}\Sigma_G \mathrm{d}P_G \mathrm{d}G. \qquad \square\end{aligned}$$

**Theorem 3** *The kinematic formula of the intersection of the lines $G$ and $L$ belonging to convex body $K$ in $\mathbb{R}^n$ is*

$$\int_{G \cap L \in K} \mathrm{d}G\mathrm{d}L = J_n O_{n-1} O_{n-2} V, \tag{8}$$

*where $J_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \mathrm{d}\theta$, $O_{n-1}$ is the surface area of the $(n-1)$-dimensional unit sphere, $O_{n-2}$ is the surface area of the $(n-2)$-dimensional unit sphere, and $V$ is the volume of $K$.*

**Proof** By formula (6), we obtain

$$\begin{aligned}\int_{G \cap L \in K} \mathrm{d}G\mathrm{d}L &= \int_{P \in K} \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}P \mathrm{d}u_{n-1} \mathrm{d}u_{n-2} \\ &= O_{n-1} O_{n-2} \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \mathrm{d}\theta \int_{P \in K} \mathrm{d}P \\ &= J_n O_{n-1} O_{n-2} V.\end{aligned}$$

Also by formule (4) and (7), we obtain

$$\begin{aligned}\int_{G \cap L \in K} \mathrm{d}G\mathrm{d}L &= \int_{G \cap L \in K} \sin^{n-1}\theta \mathrm{d}\theta \mathrm{d}P_G \mathrm{d}G \mathrm{d}\Sigma_G \\ &= \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \mathrm{d}\theta \int_{G \cap L \in K} \mathrm{d}P_G \int_{G \cap K \neq \emptyset} \mathrm{d}G \int \mathrm{d}\Sigma_G \\ &= \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \mathrm{d}\theta \int_{G \cap K \neq \emptyset} \mathrm{vol}(G \cap K) \mathrm{d}G \\ &= 2J_n O_{n-2} \int_{G \cap K \neq \emptyset} \sigma \mathrm{d}G \\ &= 2J_n O_{n-2} I_1^{(n)}(K) = J_n O_{n-1} O_{n-2} V. \qquad \square\end{aligned}$$

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