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Some Results on Sum Graph, Integral Sum Graph and Mod Sum Graph

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Abstract Let N denote the set of positive integers. The sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is said to be a sum graph if it is isomorphic to the sum graph of some $S \subset N$. By using the set Z of all integers instead of N, we obtain the definition of the integral sum graph. A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer z and a labelling, λ, of the vertices of G with distinct elements from $\{0, 1, 2, \ldots, z-1\}$ so that $uv \in E$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. In this paper, we prove that flower tree is integral sum graph. We prove that Dutch m-wind-mill (D_m) is integral sum graph and mod sum graph, and give the sum number of D_m.

Keywords sum graph; integral sum graph; mod sum graph; flower tree; Dutch m-wind-mill.

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1. Introduction

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of Ref. [1] unless otherwise specified.

Harary[2] introduced the idea of sum graphs and integral sum graphs. At first, let N denote the set of positive integers. The sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u+v \in S$. A graph G is said to be a sum graph if it is isomorphic to the sum graph of some $S \subset N$. The sum number $\sigma(G)$ of a connected graph is the least nonnegative m of isolated vertices mK_1, such that $G \cup mK_1$ is a sum graph. In the above definition, by using the set Z of all integers instead of N, we obtain the definition of the integral sum graph. Analogously, the integral sum number $\zeta(G)$ is the smallest nonnegative m such that $G \cup mK_1$ is an integral sum graph. It is easy to see that the graph G is an integral sum graph if and only if $\zeta(G) = 0$. It is obvious that $\zeta(G) \leq \sigma(G)$.

Mod sum graph was introduced by Bolland, Laskar, Yurner and Domke[11] as a generalization of sum graph. A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer z and a labelling, λ, of the vertices of G with distinct elements from $\{0, 1, 2, \ldots, z-1\}$ so that $uv \in E$

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if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. The mod sum number $\rho(G)$ of a connected graph G is the least nonnegative number r of isolated vertices rK_1 so that $G \cup rK_1$ is a mod sum graph. Note that G is a mod sum graph if and only if $\rho(G) = 0$. Any sum graph can be considered as a mod sum graph by choosing a sufficiently large modulus z. However the converse is not true.

Although some results on sum graphs, integral sum graphs and mod sum graphs were solved[3−10], a considerable number of problems still remain unsolved. One of them is the conjecture: "Every tree is an integral sum graph", which was proposed by Zhibo Chen in Ref. [6]. In Section 2, we prove that flower tree is integral sum graph. Although our result is not the final solution on integral sum trees, it improves the previous result. In section 3, we prove that Dutch m-wind-mill (D_m) is integral sum graph and mod sum graph, and give the sum number of D_m.

To simplify notations, throughout this paper, we assume that the vertices of G are already identified by their labels.

2. Flower tree

On every vertex of a path $P_n = y_1 y_2 \cdots y_n$ called the main path (when n is odd, y_n is excluded), we identify a path, whose length identified on y_{2k-1} and y_{2k} is k ($k = [\frac{n}{2}]$). And we call the obtained graph the flower tree, denoted by FT_n.

To prove that the flower tree is integral sum graph, we need the following arguments.

Let G_1 and G_2 be two graphs. Suppose $r_1 \in V(G_1)$ is a fixed vertex of G_1, called the root of G_1, and $r_2 \in V(G_2)$ is the root of G_2. Let $(G, r) \equiv (G_1, r_1) \bowtie (G_2, r_2)$ denote the graph G with root r, which is obtained from G_1 and G_2 by identifying r_1 and r_2 as one vertex r. When the vertex r is not considered as the root of the obtained graph, we simply denote the graph as $G = (G_1, r_1) \bowtie (G_2, r_2)$. It is clear that $V(G) = (V(G_1) - \{r_1\}) \cup (V(G_2) - \{r_2\}) \cup \{r\}$ and $E(G) = E(G_1) \cup E(G_2)$. We may consider G_1 and G_2 as the subgraphs of G and consider r, r_1 and r_2 as the same vertex.

Assume that f is an integral sum labelling of $V(G)$ with distinct integers. Then we have two facts:

Fact 1 For any nonzero integer m, $m*(f(x))$ is also an integral sum labelling of G (this labelling is denoted as mf.)

Fact 2 Suppose that G is a nontrivial graph. Then $f(x) \neq 0$ for every vertex x of G if and only if the maximum degree $\triangle(G) < |V(G)| - 1$.

Lemma 2.1[6] *Let (G_i, r_i) be a graph with root r_i and φ_i be its integral sum graph labelling, $i = 1, 2$. Suppose that*

(i) $\forall\ x \neq 0$, $x \in V(G_i) - \{r_i\}$, $i = 1, 2$;

(ii) $x = y$ *if and only if* $x = r_1$ *and* $y = r_2$;

(iii) $a \pm b \neq x$ *for all distinct* $a, b \in V(G_1)$ *and* $\forall\ x \in V(G_2) - \{r_2\}$; *and*

(iv) $x \pm y \neq a$ *for all distinct* $x, y \in V(G_2)$ *and* $\forall\ a \in V(G_1) - \{r_1\}$.

Then $G \equiv (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph.

Theorem 2.1 *Let $(G_1, r_1) = a_0a_1a_2 \cdots a_n$ be a path with length $n \geq 5$ and root $r_1 = a_1$. Let (G_2, r_2) be a connected graph with root r_2, which satisfies the following conditions:*

(1) The maximum degree $\triangle(G_2) < |V(G_2)| - 1$, and

(2) There is an integral sum labelling ψ of G_2 such that $x \neq -r_2$, $\forall\, x \in V(G_2) - \{r_2\}$.

Then $G \equiv (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph.

Proof We give an integer sum labelling φ of G_1, such that $a_0 = t$, $a_1 = 1$, $a_2 = 1 + t$, $a_3 = 2 + t$, $a_4 = -1 - t$, $a_5 = 3 + 2t$, and $a_k = a_{k-2} - a_{k-1}$, for all $k \geq 5$, where we assume that $t > 2\max\{|x| \,|\forall\, x \in V(G_2)\}$.

The given condition (1) implies that G_2 is nontrivial by Fact 2 and $x \neq 0$ for $\forall\, x \in V(G_2)$. By Fact 1, we define an integral sum labelling of G_1 as $\varphi_1 = m\varphi$, where $m = r_2$. We also use φ_2 to denote ψ. To show G is an integral sum graph, we only need to show that φ_1 and φ_2 satisfy the conditions (i)–(iv) in lemma 2.1. Clearly, (i) is satisfied. Note that $r_1 = ma_1 = m = r_2$. So (ii) is satisfied. From the labelling of G_1, we can see that $|x| \geq t \geq |y|$ for $\forall\, x \in V(G_1) - \{r_1\}$ and $\forall\, y \in V(G_2)$. It is obvious that (iv) is satisfied, since $|x \pm y| < t \leq a$ for all distinct $x, y \in V(G_2)$ and $\forall\, a \in V(G_1) - \{r_1\}$. So we only need to consider (iii). We can see that $a \pm b = 0, \pm m$, or $|a \pm b| \geq |m|(t-1) > |x|$, for all distinct $a, b \in V(G_1)$ and $\forall\, x \in V(G_2) - \{r_2\}$. By the condition (2), we have $x \neq 0, \pm m$, for any $x \in V(G_2) - \{r_2\}$. Then (iii) is also satisfied.

So the conclusion is right.

Lemma 2.2[6] *Let $(G_1, r_1) = a_0a_1a_2 \cdots a_n$ be a path with length $n \geq 4$ and root $r_1 = a_0$. Let (G_2, r_2) be a connected graph with root r_2, which satisfies the following conditions:*

(1) The maximum degree $\triangle(G_2) < |V(G_2)| - 1$, and

(2) There is an integral sum labelling ψ of G_2 such that $x \neq -r_2, \forall\, x \in V(G_2) - \{r_2\}$.

Then $G \equiv (G_1, r_1) \bowtie (G_2, r_2)$ is an integral sum graph.

Theorem 2.2 *Flower tree(FT_n) is integral sum graph, for any integer $n \geq 2$.*

Proof When $n \leq 7$, see Fig. 1, it is easy to confirm that.

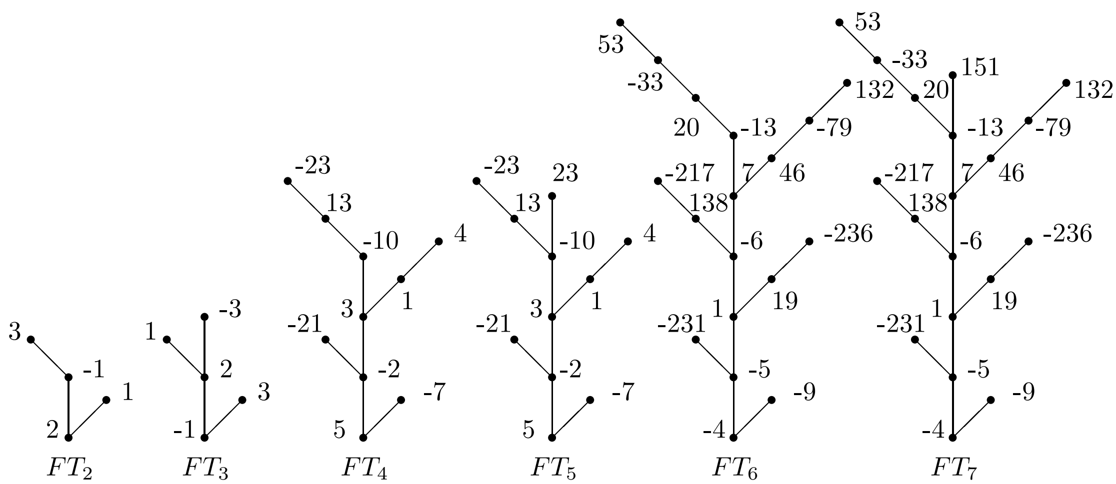

Fig. 1

When $n \geq 8$, there are two cases to consider:

Case 1. n is even. We give the path $P_k = a_0a_1a_2 \ldots a_k$ that identifies a labelling $f$to the vertices y_{n-2} and y_{n-1} of the main path. Let

$$a_0 = 1; a_1 = t; a_2 = 1 + t; a_3 = 1 + 2t; a_4 = -1 - t;$$
$$a_k = a_{k-2} - a_{k-1}, \quad \text{when} \;\; 5 \leq k \leq [\frac{n}{2}] + 1.$$

Let $(G_2, r_2) = (FT_6, y_6)$, where FT_6 is a connected graph, the maximum degree $\triangle(FT_6) < |V(FT_6) - 1|$, and $y_6 \neq -x$ for $\forall\ x \in V(FT_6) - \{y_6\}$. Let $(G_1, r_1) = (a_0a_1 \cdots a_5, a_0)$. y_6f is an integral sum labelling of G_1, where we assume that $t > 2\max\{|x| \;\big|x \in V(FT_6)\}$. They satisfy the conditions of Lemma 2.2. We obtain that FT'_7 is an integral sum graph with root $y_7 = a_1$ by Lemma 2.2, and $y_7 \neq -x$ for $\forall\ x \in V(FT'_7) - \{y_7\}$.

Next we let $(G_2, r_2) = (FT'_7, y_7)$ and $(G_1, r_1) = (a_0a_1 \cdots a_5, a_0)$. y_7f is an integral sum labelling of G_1, where we assume that $t > 2\max\{|x| \;\big|x \in V(FT'_7)\}$. By Lemma 2.2, we obtain FT_8 is an integral sum graph. By the same method we can obtain $FT_{10}, FT_{12}, \ldots$ are integral sum graphs.

Case 2. n is odd. We give the path $P_k = a_0a_1a_2 \cdots a_k$ that identifies a labelling g to the vertices y_{n-2} and y_{n-1} of the main path. Let

$$a_0 = t; a_1 = 1; a_2 = 1 + t; a_3 = 2 + t; a_4 = -1 - t;$$
$$a_k = a_{k-2} - a_{k-1}, \quad \text{when} \;\; 5 \leq k \leq [\frac{n}{2}] + 1.$$

Let $(G_2, r_2) = (FT_7, y_7)$, where FT_7 is a connected graph, the maximum degree $\triangle(FT_7) < |V(FT_7) - 1|$, and $y_7 \neq -x$ for $\forall\ x \in V(FT_7) - \{y_7\}$. Let $(G_1, r_1) = (a_0a_1 \ldots a_5, a_1)$. y_7f is an integral sum labelling of G_1, where we assume that $t > 2\max\{|x| \;\big|x \in V(FT_7)\}$. They satisfy the conditions of Lemma 2.2. We obtain that FT'_8 is an integral sum graph with root $y_8 = a_0$ by Theorem 2.1, and $y_8 \neq -x$ for $\forall\ x \in V(FT'_8) - \{y_8\}$.

Next we let $(G_2, r_2) = (FT'_8, y_8)$ and $(G_1, r_1) = (a_0a_1 \cdots a_5, a_1)$. y_8f is an integral sum labelling of G_1, where we assume that $t > 2\max\{|x| \;\big|x \in V(FT'_8)\}$. By Theorem 2.1, we obtain that FT_9 is an integral sum graph. By the same method we can obtain $FT_{11}, FT_{13}, \ldots$ are integral sum graph.

Thus, for $n \geq 2$, FT_n is integral sum graph.

3. Dutch m-wind-mill

For any integer $m \geq 2$, a Dutch m-wind-mill (D_m) is the graph defined by a pair of sets (V, E), where $V = \{c, v_1, v_2, v_3, v_4, \ldots, v_{2m-1}, v_{2m}\}$ and $E = \{(c, v_i), (v_{2n-1}, v_{2n}) \mid i = 1, 2, \ldots, 2m, n = 1, 2, \ldots, m\}$. The vertex c is called the center of the Dutch m-wind mill, each edge (c, v_i), for $i = 1, 2, \ldots, 2m$, is called a spoke, and the edge (v_{2n-1}, v_{2n}), for $n = 1, 2, \ldots, m$, is called the rim.

3.1 The result on $\sigma(D_m)$

In this section, we shall determine the value of $\sigma(D_m)$ for $m \geq 2$. Throughout this section, let $t = \sigma(D_m)$ and $S = V(D_m \cup tK_1)$. There exists a finite subset L of N such that L gives an optimal sum labelling of $D_m \cup tK_1$.

Theorem 3.1 *For any integer $m \geq 3$, $\sigma(D_m) = 2$.*

Proof First we confirm $\sigma(D_m) \geq 2$. Without loss of generality, we may consider v_1 greater than other vertices in $V - \{c\}$. So the spoke $c + v_1$ is isolated. Now we consider $v_1 + v_2$. If $v_1 + v_2 \notin V$, however $c + v_1 \neq v_1 + v_2$, so $\sigma(D_m) \geq 2$. If $v_1 + v_2 \in V$, There are two cases to consider:

Case 1. $v_1 + v_2 \in \{v_3, v_4, \ldots, v_{2m-1}, v_{2m}\}$, which contradicts the supposition that v_1 is the greatest in $V - \{c\}$.

Case 2. $v_1 + v_2 = c$. Without loss of generality, let v_3 be greater than other vertices in $V - \{c, v_1, v_2\}$. We can see that $c + v_3 \notin V - \{c\}$ and $c + v_3 \neq c + v_1$, so $\sigma(D_m) \geq 2$. Thus we obtain that $\sigma(D_m) \geq 2$.

Now we consider the following sum labelling of $D_m \cup 2K_1$:

$$c = 2; \quad v_{2n-1} = 3 + 2(n-1), \quad n = 1, 2, \ldots, m;$$

$$v_{2(m-n)} = 3 + 2(m+n), \quad n = 0, 1, 2, \ldots, m-1.$$

Let $S - V(D_m) = \{b_1, b_2\}$ and

$$b_1 = 4m + 4, \quad b_2 = 4m + 3.$$

It is easy to verify that the labelling is the sum labelling of $D_m \cup 2K_1$.

Thus $2 \leq \sigma(D_m) \leq 2$ and the result follows.

3.2 D_m is integral sum graph

Theorem 3.2 *For any integer $m \geq 2$, $\zeta(D_m) = 0$.*

Proof Let $D_m = G(S) = (\{0, 2, -2, 2^2, -2^2, \ldots, 2^m, -2^m\})$. It is easy to verify that

$$0 + v_i = v_i, \quad i = 1, 2, \ldots, 2m;$$

$$v_{2n-1} + v_{2n} = 0, \quad i = 1, 2, \ldots, m;$$

$$v_{2i-1} + v_{2j} \notin S, \quad i, j = 1, 2, \ldots, m, i \neq j.$$

So $\zeta(D_m) = 0$.

3.3 D_m is mod sum graph

Theorem 3.3 *For any integer $m \geq 2$, $\rho(D_m) = 0$.*

Proof Let $D_m = G^{m+}(S) = (\{25, 3, 22, 3+25, 22+(m-1)25, 3+2\times 25, 22+(m-2)25, \ldots, 3+(m-1)25, 22+25\})$ with modulus $z = 25m$. Thus we can see that

$$v_{2i-1} + v_{2j} (\mathrm{mod}\ z) \notin S, \quad i, j = 1, 2, \ldots, m, i \neq j;$$

$$v_{2n} + c \equiv v_{2n-2} (\text{mod } z),\ n = 2, 3, \dots, m;\quad v_2 + c \equiv v_{2m} (\text{mod } z);$$

$$v_{2n-1} + c \equiv v_{2n+1} (\text{mod } z),\ n = 1, 2, \dots, m-1;\quad v_{2m-1} + c \equiv v_1 (\text{mod } z).$$

So the labelling is a mod sum labelling of D_m. Thus $\rho(D_m) = 0$.

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