

Basic Hypergeometric Series—Quick Access to Identities

CHU Wen Chang^{1,2}, WANG Xiao Xia^{3†}

(1. Department of Applied Mathematics, Dalian University of Technology, Liaoning 116024, China;

2. Dipartimento Di Matematica, Università Degli Studi Di Lecce,

Lecce-Arnesano P. O. Box 193, 73100 Lecce, Italia;

3. Department of Mathematics, Shanghai University, Shanghai 200444, China)

(E-mail: chu.wenchang@unile.it; xiaoxiadlut@yahoo.com.cn)

Abstract The importance of basic hypergeometric series has been widely recognized. For non specialists, it is necessary to have a quick introduction to this classical but flourishing subject. An effort along this direction will be made in the present article.

Keywords Basic hypergeometric series; Heine's tranformation; Watson's transformation; Jacobi's triple product identity; Rogers-Ramanujan identities.

Document code A

MR(2000) Subject Classification 05A10; 05A19

Chinese Library Classification O157.1

For two indeterminate q and x , define the shifted factorial by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, the shifted factorial of infinite order reads as

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k) \quad \text{and} \quad (x; q)_n = \frac{(x; q)_\infty}{(xq^n; q)_\infty} \quad \text{where } n \in \mathbb{Z}.$$

Its product and fraction forms are abbreviated compactly to

$$\begin{aligned} [a, b, \dots, c; q]_n &= (a; q)_n (b; q)_n \cdots (c; q)_n, \\ \left[\begin{matrix} a, b, \dots, c \\ \alpha, \beta, \dots, \gamma \end{matrix} \middle| q \right]_n &= \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}. \end{aligned}$$

Following Bailey^[5] and Slater^[57], the unilateral and bilateral basic hypergeometric series are defined, respectively, by

$$\begin{aligned} {}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right] &= \sum_{n=0}^{\infty} z^n \left[\begin{matrix} a_0, a_1, \dots, a_r \\ q, b_1, \dots, b_s \end{matrix} \middle| q \right]_n, \\ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right] &= \sum_{n=0}^{\infty} z^n \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q \right]_n. \end{aligned}$$

Received date: 2006-06-13; **Accepted date:** 2007-01-18

Foundation item: Group GNSAGA (Italy).

[†] Corresponding author

Obviously, the unilateral series may be considered as a special case of the corresponding bilateral one. Throughout the paper, the base q will be confined to $|q| < 1$ for non-terminating series.

The study of basic hypergeometric series was essentially started by Euler (1748) with the emphasis on generating functions of partitions. Subsequently, Gauss (1813), Cauchy (1825) and Heine (1846) found several transformation and summation formulae of lower order q -series. From the end of the nineteenth century to the middle of the twentieth century, there were many great mathematicians (such as Rogers, Ramanujan, Watson, Bailey and Slater, mainly from Cambridge University) who made important contributions to basic hypergeometric series. Among them, Jackson embarked on a lifelong time program of developing the q -series theory systematically. During the “dark” period from 1950’s to 1970’s, Andrews and Askey had persistently organized several conferences and written numerous papers, convincing the mathematical public how useful the q -series are to classical partitions, number theory and other disciplines. Thanks to these two mathematicians, basic hypergeometric series has become a flourishing research area today. For a comprehensive account of the story, we refer the reader to the monumental monograph written by Gasper and Rahman^[39].

During the last two decades, as the q -series theory develops explosively, its application fields spread from other mathematical disciplines to physics and computer sciences. For non specialist, it is necessary to have a quick access to this classical but modern subject. An effort along this direction will be made in the present article. We shall try to prepare for readers a soft-landing on the q -series field by introducing several classical formulae and transformations, which have fundamental importance in the q -series theory and applications. The reader will be guided to go smoothly through the topics around q -series without having to read the huge volumes such as Ref. [39], even though it is needless to say that it is indispensable for further study on basic hypergeometric series.

1. q -binomial theorem

Define the Gaussian binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

We start with the q -binomial theorem:

$$(x; q)_n = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k. \quad (1)$$

Proof Writing $(x; q)_n$ formally as a polynomial

$$(x; q)_n = \sum_{k=0}^n A_k x^k, \quad (2)$$

where A_k is independent of variable x . It is trivial to see that

$$(1 - q^n x)(x; q)_n = (1 - x)(qx; q)_n$$

which is equivalent to

$$(1 - q^n x) \sum_{k=0}^n A_k x^k = (1 - x) \sum_{k=0}^n A_k q^k x^k.$$

Equating coefficients of x^k across the last equation, we find the relation:

$$A_k = \frac{q^n - q^{k-1}}{1 - q^k} A_{k-1}, \quad \text{where } k = 1, 2, \dots, n.$$

Iterating the above equation for k times leads us to the following closed form:

$$A_k = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} A_0 = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}},$$

where $A_0 = 1$ has been justified by setting $x = 0$ in (2). This completes the proof of the q -binomial theorem. \square

2. q -binomial expansion formula ($|z| < 1$)

$${}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix} \middle| q; z \right] = \frac{(az; q)_\infty}{(z; q)_\infty}. \quad (3)$$

This formula was found by Cauchy in 1893 and is considered as one of the most important formulae in the q -series theory.

Proof We express the right member of this formula in terms of Maclaurin series

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{+\infty} B_n z^n,$$

where $\{B_n\}$ are independent of variable z . From the product representation in the above equation, it is easy to verify that

$$(1 - z) \frac{(az; q)_\infty}{(z; q)_\infty} = (1 - az) \frac{(qaz; q)_\infty}{(qz; q)_\infty}$$

or equivalently

$$(1 - z) \sum_{n=0}^{\infty} B_n z^n = (1 - az) \sum_{n=0}^{\infty} B_n q^n z^n.$$

Equating the coefficients of z^n on both sides of the last equation, we get the following recurrence relation:

$$B_n = B_{n-1} \frac{1 - q^{n-1}a}{1 - q^n}.$$

Iterating this recurrence relation for n times, we obtain

$$B_n = \frac{(a; q)_n}{(q; q)_n} B_0 = \frac{(a; q)_n}{(q; q)_n},$$

where $B_0 = 1$ is confirmed for the same reason as that for $A_0 = 1$. \square

3. The Jacobi triple product identity and the limiting form

$$\sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n = [q, x, q/x; q]_{\infty}, \quad (4a)$$

$$\sum_{n=0}^{+\infty} (-1)^n \{1 + 2n\} q^{\binom{1+n}{2}} = (q; q)_{\infty}^3. \quad (4b)$$

This identity was found by Jacobi in 1829. The proof we are going to present here is the simplest one^[3, §10.4] due to Cauchy (1843) and Gauss (1866). For the different proofs through Durfee rectangles and iteration method, see Chu^[21] and Hardy-Wright^[43, §19.8].

Proof Performing the replacements $n \rightarrow m + n$ and $x \rightarrow xq^{-n}$ in the q -binomial Theorem 1, we obtain the following identity:

$$(q^{-n}x; q)_{m+n} = \sum_{k=0}^{m+n} (-1)^k \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\binom{k}{2} - kn} x^k. \quad (5)$$

By means of the almost trivial equation

$$(q^{-n}x; q)_{m+n} = (q^{-n}x; q)_n (x; q)_m = (-1)^n x^n q^{-\binom{n+1}{2}} (q/x; q)_n (x; q)_m,$$

the identity (5) can be restated as:

$$(x; q)_m (q/x; q)_n = \sum_{k=0}^{m+n} (-1)^{k+n} \begin{bmatrix} m+n \\ k \end{bmatrix} q^{\binom{k-n}{2}} x^{k-n} = \sum_{k=-n}^m (-1)^k \begin{bmatrix} m+n \\ k+n \end{bmatrix} q^{\binom{k}{2}} x^k, \quad (6)$$

where we have performed the replacement $k \rightarrow k + n$ for the second expression. On account of the fact that

$$\lim_{n, m \rightarrow \infty} \begin{bmatrix} m+n \\ k+n \end{bmatrix} = \lim_{n, m \rightarrow \infty} \frac{(q; q)_{m+n}}{(q; q)_{k+n} (q; q)_{m-k}} = \frac{1}{(q; q)_{\infty}},$$

the limit case of $m, n \rightarrow \infty$ in the identity (6) results in

$$(x; q)_{\infty} (q/x; q)_{\infty} = \frac{1}{(q; q)_{\infty}} \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} x^k$$

which is equivalent to Jacobi's triple product identity (4a).

Splitting the sum displayed in (4a) into two parts, we can proceed

$$\begin{aligned} (1-x)[q, qx, q/x; q]_{\infty} &= \sum_{n=1}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n + \sum_{n=-\infty}^0 (-1)^n q^{\binom{n}{2}} x^n \\ &= \sum_{n=0}^{+\infty} (-1)^{n+1} q^{\binom{n+1}{2}} x^{n+1} + \sum_{n=0}^{+\infty} (-1)^n q^{\binom{n+1}{2}} x^{-n} \\ &= \sum_{n=0}^{+\infty} (-1)^n q^{\binom{n+1}{2}} x^{-n} \{1 - x^{2n+1}\}, \end{aligned}$$

where the replacements $n \rightarrow n + 1$ and $n \rightarrow -n$ have been performed respectively for the first and the second sum in the middle line. Dividing by $1 - x$ the extreme members of the last equations and then applying L'Hôpital's rule to the limit $x \rightarrow 1$, we obtain (4b). \square

4. Quintuple product identity and limiting forms

The typical expressions of the quintuple product identity may be reproduced as

$$[q, z, q/z; q]_{\infty} [qz^2, q/z^2; q^2]_{\infty} = \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - zq^n\} (qz^3)^n \quad (7a)$$

$$= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - z^{1+6n}\} (q^2/z^3)^n \quad (7b)$$

$$= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - (q/z^2)^{1+3n}\} (qz^3)^n \quad (7c)$$

$$= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - (qz^2)^{2+3n}\} (q^2/z^3)^n \quad (7d)$$

which has two different limiting cases:

$$\sum_{k=-\infty}^{+\infty} (1 + 6k) q^{3\binom{k}{2} + 2k} = [q, q, q; q]_{\infty} [q, q; q^2]_{\infty}, \quad (8a)$$

$$\sum_{k=-\infty}^{+\infty} (1 + 3k) q^{3\binom{k}{2} + \frac{5}{2}k} = [q, q^{1/2}, q^{1/2}; q]_{\infty} [q^2, q^2; q^2]_{\infty}. \quad (8b)$$

There are many proofs for this important result. For the historical account, see Cooper's recent survey paper^[36] and Carlitz-Subbarao^[13], where the identity has been verified by multiplying two triple products. Recently Chen, Chu and Gu^[14] find a finite form of it. Here we present the proof due to Bhargava^[11] via iteration method.

Proof With the same method we have used in proving the q -binomial formula, we first define $f(z)$ and express it as a Laurent series

$$f(z) = [q, z, q/z; q]_{\infty} [q^2, qz^2, q/z^2; q^2]_{\infty} = \sum_{n=-\infty}^{+\infty} \Omega_n z^n,$$

where Ω_n is independent of variable z . According to the product representation in the definition of $f(z)$, we can readily verify that

$$f(zq) = q^{-1} z^{-3} f(z) \quad \text{and} \quad f(q/z) = -q^{-1} z^2 f(z)$$

or equivalently

$$\sum_{n=-\infty}^{+\infty} \Omega_n z^n = \sum_{n=-\infty}^{+\infty} \Omega_{n-3} q^{n-2} z^n \quad \text{and} \quad \sum_{n=-\infty}^{+\infty} \Omega_n z^n = - \sum_{n=-\infty}^{+\infty} \Omega_{-n-2} q^{-1-n} z^n.$$

Equating the coefficients of z^n on both sides of the last two equations respectively, we get the following recurrence relations:

$$\Omega_n = q^{n-2} \Omega_{n-3}, \quad (9a)$$

$$\Omega_n = -q^{-n-1} \Omega_{-n-2}. \quad (9b)$$

Iterating the recurrence relation (9a), we get further three expressions:

$$\begin{aligned}\Omega_{3n} &= q^{3n-2}\Omega_{3(n-1)} = \cdots = \Omega_0 q^{3\binom{n}{2}+n}, \\ \Omega_{3n+1} &= q^{3n-1}\Omega_{3(n-1)+1} = \cdots = \Omega_1 q^{3\binom{n}{2}+2n}, \\ \Omega_{3n-1} &= q^{3n-3}\Omega_{3(n-1)-1} = \cdots = \Omega_{-1} q^{3\binom{n}{2}}.\end{aligned}$$

Letting $n = -1$ in (9b), we derive $\Omega_{-1} = 0$. Alternatively, combining the case $n = 0$ of (9b) with the case $n = 1$ of (9a), we get

$$\Omega_0 = -q^{-1}\Omega_{-2} = -\Omega_1.$$

Consequently, $f(z)$ can be restated as follows:

$$f(z) = \sum_{n=-\infty}^{+\infty} \left\{ \Omega_0 q^{3\binom{n}{2}+n} z^{3n} + \Omega_1 q^{3\binom{n}{2}+2n} z^{3n+1} \right\} \quad (10a)$$

$$= \Omega_0 \sum_{n=-\infty}^{+\infty} \{1 - q^n z\} q^{3\binom{n}{2}} (qz^3)^n. \quad (10b)$$

In order to confirm (7a), we have to evaluate Ω_0 . Recalling the Jacobi triple product identity (4a), we can expand $f(z)$ as follows

$$f(z) = \sum_{i=-\infty}^{+\infty} (-1)^i q^{\binom{i}{2}} z^i \sum_{k=-\infty}^{+\infty} (-1)^k q^{k^2} z^{2k}.$$

By invoking (4a) again, we get the following product expression:

$$\begin{aligned}\Omega_0 = [z^0]f(z) &= \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{-2k}{2}+k^2} = \sum_{k=-\infty}^{+\infty} (-1)^k q^{6\binom{k}{2}+4k} \\ &= [q^6, q^2, q^4; q^6]_{\infty} = (q^2; q^2)_{\infty}.\end{aligned}$$

Substituting it into (10b) and then simplifying the result, we get the identity (7a).

Splitting the summation of (7a) into two parts and then reversing the summation order by $n \rightarrow -n$ for the first part, we confirm (7b) as follows:

$$\begin{aligned}&[q, z, q/z; q]_{\infty} [qz^2, q/z^2; q^2]_{\infty} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - q^n z\} (qz^3)^n \\ &= \sum_{n=-\infty}^{+\infty} \left\{ z^{3n} q^{n+3\binom{n}{2}} - z^{1+3n} q^{2n+3\binom{n}{2}} \right\} \\ &= \sum_{n=-\infty}^{+\infty} \left\{ z^{-3n} q^{2n+3\binom{n}{2}} - z^{1+3n} q^{2n+3\binom{n}{2}} \right\} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \{1 - z^{1+6n}\} (q^2/z^3)^n.\end{aligned}$$

Instead, if we reverse by $n \rightarrow -n - 1$ for the second sum displayed in the last middle line, then (7c) will be proved similarly:

$$\begin{aligned} & [q, z, q/z; q]_{\infty} [qz^2, q/z^2; q^2]_{\infty} \\ &= \sum_{n=-\infty}^{+\infty} \left\{ z^{3n} q^{n+3\binom{n}{2}} - z^{-2-3n} q^{1+4n+3\binom{n}{2}} \right\} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \left\{ 1 - (q/z^2)^{1+3n} \right\} (qz^3)^n. \end{aligned}$$

Splitting the formula (7b) into two parts, performing the replacement $n \rightarrow n + 1$ for the second sum and then simplifying the result, we have

$$\begin{aligned} & \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \left\{ 1 - z^{1+6n} \right\} (q^2/z^3)^n \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} (q^2/z^3)^n - \sum_{n=-\infty}^{+\infty} z^{1+6(n+1)} q^{3\binom{n+1}{2}} (q^2/z^3)^{n+1} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} (q^2/z^3)^n - \sum_{n=-\infty}^{+\infty} z^{4+3n} q^{3\binom{n}{2}+2+5n} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \left\{ 1 - (qz^2)^{2+3n} \right\} (q^2/z^3)^n. \end{aligned}$$

This is exactly the formula displayed in (7d).

Finally dividing by $1 - z$ both sides of the equation (7b), we obtain

$$[q, qz, q/z; q]_{\infty} [qz^2, q/z^2; q^2]_{\infty} = \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \left\{ \frac{1 - z^{1+6n}}{1 - z} \right\} (q^2/z^3)^n.$$

Letting $z \rightarrow 1$ and applying L'Hôpital's rule, we get the limiting form (8a). Another limiting form (8b) can be shown analogously by dividing by $1 - q/z^2$ both sides of (7c):

$$[q, z, q/z; q]_{\infty} [qz^2, q^3/z^2; q^2]_{\infty} = \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}} \left\{ \frac{1 - (q/z^2)^{1+3n}}{1 - q/z^2} \right\} (qz^3)^n.$$

5. Heine's q -Euler transformations

In 1878, Heine discovered the following important transformation formulae:

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; z \right] = \frac{[b, az; q]_{\infty}}{[c, z; q]_{\infty}} \times {}_2\phi_1 \left[\begin{matrix} c/b, & z \\ & az \end{matrix} \middle| q; b \right] \quad (11a)$$

$$= \frac{[c/b, bz; q]_{\infty}}{[c, z; q]_{\infty}} \times {}_2\phi_1 \left[\begin{matrix} abz/c, & b \\ & bz \end{matrix} \middle| q; c/b \right] \quad (11b)$$

$$= \frac{[abz/c, q]_{\infty}}{[z; q]_{\infty}} \times {}_2\phi_1 \left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \middle| q; abz/c \right] \quad (11c)$$

Proof It suffices to show transformation (11a) because the two other identities (11b) and (11c) follow from iterating (11a).

Recalling q -binomial formula (3):

$$\frac{(b; q)_n}{(c; q)_n} = \frac{(b; q)_\infty (q^n c; q)_\infty}{(c; q)_\infty (q^n b; q)_\infty} = \frac{(b; q)_\infty}{(c; q)_\infty} {}_1\phi_0 \left[\begin{matrix} c/b \\ - \end{matrix} \middle| q; q^n b \right] = \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} (q^n b)^k$$

we may manipulate basic hypergeometric series as follows:

$$\begin{aligned} {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; z \right] &= \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n \\ &= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \frac{(q^n c; q)_\infty}{(q^n b; q)_\infty} \\ &= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} (q^n b)^k. \end{aligned}$$

Exchanging the order of the last double sum and then applying the q -binomial formula again, we prove (11a) as follows:

$$\begin{aligned} {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; z \right] &= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} b^k \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (q^k z)^n \\ &= \frac{(b; q)_\infty}{(c; q)_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} \frac{(q^k a z; q)_\infty}{(q^k z; q)_\infty} b^k \\ &= \frac{[b, a z; q]_\infty}{[c, z; q]_\infty} \sum_{k=0}^{\infty} \frac{(c/b; q)_k}{(q; q)_k} \frac{(z; q)_k}{(a z; q)_k} b^k \\ &= \frac{[b, a z; q]_\infty}{[c, z; q]_\infty} {}_2\phi_1 \left[\begin{matrix} c/b, & z \\ & a z \end{matrix} \middle| q; b \right]. \end{aligned}$$

From the transformation (11a), the ${}_2\phi_1$ -series in the right side of (11a) can be transformed as

$${}_2\phi_1 \left[\begin{matrix} c/b, & z \\ & a z \end{matrix} \middle| q; b \right] = \frac{[c/b, b z; q]_\infty}{[a z, b; q]_\infty} {}_2\phi_1 \left[\begin{matrix} a b z / c, & b \\ & b z \end{matrix} \middle| q; c/b \right].$$

Substituting it into the right member of (11a), we get the identity stated in (11b).

Iterating (11a) once again, we obtain

$${}_2\phi_1 \left[\begin{matrix} a b z / c, & b \\ & b z \end{matrix} \middle| q; c/b \right] = \frac{[a b z / c, c; q]_\infty}{[b z, c/b; q]_\infty} {}_2\phi_1 \left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \middle| q; a b z / c \right].$$

Substituting it into the right side of the equation (11b) leads us to (11c). This completes the proof of Heine's q -Euler transformations. \square

6. q -Gauss summation formula ($|c/ab| < 1$)

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; c/ab \right] = \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}. \quad (12)$$

Proof For this very useful theorem in basic hypergeometric series, there are several proofs. The asymptotic approach via Cesàro theorem can be found in Chu^[28]. Here we derive it directly by putting $z = c/ab$ in Heine's transformation (11b):

$$\begin{aligned} {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; c/ab \right] &= \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty} {}_2\phi_1 \left[\begin{matrix} 1, & b \\ & c/a \end{matrix} \middle| q; c/b \right] \\ &= \frac{(c/a; q)_\infty (c/b; q)_\infty}{(c; q)_\infty (c/ab; q)_\infty}. \end{aligned}$$

We remark that one can also provide an independent derivation for it, following exactly the same proof of (11b). \square

7. Ramanujan's bilateral ${}_1\psi_1$ -series identity ($|c/a| < |z| < 1$)

$${}_1\psi_1 \left[\begin{matrix} a \\ c \end{matrix} \middle| q; z \right] = \left[\begin{matrix} q, & az, & q/az, & c/a \\ c, & z, & c/az, & q/a \end{matrix} \middle| q \right]_\infty.$$

This result was found by Ramanujan in his notebook, which has been considered as one of the most important identities in bilateral basic hypergeometric series. For more information related to this identity see [10, Part III: Entry 17]. Application to the representation of natural numbers by square-sums can be found in Milne^[50].

Proof We prove this identity again by iteration method, which has extensively been investigated by Fine^[37]. For this purpose, we define $F(z)$ and express it in terms of Laurent series:

$$F(z) = \left[\begin{matrix} az, & q/az \\ z, & c/az \end{matrix} \middle| q \right]_\infty = \sum_{n=-\infty}^{+\infty} \Omega_n z^n, \quad (13)$$

where Ω_n is independent of variable z . From the product representation in (13), we can verify, without difficulty, the functional equation

$$(1-z)qF(z) = (c-qaz)F(qz)$$

which is equivalent to the following:

$$(1-z)q \sum_{n=-\infty}^{+\infty} \Omega_n z^n = (c-qaz) \sum_{n=-\infty}^{+\infty} \Omega_n q^n z^n. \quad (14)$$

Equating the coefficients of z^n across equation (14), we get the recurrence relation:

$$\Omega_n = \frac{1-q^{n-1}a}{1-q^{n-1}c} \Omega_{n-1} = \frac{(a; q)_n}{(c; q)_n} \Omega_0,$$

where the last expression is obtained by iterating the first one for n times.

In view of the q -binomial formula (3), we have the following expansion:

$$\left[\begin{matrix} az, & q/az \\ z, & c/az \end{matrix} \middle| q \right]_\infty = {}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix} \middle| q; z \right] {}_1\phi_0 \left[\begin{matrix} q/c \\ - \end{matrix} \middle| q; \frac{c}{az} \right]$$

$$= \sum_{i=0}^{+\infty} \frac{(a; q)_i}{(q; q)_i} z^i \sum_{j=0}^{+\infty} \frac{(q/c; q)_j}{(q; q)_j} \left(\frac{c}{az}\right)^j.$$

By means of the definition Ω_n and the q -Gauss summation formula (12), we have

$$\begin{aligned} \Omega_0 &= [z^0] \left[\begin{matrix} az, & q/az \\ z, & c/az \end{matrix} \middle| q \right]_{\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k (q/c; q)_k}{(q; q)_k (q; q)_k} \left(\frac{c}{a}\right)^k \\ &= {}_2\phi_1 \left[\begin{matrix} a, & q/c \\ & q \end{matrix} \middle| q; \frac{c}{a} \right] = \left[\begin{matrix} q/a, & c \\ q, & c/a \end{matrix} \middle| q \right]_{\infty} \end{aligned}$$

which leads us to the following relation:

$$\left[\begin{matrix} az, & q/az \\ z, & c/az \end{matrix} \middle| q \right]_{\infty} = \Omega_0 \sum_{n=-\infty}^{+\infty} \frac{(a; q)_n}{(c; q)_n} z^n = \left[\begin{matrix} q/a, & c \\ q, & c/a \end{matrix} \middle| q \right]_{\infty} {}_1\psi_1 \left[\begin{matrix} a \\ c \end{matrix} \middle| q; z \right].$$

This completes the proof of Ramanujan's ${}_1\psi_1$ -summation formula. \square

8. The q -analogue of Chu-Vandermonde convolution

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; q^n c/b \right] = \frac{(c/b; q)_n}{(c; q)_n}, \quad (15a)$$

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; q \right] = \frac{(c/b; q)_n}{(c; q)_n} b^n, \quad (15b)$$

$$\sum_{k=0}^n \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} = \begin{bmatrix} x+y \\ n \end{bmatrix}. \quad (15c)$$

There are numerous combinatorial interpretations of q -binomial convolutions. We refer to Chu^[17] and Gessel^[40] respectively for lattice path counting and Durfee rectangle method^[1, §2.3].

Proof It is easy to see that identity (15a) is the terminating case $a = q^{-n}$ of the q -Gauss summation formula (12):

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; q^n c/b \right] = \left[\begin{matrix} q^n c, & c/b \\ c, & q^n c/b \end{matrix} \middle| q \right]_{\infty} = \frac{(c/b; q)_n}{(c; q)_n}.$$

According to the definition of unilateral q -series, expanding the ${}_2\phi_1$ -series stated in the identity (15a) and then reversing the summation index, we have

$$\begin{aligned} {}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; q^n c/b \right] &= \sum_{k=0}^n \left[\begin{matrix} q^{-n}, & b \\ q, & c \end{matrix} \middle| q \right]_{n-k} (q^n c/b)^{n-k} \\ &= (q^n c/b)^n \left[\begin{matrix} q^{-n}, & b \\ q, & c \end{matrix} \middle| q \right]_n {}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \middle| q; q \right]. \end{aligned}$$

Comparing (15a) with the last identity leads us to

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{1-n}/c \\ & q^{1-n}/b \end{matrix} \middle| q; q \right] = \left[\begin{matrix} q, & c/b \\ q^{-n}, & b \end{matrix} \middle| q \right]_n (q^{-n} b/c)^n.$$

Under the replacements $c \rightarrow q^{1-n}/b$ and $b \rightarrow q^{1-n}/c$, the last identity becomes the identity stated in (15b).

From the definition of the Gaussian binomial coefficient, we have the following two relations:

$$\begin{aligned} \begin{bmatrix} x \\ k \end{bmatrix} &= \frac{(q^{x-k+1}; q)_k}{(q; q)_k} = (-1)^k q^{kx - \binom{k}{2}} \frac{(q^{-x}; q)_k}{(q; q)_k}, \\ \begin{bmatrix} y \\ n-k \end{bmatrix} &= \frac{(q^{y-n+k+1}; q)_{n-k}}{(q; q)_{n-k}} = (-1)^k q^{nk - \binom{k}{2}} \frac{(q^{1+y-n}; q)_n (q^{-n}; q)_k}{(q; q)_n (q^{1+y-n}; q)_k}. \end{aligned}$$

Then the left side of (15c) can be reformulated as:

$$\begin{aligned} \sum_{k=0}^n \begin{bmatrix} x \\ k \end{bmatrix} \begin{bmatrix} y \\ n-k \end{bmatrix} q^{(x-k)(n-k)} &= \frac{(q^{1+y-n}; q)_n}{(q; q)_n} q^{nx} {}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{-x} \\ q^{1+y-n} \end{matrix} \middle| q; q \right] \\ &= \frac{(q^{1+x+y-n}; q)_n}{(q; q)_n} = \begin{bmatrix} x+y \\ n \end{bmatrix}, \end{aligned}$$

where the ${}_2\phi_1$ -series has been evaluated by (15b) as:

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, & q^{-x} \\ q^{1+y-n} \end{matrix} \middle| q; q \right] = q^{-nx} \frac{(q^{1+x+y-n}; q)_n}{(q^{1+y-n}; q)_n}.$$

This completes the proof of the q -analogue of Chu-Vandermonde convolution. \square

9. Inverse series relations due to Carlitz (1973)

Let $\{a_i\}$ and $\{b_j\}$ be two complex sequences such that the polynomials defined by

$$\phi(x; 0) = 1 \quad \text{and} \quad \phi(x; n) = \prod_{k=0}^{n-1} (a_k + xb_k) \quad \text{for } n = 1, 2, \dots$$

differ from zero for $x = q^n$ with n being non-negative integers. Then we have the following inverse series relations due to Carlitz^[12]:

$$F(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k; n) G(k), \quad (16a)$$

$$G(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{a_k + q^k b_k}{\phi(q^n; k+1)} F(k). \quad (16b)$$

The Carlitz inversions may be considered as q -analogue of the inverse series relations due to Gould-Hsu^[41], which read as:

$$f(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \phi(k; n) g(k) \iff g(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\phi(n; k+1)} f(k).$$

For $a_k \equiv 1$ and $b_k \equiv 0$, we have $\phi(x; n) \equiv 1$ and then corresponding classical delta-inversion:

$$f(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} g(k) \iff g(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} f(k).$$

Taking account of the importance of the Carlitz inversions, we present two different proofs.

Proof To prove the dual implications (16a) \Rightarrow (16b), it is sufficient to verify one implication because one system of equations with $F(n)$ in terms of $G(k)$ can be considered as the (unique) solution of another system with $G(n)$ in terms of $F(k)$, and vice versa.

\Leftarrow . We first reproduce the original proof due to Carlitz^[12]. Suppose that the relations of $G(n)$ in terms of $F(k)$ are valid. We have to verify the relations of $F(n)$ in terms of $G(k)$.

Substituting the relations of $G(n)$ in terms of $F(k)$ into the right hand side of those of $F(n)$ in terms of $G(k)$ and observing that

$$\begin{bmatrix} n \\ k \end{bmatrix} \times \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} \times \begin{bmatrix} n-i \\ k-i \end{bmatrix},$$

we get the double sum expression

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k; n) G(k) \\ &= \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} \phi(q^k; n) \sum_{i=0}^k (-1)^i \begin{bmatrix} k \\ i \end{bmatrix} \frac{a_i + q^i b_i}{\phi(q^k; i+1)} F(i) \\ &= \sum_{i=0}^n (a_i + q^i b_i) \begin{bmatrix} n \\ i \end{bmatrix} F(i) \sum_{k=i}^n (-1)^{k+i} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^k; n)}{\phi(q^k; i+1)} q^{\binom{n-k}{2}}. \end{aligned}$$

Let $S(i, n)$ stand for the inner sum with respect to k . It is trivial to see that

$$S(n, n) = \frac{\phi(q^n; n)}{\phi(q^n; n+1)} = \frac{1}{a_n + q^n b_n}$$

which implies that the double sum reduces to $F(n)$ when $i = n$.

In order to prove that the double sum is equal to $F(n)$, we have to verify that $S(i, n) = 0$ for $0 \leq i < n$.

Noting that $\frac{\phi(q^k; n)}{\phi(q^k; i+1)}$ is a polynomial of degree $n-i-1$ in q^k , we can write it formally as

$$\frac{\phi(q^k; n)}{\phi(q^k; i+1)} = \sum_{j=i+1}^n \beta_j q^{k(n-j)},$$

where $\{\beta_j\}$ are constants independent of k . Therefore $S(i, n)$ can be reformulated as follows:

$$\begin{aligned} S(i, n) &= \sum_{k=i}^n (-1)^{k+i} \begin{bmatrix} n-i \\ k-i \end{bmatrix} q^{\binom{n-k}{2}} \sum_{j=i+1}^n \beta_j q^{k(n-j)} \\ &= \sum_{j=i+1}^n \beta_j q^{\binom{n-i}{2} + i(n-j)} \sum_{k=i}^n (-1)^{k+i} \begin{bmatrix} n-i \\ k-i \end{bmatrix} q^{\binom{k-i}{2} + (1+i-j)(k-i)}, \end{aligned}$$

where we have applied the binomial relation

$$\binom{n-k}{2} = \binom{(n-i) - (k-i)}{2} = \binom{n-i}{2} + \binom{k-i}{2} + (1-n+i)(k-i).$$

By means of Euler's q -difference formula (1), evaluate the inner sum as

$$\sum_{k=i}^n (-1)^{k+i} \begin{bmatrix} n-i \\ k-i \end{bmatrix} q^{\binom{k-i}{2} + (1+i-j)(k-i)} = (q^{1+i-j}; q)_{n-i}.$$

It vanishes for $i < j < n$, which implies $S(i, n) = 0$ for $0 \leq i < n$.

\Rightarrow . An alternative proof is based on telescoping method, which is the q -analogue of the proof presented by Chu^[15]. Assuming that (16a) is true for all $n \in \mathbb{N}_0$, we should verify the truth of (16b).

In fact, substituting the first relation into the second, we reduce the question to the confirmation of the following orthogonal relation:

$$\sum_{k=i}^n (-1)^{k+i} \{a_k + q^k b_k\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i; k)}{\phi(q^n; k+1)} q^{\binom{k-i}{2}} = \begin{cases} 1, & i = n; \\ 0, & i \neq n. \end{cases} \quad (17)$$

It is obvious that the relation is valid for $i = n$. We therefore need to verify it only when $i < n$. For that purpose, we introduce the sequence

$$\tau_k := \begin{bmatrix} n-i-1 \\ k-i-1 \end{bmatrix} \frac{\phi(q^i; k)}{\phi(q^n; k)} q^{\binom{k-i}{2}}.$$

Then it is not hard to check that the summand in (17) can be expressed as follows:

$$\tau_k + \tau_{k+1} = \{a_k + q^k b_k\} \begin{bmatrix} n-i \\ k-i \end{bmatrix} \frac{\phi(q^i; k)}{\phi(q^n; k+1)} q^{\binom{k-i}{2}}.$$

Separating the two extreme terms indexed with $k = i$ and $k = n$ from the sum displayed in (17)

$$\tau_{i+1} = \frac{\phi(q^i; i+1)}{\phi(q^n; i+1)} \quad \text{and} \quad \tau_n = \frac{\phi(q^i; n)}{\phi(q^n; n)} q^{\binom{n-i}{2}}$$

and then appealing to the telescoping method^[42, §2.6], we find that the left member of (17) equals

$$\begin{aligned} & \tau_{i+1} + (-1)^{n+i} \tau_n + \sum_{i < k < n} (-1)^{k+i} \{\tau_k + \tau_{k+1}\} \\ &= \{\tau_{i+1} + (-1)^{n+i} \tau_n\} - \{\tau_{i+1} + (-1)^{n+i} \tau_n\} = 0. \end{aligned}$$

This completes the proof of (17). \square

The inversion technique has been extensively explored by Chu^[20,23] systematically to prove combinatorial identities, which has been shown to be powerful and great potential to deal with terminating basic hypergeometric series identities.

10. The q -Pfaff-Saalschütz summation theorem

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \middle| q; q \right] = \left[\begin{matrix} c/a, & c/b \\ & c, & c/ab \end{matrix} \middle| q \right]_n. \quad (18)$$

This identity was first found by Saalschütz in 1890. Sears^[54] derived several important transformation formulae for the balanced series. For symmetric extensions and applications through

chain reactions to multiple series, we refer to Chu^[27,31,33].

Proof Recall Heine's q -Euler transformation (11c):

$${}_2\phi_1 \left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \middle| q; abz/c \right] = \frac{(z; q)_\infty}{(abz/c; q)_\infty} \times {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; z \right]$$

which can be reformulated through the q -binomial theorem (3), as a product of two basic hypergeometric series:

$${}_2\phi_1 \left[\begin{matrix} c/a, & c/b \\ & c \end{matrix} \middle| q; abz/c \right] = {}_1\phi_0 \left[\begin{matrix} c/ab \\ - \end{matrix} \middle| q; abz/c \right] \times {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; z \right].$$

Extracting the coefficient of z^n across the last equation, we have

$$\left[\begin{matrix} c/a, & c/b \\ q, & c \end{matrix} \middle| q \right]_n (ab/c)^n = \sum_{k=0}^n \left[\begin{matrix} a, & b \\ q, & c \end{matrix} \middle| q \right]_k \left[\begin{matrix} c/ab \\ q \end{matrix} \middle| q \right]_{n-k} (ab/c)^{n-k}$$

which can be restated as:

$$\begin{aligned} \left[\begin{matrix} c/a, & c/b \\ q, & c \end{matrix} \middle| q \right]_n &= \left[\begin{matrix} c/ab \\ q \end{matrix} \middle| q \right]_n \sum_{k=0}^n \left[\begin{matrix} a, & b, & q^{-n} \\ q, & c, & q^{1-n}ab/c \end{matrix} \middle| q \right]_k q^k \\ &= \left[\begin{matrix} c/ab \\ q \end{matrix} \middle| q \right]_n {}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \middle| q; q \right]. \end{aligned}$$

This is equivalent to the q -Pfaff-Saalschütz formula (18).

The q -Pfaff-Saalschütz summation theorem (18) can also be proved through series composition method. Recalling the q -analogue of Chu-Vandermonde convolution (15a), we have

$$\frac{(a; q)_k}{(c; q)_k} = {}_2\phi_1 \left[\begin{matrix} q^{-k}, & c/a \\ & c \end{matrix} \middle| q; q^k a \right].$$

Therefore the left side of (18) can be rewritten as:

$$\sum_{k=0}^n \left[\begin{matrix} q^{-n}, & b \\ q, & q^{1-n}ab/c \end{matrix} \middle| q \right]_k q^k \sum_{i=0}^k q^{ki} \left[\begin{matrix} q^{-k}, & c/a \\ q, & c \end{matrix} \middle| q \right]_i a^i.$$

Changing the summation order and then setting $k = i + j$, we can reduce the last double sum as

$$\begin{aligned} &\sum_{i=0}^n (-a)^i \left[\begin{matrix} c/a, b, q^{-n} \\ q, c, q^{1-n}ab/c \end{matrix} \middle| q \right]_i q^{\binom{i+1}{2}} {}_2\phi_1 \left[\begin{matrix} q^{-n+i}, q^i b \\ q^{1+i-n}ab/c \end{matrix} \middle| q; q \right] \\ &= \sum_{i=0}^n (-a)^i \left[\begin{matrix} c/a, b, q^{-n} \\ q, c, q^{1-n}ab/c \end{matrix} \middle| q \right]_i q^{\binom{i+1}{2}} \frac{(q^{1-n}a/c; q)_{n-i}}{(q^{1+i-n}ab/c; q)_{n-i}} (q^i b)^{n-i} \\ &= \left[\begin{matrix} c/a \\ c/ab \end{matrix} \middle| q \right]_n {}_2\phi_1 \left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix} \middle| q; q^n c/b \right] = \left[\begin{matrix} c/a, & c/b \\ c, & c/ab \end{matrix} \middle| q \right]_n, \end{aligned}$$

where the q -Chu-Vandermonde formulae (15a) and (15b) have been applied to evaluating the both ${}_2\phi_1$ -series. \square

11. The terminating q -Dixon formula

$${}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, q^{-n} \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, q^{1+n}a \end{matrix} \middle| q; \frac{q^{1+n}a}{bc} \right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_n. \quad (19)$$

Proof Performing the replacements $a \rightarrow aq^n$, $b \rightarrow qa/bc$ and $c \rightarrow qa/c$ in the q -Pfaff-Saalschütz formula (18), we get

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & q^n a, & qa/bc \\ & qa/b, & qa/c \end{matrix} \middle| q; q \right] = \left[\begin{matrix} b, & c \\ qa/b, & qa/c \end{matrix} \middle| q \right]_n \left(\frac{qa}{bc} \right)^n.$$

This can be reformulated equivalently to the following identity:

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}} (q^k a; q)_n \left[\begin{matrix} a, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_k = q^{\binom{n}{2}} \left[\begin{matrix} a, b, c \\ qa/b, qa/c \end{matrix} \middle| q \right]_n \left(\frac{qa}{bc} \right)^n.$$

The last equation matches perfectly to (16a) under the parameter specifications $a_i \equiv 1$ and $b_i = -aq^i$ as well as

$$F(k) = q^{\binom{k}{2}} \left[\begin{matrix} a, b, c \\ qa/b, qa/c \end{matrix} \middle| q \right]_k \left(\frac{qa}{bc} \right)^k \quad \text{and} \quad G(k) = \left[\begin{matrix} a, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_k.$$

Then the dual relation (16b) from the Carlitz inversions

$$G(n) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - q^{2k}a}{(q^n a; q)_{k+1}} F(k)$$

reads explicitly as

$$\begin{aligned} \left[\begin{matrix} a, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_n &= \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{1 - q^{2k}a}{(q^n a; q)_{k+1}} q^{\binom{k}{2}} \left[\begin{matrix} q, b, c \\ qa/b, qa/c \end{matrix} \middle| q \right]_k \left(\frac{qa}{bc} \right)^k \\ &= \sum_{k=0}^n \frac{1 - q^{2k}a}{1 - q^n a} \left[\begin{matrix} a, & b, & c, & q^{-n} \\ q, & qa/b, & qa/c, & q^{n+1}a \end{matrix} \middle| q \right]_k \left(\frac{q^{1+n}a}{bc} \right)^k. \end{aligned}$$

Multiplying the both sides by the quotient $(1 - q^n a)/(1 - a)$, we get the terminating q -Dixon formula (19).

From this proof, we see that the inversion techniques are really powerful tools to deal with terminating hypergeometric series identities. One can consult Chu^[16,19,22,25,26] for more examples on this subject.

12. Watson's q -Whipple transformation

$${}_8\phi_7 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \middle| q; \frac{q^{2+n}a^2}{bcde} \right] \quad (20a)$$

$$= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_n {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \middle| q; q \right]. \quad (20b)$$

This transformation was discovered by Watson^[58].

Proof Rewriting the q -Pfaff-Saalschütz formula (18)

$${}_3\phi_2 \left[\begin{matrix} q^{-k}, & q^k a, & qa/de \\ & qa/d, & qa/e \end{matrix} \middle| q; q \right] = \left[\begin{matrix} d, & e \\ qa/d, & qa/e \end{matrix} \middle| q \right]_k \left(\frac{qa}{de} \right)^k,$$

we can express the ${}_8\phi_7$ -series in (20a) explicitly as follows:

$$\begin{aligned} & {}_8\phi_7 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \middle| q; \frac{q^{2+n}a^2}{bcde} \right] \\ &= \sum_{k=0}^n \frac{1-q^{2k}a}{1-a} \left[\begin{matrix} a, b, c, q^{-n} \\ q, qa/b, qa/c, q^{1+n}a \end{matrix} \middle| q \right]_k \left(\frac{q^{1+n}a}{bc} \right)^k {}_3\phi_2 \left[\begin{matrix} q^{-k}, q^k a, qa/de \\ qa/d, qa/e \end{matrix} \middle| q; q \right] \\ &= \sum_{k=0}^n \frac{1-q^{2k}a}{1-a} \left[\begin{matrix} a, b, c, q^{-n} \\ q, qa/b, qa/c, q^{1+n}a \end{matrix} \middle| q \right]_k \left(\frac{q^{1+n}a}{bc} \right)^k \sum_{i=0}^k \left[\begin{matrix} q^{-k}, q^k a, qa/de \\ qa/d, qa/e \end{matrix} \middle| q \right]_k q^i. \end{aligned}$$

Interchanging the summation order, performing the replacement $k \rightarrow i + \iota$ on summation index and then simplifying the result, we can further reformulate the last double sum as follows:

$$\begin{aligned} \text{Eq(20a)} &= \sum_{i=0}^n \left[\begin{matrix} qa/de \\ q, qa/d, qa/e \end{matrix} \middle| q \right]_i \sum_{k=i}^n \frac{1-q^{2k}a}{1-a} \frac{(q^{-k}; q)_i}{(q; q)_k} (a; q)_{k+i} \times \\ &\quad \left[\begin{matrix} b, & c, & q^{-n} \\ qa/b, & qa/c, & q^{1+n}a \end{matrix} \middle| q \right]_k \left(\frac{q^{1+n}a}{bc} \right)^k \\ &= \sum_{i=0}^n (-1)^i (qa; q)_{2i} \left[\begin{matrix} q^{-n}, b, c, qa/de \\ q, q^{1+n}a, qa/b, qa/c, qa/d, qa/e \end{matrix} \middle| q \right]_i \left(\frac{q^{1+n}a}{bc} \right)^i q^{-\binom{i}{2}} \times \\ &\quad \sum_{\iota=0}^{n-i} \frac{1-q^{2(i+\iota)a}}{1-q^{2i}a} \left[\begin{matrix} q^{2i}a, q^i b, q^i c, q^{-n+i} \\ q, q^{1+i}a/b, q^{1+i}a/c, q^{1+n+i}a \end{matrix} \middle| q \right]_{\iota} \left(\frac{q^{1+n-i}a}{bc} \right)^{\iota}. \end{aligned}$$

By means of the terminating q -Dixon formula (19), the last sum with respect to ι may be evaluated as:

$$\begin{aligned} & {}_6\phi_5 \left[\begin{matrix} q^{2i}a, & q^{1+i}\sqrt{a}, & -q^{1+i}\sqrt{a}, & q^i b, & q^i c, & q^{-(n-i)} \\ & q^i\sqrt{a}, & -q^i\sqrt{a}, & q^{1+i}a/b, & q^{1+i}a/c, & q^{1+n+i}a \end{matrix} \middle| q; \frac{q^{1+n-i}a}{bc} \right] \\ &= \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix} \middle| q \right]_{n-i} \end{aligned}$$

which leads to the following simplified expression:

$$\begin{aligned} \text{Eq(20a)} &= \sum_{i=0}^n (qa; q)_{2i} \left[\begin{matrix} q^{-n}, b, c, qa/de \\ q, q^{1+n}a, qa/b, qa/c, qa/d, qa/e \end{matrix} \middle| q \right]_i \times \\ &\quad \left[\begin{matrix} q^{1+2i}a, & qa/bc \\ q^{1+i}a/b, & q^{1+i}a/c \end{matrix} \middle| q \right]_{n-i} \left(-\frac{q^{1+n}a}{bc} \right)^i q^{-\binom{i}{2}} \end{aligned}$$

$$\begin{aligned}
&= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_n \sum_{i=0}^n \left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ q, & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \middle| q \right]_i q^i \\
&= \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \middle| q \right]_n {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \middle| q; q \right] = \text{Eq(20b)}.
\end{aligned}$$

This completes the proof of Watson's transformation. \square

13. The Rogers-Ramanujan identities

$$\sum_{m=0}^{+\infty} \frac{q^{m^2}}{(q; q)_m} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}, \quad (21a)$$

$$\sum_{m=0}^{+\infty} \frac{q^{m^2+m}}{(q; q)_m} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}. \quad (21b)$$

Up to now, there are a dozen proofs^[2,§1] for this beautiful pair of identities. The most recent ones are, respectively, due to Baxter^[9] based on the statistical mechanics and Lepowsky-Milne^[49] through the character formula on infinite dimensional Lie algebra^[47,§13].

Proof The limiting case $b, c, d, e, n \rightarrow \infty$ of transformation (20a–20b) reads as:

$$\sum_{m=0}^{+\infty} \frac{q^{m^2} a^m}{(q; q)_m} = \frac{1}{(qa; q)_\infty} \sum_{k=0}^{+\infty} (-1)^k \frac{1 - q^{2k} a}{1 - a} \frac{(a; q)_k}{(q; q)_k} q^{5\binom{k}{2} + 2k} a^{2k}. \quad (22)$$

For the last equation, separating the initial term corresponding to $k = 0$ from the sum on the right hand side, setting $a = 1$ and then applying the Jacobi triple product identity (4a), we get

$$\begin{aligned}
\sum_{m=0}^{+\infty} \frac{q^{m^2}}{(q; q)_m} &= \frac{1}{(q; q)_\infty} \left\{ 1 + \sum_{k=1}^{+\infty} (-1)^k (1 + q^k) q^{5\binom{k}{2} + 2k} \right\} \\
&= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^{5\binom{k}{2} + 2k} \\
&= \frac{[q^5, q^2, q^3; q^5]_\infty}{(q; q)_\infty} \frac{1}{[q, q^4; q^5]_\infty}.
\end{aligned}$$

Similarly, the case $a = q$ of (22) results in another identity due to Rogers-Ramanujan:

$$\begin{aligned}
\sum_{m=0}^{+\infty} \frac{q^{m+m^2}}{(q; q)_m} &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{+\infty} (-1)^k \{1 - q^{1+2k}\} q^{5\binom{k}{2} + 4k} \\
&= \frac{1}{(q; q)_\infty} \sum_{k=-\infty}^{+\infty} (-1)^k q^{5\binom{k}{2} + 4k} \\
&= \frac{[q, q^4, q^5; q^5]_\infty}{(q; q)_\infty} = \frac{1}{[q^2, q^3; q^5]_\infty}.
\end{aligned}$$

This completes the proofs of both identities displayed in (21a) and (21b). \square

14. Jackson's q -Dougall-Dixon formula

$${}_8\phi_7 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \middle| q; q \right] \quad (23a)$$

$$= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix} \middle| q \right]_n, \quad \text{where } q^{n+1}a^2 = bcde. \quad (23b)$$

This general result was first found by Jackson^[44]. As an immediate consequence, we can derive it from Watson's transformation.

Proof When $e = q^{1+n}a^2/bcd$ or equivalently $qa/de = q^{-n}bc/a$, the ${}_4\phi_3$ -series in Watson's transformation (20a–20b) reduces to a balanced ${}_3\phi_2$ -series, which can be evaluated by means of the q -Pfaff-Saalschütz formula (18). Therefore, we have in this case the following closed form:

$$\begin{aligned} & {}_8\phi_7 \left[\begin{matrix} q, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \middle| q; q \right] \\ &= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} \middle| q \right]_n {}_3\phi_2 \left[\begin{matrix} q^{-n}, & b, & c \\ & qa/d, & qa/e \end{matrix} \middle| q; q \right] \\ &= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} \middle| q \right]_n \times \left[\begin{matrix} qa/bd, & qa/cd \\ qa/d, & qa/bcd \end{matrix} \middle| q \right]_n. \end{aligned}$$

This proves Jackson's very well-poised ${}_8\phi_7$ -series identity. \square

15. The nonterminating ${}_6\phi_5$ -series identity

$${}_6\phi_5 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix} \middle| q; \frac{qa}{bcd} \right] \quad (24a)$$

$$= \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix} \middle| q \right]_\infty, \quad \text{where } \left| \frac{qa}{bcd} \right| < 1. \quad (24b)$$

Proof Making substitution $e = q^{1+n}a^2/bcd$ in Jackson's q -Dougall-Dixon formula

$${}_8\phi_7 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & q^{1+n}a^2/bcd, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & q^{-n}bcd/a, & q^{n+1}a \end{matrix} \middle| q; q \right] \quad (25a)$$

$$= \left[\begin{matrix} qa, & qa/bc, & qa/bd, & qa/cd \\ qa/b, & qa/c, & qa/d, & qa/bcd \end{matrix} \middle| q \right]_n \quad (25b)$$

and then noting that two limiting relations

$$\lim_{n \rightarrow \infty} \frac{(q^{1+n}a^2/bcd; q)_k}{(q^{n+1}a; q)_k} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{(q^{-n}; q)_k}{(q^{-n}bcd/a; q)_k} = \left(\frac{a}{bcd} \right)^k,$$

we have no difficulty to verify that the limiting case $n \rightarrow \infty$ of (25a–25b) yields the non-terminating ${}_6\phi_5$ -summation formula (24a–24b). We remark that when $d = q^{-n}$, the formula (24a–24b) reduces to the terminating q -Dixon formula (19). \square

16. Two well-poised bilateral series identities ($|q^{\pm 1}a/bcd| < 1$)

$${}_4\psi_4 \left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix} \middle| q \right]_{\infty}, \quad (26a)$$

$${}_5\psi_5 \left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{q^{-1}}{bcd} \right] = \left[\begin{matrix} q, & 1/bc, & 1/bd, & 1/cd \\ q/b, & q/c, & q/d, & q^{-1}/bcd \end{matrix} \middle| q \right]_{\infty} \times \frac{1 - 1/quv}{(1 - 1/u)(1 - 1/v)}. \quad (26b)$$

These two well-poised bilateral series identities can be considered as linear combinations of the ${}_3\psi_3$ -series identities due to Bailey^[7], which are reproduced below.

Proof For the nonterminating very well-poised ${}_6\phi_5$ -series formula (24a–24b), the case $a \rightarrow 1$ reads as

$$1 + \sum_{k=1}^{+\infty} (1 + q^k) \left[\begin{matrix} b, c, d \\ q/b, q/c, q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bcd} \right)^k = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (27)$$

Splitting the last sum into two parts and then reversing the summation order by $k \rightarrow -k$ for the second summation, we have

$$\begin{aligned} & 1 + \sum_{k=1}^{+\infty} (1 + q^k) \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bcd} \right)^k \\ &= 1 + \sum_{k=1}^{+\infty} \left[\begin{matrix} b, c, d \\ q/b, q/c, q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bcd} \right)^k + \sum_{k=1}^{+\infty} \left[\begin{matrix} b, c, d \\ q/b, q/c, q/d \end{matrix} \middle| q \right]_k \left(\frac{q^2}{bcd} \right)^k \\ &= \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bcd} \right)^k. \end{aligned}$$

This leads to the following identity:

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (28)$$

By reversing the summation index $k \rightarrow -k$, it is trivial to see that

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = {}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q^2}{bcd} \right].$$

We therefore get a reversal variant of (28):

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q^2}{bcd} \right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (29)$$

Subtracting (29) times w from (28), we derive the identity stated in (26a):

$${}_4\psi_4 \left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix} \middle| q \right]_{\infty}.$$

Instead, letting $a = q$ in the nonterminating very well-poised ${}_6\phi_5$ -series identity (24a–24b) and then multiplying both sides by $1 - q$, we have alternatively the following equation:

$$\sum_{k=0}^{+\infty} \{1 - q^{1+2k}\} \left[\begin{matrix} b, c, d \\ q^2/b, q^2/c, q^2/d \end{matrix} \middle| q \right]_k \left(\frac{q^2}{bcd} \right)^k = \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix} \middle| q \right]_{\infty}.$$

By means of the same method employed for (27), the left member of the above identity can be reformulated as:

$$\begin{aligned} & \sum_{k=0}^{+\infty} \{1 - q^{1+2k}\} \left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix} \middle| q \right]_k \left(\frac{q^2}{bcd} \right)^k \\ &= \sum_{k=0}^{+\infty} \left[\begin{matrix} b, c, d \\ q^2/b, q^2/c, q^2/d \end{matrix} \middle| q \right]_k \left(\frac{q^2}{bcd} \right)^k + \sum_{k=1}^{+\infty} \left[\begin{matrix} b, c, d \\ q^2/b, q^2/c, q^2/d \end{matrix} \middle| q \right]_{-k} \left(\frac{q^2}{bcd} \right)^{-k} \\ &= \sum_{k=-\infty}^{+\infty} \left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix} \middle| q \right]_k \left(\frac{q^2}{bcd} \right)^k. \end{aligned}$$

This gives rise to the following identity:

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix} \middle| q; \frac{q^2}{bcd} \right] = \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (30)$$

Writing explicitly the last ${}_3\psi_3$ -series in terms of bilateral sum, shifting the summation index $k \rightarrow k - 1$ and then performing the replacements $b \rightarrow qb$, $c \rightarrow qc$ and $d \rightarrow qd$, we obtain the following equivalent formula:

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{q^{-1}}{bcd} \right] = -\frac{1}{q} \left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (31)$$

Its reversal can be stated, after some little modification, as

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = \left[\begin{matrix} q, 1/bc, 1/bd, 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix} \middle| q \right]_{\infty}. \quad (32)$$

When the variable of ${}_3\psi_3$ -series is situated between those of (31) and (32), there holds the following reduced formula

$${}_3\psi_3 \left[\begin{matrix} b, & c, & d \\ 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{1}{bcd} \right] = 0. \quad (33)$$

We are going to prove an even more general identity:

$${}_{1+2\ell}\psi_{1+2\ell} \left[\begin{matrix} a_1, & \dots, & a_{1+2\ell} \\ 1/a_1, & \dots, & 1/a_{1+2\ell} \end{matrix} \middle| q; \frac{1}{a_1 a_2 \cdots a_{1+2\ell}} \right] = 0. \quad (34)$$

Denote by Θ the last bilateral series. Reversing the series by $k \rightarrow -k$ and then shifting the summation index by $k \rightarrow k - 1$, we can show $\Theta = 0$ as follows:

$$\Theta = \sum_{k=-\infty}^{\infty} \left[\begin{matrix} a_1, & \dots, & a_{1+2\ell} \\ 1/a_1, & \dots, & 1/a_{1+2\ell} \end{matrix} \middle| q \right]_k \left(\frac{1}{a_1 a_2 \cdots a_{1+2\ell}} \right)^k$$

$$\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \left[\begin{matrix} qa_1, & \dots, & qa_{1+2\ell} \\ q/a_1, & \dots, & q/a_{1+2\ell} \end{matrix} \middle| q \right]_k \left(\frac{1}{a_1 a_2 \cdots a_{1+2\ell}} \right)^k \\
&= \prod_{i=1}^{1+2\ell} \frac{1-1/a_i}{1-a_i} \sum_{k=-\infty}^{\infty} \left[\begin{matrix} a_1, & \dots, & a_{1+2\ell} \\ 1/a_1, & \dots, & 1/a_{1+2\ell} \end{matrix} \middle| q \right]_k \left(\frac{1}{a_1 a_2 \cdots a_{1+2\ell}} \right)^{k-1} \\
&= (-1)^{1+2\ell} \Theta = -\Theta = 0.
\end{aligned}$$

Considering the linear combination

$$\frac{\text{Eq(31)}}{(1-u)(1-v)} + \frac{uv \text{Eq(32)}}{(1-u)(1-v)} - \frac{(u+v) \text{Eq(33)}}{(1-u)(1-v)}$$

and simplifying the result, we get

$$\begin{aligned}
&{}_5\psi_5 \left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{q^{-1}}{bcd} \right] \\
&= \left[\begin{matrix} q, & 1/bc, & 1/bd, & 1/cd \\ q/b, & q/c, & q/d, & q^{-1}/bcd \end{matrix} \middle| q \right]_{\infty} \frac{1-1/quv}{(1-1/u)(1-1/v)}.
\end{aligned}$$

This confirms the bilateral series identity stated in (26b). \square

Further bilateral identities of this type and applications can be found in Chu^[29], where a systematic treatment of basic almost-poised hypergeometric series has been fulfilled. The partial fraction decomposition method has been employed by Chu^[24,30] to derive bilateral series identities with integer parameter differences. See Gasper^[38], Karlsson^[48] and Minton^[51] for additional information.

17. q -analogue of Dixon's theorem

$$\sum_{k=-n}^n (-1)^k \left[\begin{matrix} 2n \\ n+k \end{matrix} \right]^3 q^{k(3k-1)/2} = \frac{(q; q)_{3n}}{(q; q)_n^3}, \quad (35a)$$

$$\sum_{k=-n}^{1+n} (-1)^k \left[\begin{matrix} 1+2n \\ n+k \end{matrix} \right]^3 q^{k(3k-1)/2} = \frac{(q; q)_{3n+1}}{(q; q)_n^3}. \quad (35b)$$

These formulae were found by Jackson^[41], which are the q -analogue of the following well-known Dixon theorem:

$$\sum_{k=-n}^{n+\delta} (-1)^k \binom{2n+\delta}{n+k}^3 = \begin{cases} \binom{3n}{n, n, n}, & \delta = 0; \\ 0, & \delta = 1. \end{cases}$$

Proof Setting $b = c = d = q^{-n}$ in the identity (28), we can reformulate the left member of (28) as

$$\begin{aligned}
&{}_3\psi_3 \left[\begin{matrix} q^{-n}, & q^{-n}, & q^{-n} \\ q^{1+n}, & q^{1+n}, & q^{1+n} \end{matrix} \middle| q; q^{1+3n} \right] = \sum_{k=-n}^n \frac{(q^{-n}; q)_k^3}{(q^{n+1}; q)_k^3} q^{(1+3n)k} \\
&= \sum_{k=-n}^n (-1)^k \left\{ \frac{(q; q)_n (q; q)_n}{(q; q)_{n-k} (q; q)_{n+k}} \right\}^3 q^{k^2 + \binom{k}{2}}
\end{aligned}$$

while the corresponding right member of (28) becomes

$$\left[\begin{array}{cccc} q, & q^{1+2n}, & q^{1+2n}, & q^{1+2n} \\ q^{1+n}, & q^{1+n}, & q^{1+n}, & q^{1+3n} \end{array} \middle| q \right]_{\infty} = \frac{(q; q)_{3n} (q; q)_n^3}{(q; q)_{2n}^3}.$$

This leads us to the first identity (35a).

If we put $b = c = d = q^{-n}$ in (30) instead, then the left member of (30) may be rewritten as

$$\begin{aligned} {}_3\psi_3 \left[\begin{array}{ccc} q^{-n}, & q^{-n}, & q^{-n} \\ q^{2+n}, & q^{2+n}, & q^{2+n} \end{array} \middle| q; q^{2+3n} \right] &= \sum_{k=-(n+1)}^n \frac{(q^{-n}; q)_k^3}{(q^{n+2}; q)_k^3} q^{(2+3n)k} \\ &= \sum_{k=-n-1}^n (-1)^k \left\{ \frac{(q; q)_n (q; q)_{n+1}}{(q; q)_{n-k} (q; q)_{n+k+1}} \right\}^3 q^{2k+3\binom{k}{2}} \end{aligned}$$

while the corresponding right member reads as

$$\left[\begin{array}{cccc} q, & q^{2+2n}, & q^{2+2n}, & q^{2+2n} \\ q^{2+n}, & q^{2+n}, & q^{2+n}, & q^{2+3n} \end{array} \middle| q \right]_{\infty} = \frac{(q; q)_{3n+1} (q; q)_{n+1}^3}{(q; q)_{2n+1}^3}.$$

Equating the last two expressions results in the identity

$$\sum_{k=-n-1}^n (-1)^k \left[\begin{array}{c} 1+2n \\ n-k \end{array} \right]^3 q^{2k+3\binom{k}{2}} = \frac{(q; q)_{3n+1}}{(q; q)_n^3}.$$

With replacement $k \rightarrow -k$, this identity gives the second one (35b). \square

18. Bailey's transformation on terminating very well-poised $_{10}\phi_9$ -series

This most general transformation due to Bailey^[4] (cf. [39, III-28] and [57, §3.4.2]) reads as

$${}_{10}\phi_9 \left[\begin{array}{cccccccccc} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & \alpha, & \beta, & \gamma, & q^{-m} \\ \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}A \end{array} \middle| q; q \right] \quad (36a)$$

$$= \left[\begin{array}{cccc} qA, & qA/BC, & qA/BD, & qA/CD \\ qA/B, & qA/C, & qA/D, & qA/BCD \end{array} \middle| q \right]_m \times \quad (36b)$$

$${}_{10}\phi_9 \left[\begin{array}{cccccccccc} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & B, & C, & D, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A, & q^{-m} \\ \sqrt{\lambda}, & -\sqrt{\lambda}, & q\lambda/B, & q\lambda/C, & q\lambda/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}\lambda \end{array} \middle| q; q \right] \quad (36c)$$

with the parameters subject to restrictions $\lambda = qA^2/\alpha\beta\gamma$ and $q^{1+m}\lambda A/BCD = 1$.

Letting $\beta = A/\lambda$ in this transformation, we recover Jackson's ${}_8\phi_7$ -series identity (23a–23b). Interestingly, Bailey's $_{10}\phi_9$ -series transformation can also be proved by iterating Jackson's q -Dougall-Dixon summation formula.

Proof Rewriting Jackson's q -Dougall-Dixon formula (23a–23b):

$$\begin{aligned} {}_8\phi_7 \left[\begin{array}{ccccccc} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A, & q^k A \\ \sqrt{\lambda}, & -\sqrt{\lambda}, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1-k}\lambda/A, & q^{k+1}\lambda \end{array} \middle| q; q \right] \\ = \left[\begin{array}{cccc} \alpha, & \beta, & \gamma, & q\lambda \\ qA/\alpha, & qA/\beta, & qA/\gamma, & A/\lambda \end{array} \middle| q \right]_k, \quad \text{where } \lambda = qA^2/\alpha\beta\gamma, \end{aligned}$$

we can reformulate (36a) as a double sum:

$$\begin{aligned}
 & {}_{10}\phi_9 \left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & \alpha, & \beta, & \gamma, & q^{-m} \\ & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}A \end{matrix} \middle| q; q \right] \\
 &= \sum_{k=0}^m q^k \left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & A/\lambda, & q^{-m} \\ q, & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & q\lambda, & q^{1+m}A \end{matrix} \middle| q \right]_k \times \\
 &\quad \left[\begin{matrix} \alpha, & \beta, & \gamma, & q\lambda \\ qA/\alpha, & qA/\beta, & qA/\gamma, & A/\lambda \end{matrix} \middle| q \right]_k \\
 &= \sum_{k=0}^m q^k \left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & A/\lambda, & q^{-m} \\ q, & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & q^{1+m}A, & q\lambda \end{matrix} \middle| q \right]_k \times \\
 &\quad \sum_{i=0}^k q^i \left[\begin{matrix} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A, & q^k A, & q^{-k} \\ q, & \sqrt{\lambda}, & -\sqrt{\lambda}, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1-k}\lambda/A, & q^{k+1}\lambda \end{matrix} \middle| q \right]_k.
 \end{aligned}$$

With the summation order being exchanged, the last expression becomes

$$\begin{aligned}
 & {}_{10}\phi_9 \left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & \alpha, & \beta, & \gamma, & q^{-m} \\ & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}A \end{matrix} \middle| q; q \right] \\
 &= \sum_{i=0}^m \left[\begin{matrix} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A \\ q, & \sqrt{\lambda}, & -\sqrt{\lambda}, & qA/\alpha, & qA/\beta, & qA/\gamma \end{matrix} \middle| q \right]_i \times \\
 &\quad \frac{(qA; q)_{2i}}{(q\lambda; q)_{2i}} \left[\begin{matrix} B, & C, & D, & q^{-m} \\ qA/B, & qA/C, & qA/D, & q^{1+m}A \end{matrix} \middle| q \right]_i \left(\frac{qA}{\lambda} \right)^i \times \\
 &\quad {}_8\phi_7 \left[\begin{matrix} q^{2i}A, & q^{1+i}\sqrt{A}, & -q^{1+i}\sqrt{A}, & q^iB, & q^iC, & q^iD, & A/\lambda, & q^{-m+i} \\ & q^i\sqrt{A}, & -q^i\sqrt{A}, & q^{1+i}A/B, & q^{1+i}A/C, & q^{1+i}A/D, & q^{1+2i}\lambda, & q^{1+m+i}A \end{matrix} \middle| q; q \right].
 \end{aligned}$$

Evaluating the last ${}_8\phi_7$ -series through Jackson's q -Dougall-Dixon formula (23a–23b)

$$\begin{aligned}
 & \left[\begin{matrix} q^{1+2i}A, & qA/BC, & qA/BD, & qA/CD \\ q^{1+i}A/B, & q^{1+i}A/C, & q^{1+i}A/D, & q^{1-i}A/BCD \end{matrix} \middle| q \right]_{m-i} \\
 &= \left[\begin{matrix} qA, & qA/BC, & qA/BD, & qA/CD \\ qA/B, & qA/C, & qA/D, & qA/BCD \end{matrix} \middle| q \right]_m \times \\
 &\quad \frac{(q\lambda; q)_{2i}}{(qA; q)_{2i}} \left[\begin{matrix} qA/B, & qA/C, & qA/D, & q^{1+m}A \\ q\lambda/B, & q\lambda/C, & q\lambda/D, & q^{1+m} \end{matrix} \middle| q \right]_i \left(\frac{\lambda}{A} \right)^i
 \end{aligned}$$

and then making some routine simplifications, we get

$$\begin{aligned}
 & {}_{10}\phi_9 \left[\begin{matrix} A, & q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & \alpha, & \beta, & \gamma, & q^{-m} \\ & \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}A \end{matrix} \middle| q; q \right] \\
 &= \left[\begin{matrix} qA, & qA/BC, & qA/BD, & qA/CD \\ qA/B, & qA/C, & qA/D, & qA/BCD \end{matrix} \middle| q \right]_m \times
 \end{aligned}$$

$$\begin{aligned}
& \sum_{i=0}^m q^i \left[\begin{matrix} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A, & B, & C, & D, & q^{-m} \\ q, & \sqrt{\lambda}, & -\sqrt{\lambda}, & qA/\alpha, & qA/\beta, & qA/\gamma, & q\lambda/B, & q\lambda/C, & q\lambda/D, & q^{1+m}\lambda \end{matrix} \middle| q \right]_i \\
&= \left[\begin{matrix} qA, & qA/BC, & qA/BD, & qA/CD \\ qA/B, & qA/C, & qA/D, & qA/BCD \end{matrix} \middle| q \right]_m \times \\
& {}_{10}\phi_9 \left[\begin{matrix} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & B, & C, & D, & \alpha\lambda/A, & \beta\lambda/A, & \gamma\lambda/A, & q^{-m} \\ & \sqrt{\lambda}, & -\sqrt{\lambda}, & q\lambda/B, & q\lambda/C, & q\lambda/D, & qA/\alpha, & qA/\beta, & qA/\gamma, & q^{1+m}\lambda \end{matrix} \middle| q; q \right]
\end{aligned}$$

which is exactly Bailey's transformation on very well-poised ${}_{10}\phi_9$ -series. \square

19. Bailey's bilateral ${}_6\psi_6$ -series identity ($|qa^2/bcde| < 1$)

$${}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right] \quad (37a)$$

$$= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \middle| q \right]_{\infty}. \quad (37b)$$

This is one of the deepest basic hypergeometric formulae found by Bailey^[6]. There are several proofs of this important identity up to now. For the most elementary proof, we refer to the recent paper by Chu^[34], where the modified Abel lemma on summation by parts has been employed. Here we reproduce a proof provided by Jouhet and Schlosser^[46]. Schlosser^[53] also gave another proof for this important result.

Proof For $\lambda = qa^2/cev$, we may restate Bailey's transformation on terminating very well-poised ${}_{10}\phi_9$ -series (36a–36c) as

$${}_{10}\phi_9 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & d, & u, & c, & e, & v, & q^{-m} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/d, & qa/u, & qa/c, & qa/e, & qa/v, & q^{1+m}a \end{matrix} \middle| q; q \right] \quad (38a)$$

$$= \left[\begin{matrix} qa, & qa/bd, & qa/bu, & qa/du \\ qa/b, & qa/d, & qa/u, & qa/bdu \end{matrix} \middle| q \right]_m \times \quad (38b)$$

$${}_{10}\phi_9 \left[\begin{matrix} \lambda, & q\sqrt{\lambda}, & -q\sqrt{\lambda}, & b, & d, & u, & \lambda c/a, & \lambda e/a, & \lambda v/a, & q^{-m} \\ & \sqrt{\lambda}, & -\sqrt{\lambda}, & q\lambda/b, & q\lambda/d, & q\lambda/u, & qa/c, & qa/e, & qa/v, & q^{1+m}\lambda \end{matrix} \middle| q; q \right]. \quad (38c)$$

Perform the replacements $m \rightarrow 2n$, $a \rightarrow q^{-2n}a$, $b \rightarrow q^{-n}b$, $c \rightarrow q^{-n}c$, $d \rightarrow q^{-n}d$ and $e \rightarrow q^{-n}e$ in the last transformation. With the summation index k being shifted to $k + n$, the ${}_{10}\phi_9$ -series in (38a) becomes

$$\begin{aligned}
& \sum_{k=-n}^n q^k \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n}a, & q^n u, & q^n v, & q^{-n} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1+n}a, & q^{1-n}a/u, & q^{1-n}a/v, & q^{1+n} \end{matrix} \middle| q \right]_k \times \\
& \left[\begin{matrix} q, 1/a \\ u/a, v/a \end{matrix} \middle| q \right]_{2n} \left[\begin{matrix} q/b, q/c, q/d, q/e, u, v, u/a, v/a \\ b/a, c/a, d/a, e/a, q, q, qa, q/a \end{matrix} \middle| q \right]_n \left(\frac{b^2 c^2 d^2 e^2 uv}{q^3 a^5} \right)^n.
\end{aligned}$$

The corresponding expression displayed in (38b–38c) reads accordingly as

$$\sum_{k=-n}^n q^k \left[\begin{matrix} \sqrt{\lambda}, & -q\sqrt{\lambda}, & b, & d, & \lambda c/a, & \lambda e/a, & q^{-n}\lambda, & q^n u, & q^n \lambda v/a, & q^{-n} \\ \sqrt{\lambda}, & -\sqrt{\lambda}, & q\lambda/b, & q\lambda/d, & qa/c, & qa/e, & q^{1+n}, & q^{1-n}\lambda/u, & q^{1-n}a/v, & q^{1+n}\lambda \end{matrix} \middle| q \right]_k \times$$

$$\left[\begin{matrix} q, & 1/a \\ u/a, & v/a \end{matrix} \middle| q \right]_{2n} \left[\begin{matrix} q/b, q/d, q\lambda/b, q\lambda/d, u, v/a, qa/bd, qa/ce, qa/c\lambda, qa/e\lambda \\ q, q, q\lambda, q/\lambda, qa/b, qa/d, b/a, c/a, d/a, e/a \end{matrix} \middle| q \right]_n \left(\frac{b^2 c^2 d^2 e^2 uv}{q^3 a^5} \right)^n.$$

Equating the last two expressions, canceling the common factors and then letting $n \rightarrow \infty$, we get the transformation on nonterminating very well-poised ${}_6\psi_6$ -series:

$${}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right] \quad (39a)$$

$$= {}_6\psi_6 \left[\begin{matrix} q\sqrt{\lambda}, & -q\sqrt{\lambda}, & b, & d, & \lambda c/a, & \lambda e/a \\ \sqrt{\lambda}, & -\sqrt{\lambda}, & q\lambda/b, & q\lambda/d, & qa/c, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right] \times \quad (39b)$$

$$\left[\begin{matrix} qa, q/a, qa/bd, qa/ce, q\lambda/b, q\lambda/d, qa/c\lambda, qa/e\lambda \\ q\lambda, q/\lambda, qa/b, qa/d, q/c, q/e, q\lambda/bd, qa^2/ce\lambda \end{matrix} \middle| q \right]_{\infty}. \quad (39c)$$

Note that the right member of this transformation has an extra parameter λ or v . Let $\lambda = d$ or equivalently $v = qa^2/cde$, the ${}_6\psi_6$ -series displayed in (39b) turns out to be a ${}_6\phi_5$ -series. Under the convergence condition $|qa^2/bcde| < 1$, this ${}_6\phi_5$ -series can be evaluated, by means of the nonterminating q -Dixon formula (24a–24b), as follows:

$${}_6\phi_5 \left[\begin{matrix} d, & q\sqrt{d}, & -q\sqrt{d}, & b, & cd/a, & de/a \\ \sqrt{d}, & -\sqrt{d}, & qd/b, & qa/c, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right]$$

$$= \left[\begin{matrix} qa/bc, & qa/be, & qd, & qa^2/cde \\ qa/c, & qa/e, & qd/b, & qa^2/bcde \end{matrix} \middle| q \right]_{\infty}.$$

Substituting this into (39b), we infer that transformation (39a–39b–39c) with $\lambda = d$ reduces to the following product

$${}_6\psi_5 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right]$$

$$= \left[\begin{matrix} qa/bc, qa/be, qd, qa^2/cde \\ qa/c, qa/e, qd/b, qa^2/bcde \end{matrix} \middle| q \right]_{\infty} \times \left[\begin{matrix} qa, q/a, qa/bd, qa/ce, qd/b, q, qa/cd, qa/ed \\ qd, q/d, qa/b, qa/d, q/c, q/e, q/b, qa^2/ced \end{matrix} \middle| q \right]_{\infty}$$

$$= \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \middle| q \right]_{\infty}$$

which is the celebrated bilateral ${}_6\psi_6$ -series identity discovered by Bailey (1936). \square

20. The telescoping method: Theta function identity

For five complex parameters A, b, c, d, e satisfying $A^2 = bcde$, there holds theta function identity:

$$\langle A/b, A/c, A/d, A/e; q \rangle_{\infty} - \langle b, c, d, e; q \rangle_{\infty} = b \langle A, A/bc, A/bd, A/be; q \rangle_{\infty}, \quad (40)$$

where the modified Jacobi theta function is defined by

$$\langle x; q \rangle_{\infty} = (x; q)_{\infty} (q/x; q)_{\infty} = \prod_{n=0}^{\infty} (1 - xq^n)(1 - q^{1+n}/x)$$

and the corresponding product form by

$$\langle a, b, \dots, c; q \rangle_{\infty} = \langle a; q \rangle_{\infty} \langle b; q \rangle_{\infty} \cdots \langle c; q \rangle_{\infty}.$$

This identity appeared explicitly for the first time in Ref. [18]. It contains several surprising theta function identities as special cases and has been shown very useful in dealing with the congruences on partition function discovered by Ramanujan^[52] including Winquist's proof^[60] corresponding to modulo 11. The details can be found in the recent work due to Chu^[32] and Chu-Diclaudio^[35,H]. For more theta function identities, we refer to the well-known historical monograph written by Whittaker and Watson^[59,§21].

For $bcde = A^2$, define the factorial fraction by

$$T_k := \frac{(b; q)_k (c; q)_k (d; q)_k (e; q)_k}{(A/b; q)_k (A/c; q)_k (A/d; q)_k (A/e; q)_k}.$$

Then it is easy to check the differences

$$\begin{aligned} \nabla T_k = T_k - T_{k+1} &= b \frac{(1-A)(1-A/bc)(1-A/bd)(1-A/be)}{(1-A/b)(1-A/c)(1-A/d)(1-A/e)} \times \\ &\quad \frac{1-Aq^{2k}}{1-A} \frac{(b; q)_k (c; q)_k (d; q)_k (e; q)_k}{(qA/b; q)_k (qA/c; q)_k (qA/d; q)_k (qA/e; q)_k} q^k. \end{aligned}$$

In view of the definition of the bilateral series, the ${}_6\psi_6$ -series identity can accordingly be telescoped, under condition $bcde = A^2$, as follows:

$$\begin{aligned} {}_6\psi_6 \left[\begin{matrix} qA^{1/2}, & -qA^{1/2}, & b, & c, & d, & e \\ A^{1/2}, & -A^{1/2}, & qA/b, & qA/c, & qA/d, & qA/e \end{matrix} \middle| q; q \right] \\ &= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \sum_{k=-\infty}^{+\infty} \nabla T_k \\ &= \frac{(1-A/b)(1-A/c)(1-A/d)(1-A/e)}{b(1-A)(1-A/bc)(1-A/bd)(1-A/be)} \{T_{-\infty} - T_{+\infty}\} \\ &= \frac{[qA, q/A, qA/bc, qA/bd, qA/be, qA/cd, qA/ce, qA/de; q]_{\infty}}{[qA/b, qA/c, qA/d, qA/e, q/b, q/c, q/d, q/e; q]_{\infty}}. \end{aligned}$$

Keeping in mind that $bcde = A^2$ and substituting two infinite factorial fractions

$$\begin{aligned} T_{+\infty} &= \frac{(b; q)_{\infty} (c; q)_{\infty} (d; q)_{\infty} (e; q)_{\infty}}{(A/b; q)_{\infty} (A/c; q)_{\infty} (A/d; q)_{\infty} (A/e; q)_{\infty}}, \\ T_{-\infty} &= \frac{(qb/A; q)_{\infty} (qc/A; q)_{\infty} (qd/A; q)_{\infty} (qe/A; q)_{\infty}}{(q/b; q)_{\infty} (q/c; q)_{\infty} (q/d; q)_{\infty} (q/e; q)_{\infty}}; \end{aligned}$$

into the last identity, we find the theta function identity displayed in (40).

For example, the well-known Jacobi identity^[59,P470]

$$(-q; q^2)_{\infty}^8 - (q; q^2)_{\infty}^8 = 16q(-q^2; q^2)_{\infty}^8$$

results from the very particular case of identity (40) specified with $b = c = d = e \rightarrow q$, $A \rightarrow -q^2$ and $q \rightarrow q^2$. Further examples can be found in Bailey^[8] and Slater^[55,56].

References

- [1] ANDREWS G E. *The Theory of Partitions* [M]. Cambridge University Press, 1976.
- [2] ANDREWS G E. *Q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra* [M]. CBMS Regional Conference Lectures Series 66, Amer. Math. Soc., Providence, RI, 1986.
- [3] ANDREWS G E. ASKEY R, ROY R. *Special Functions* [M]. Cambridge University Press, 1999.
- [4] BAILEY W N. An identity involving Heine's basic hypergeometric series [J]. J. London Math. Society, 1929, **4**: 254–257.
- [5] BAILEY W N. *Generalized Hypergeometric Series* [M]. Cambridge University Press, Cambridge, 1935.
- [6] BAILEY W N. Series of hypergeometric type which are infinite in both directions [J]. Quart. J. Math., Oxford, 1936, **7**: 105–115.
- [7] BAILEY W N. On the analogue of Dixon's theorem for bilateral basic hypergeometric series [J]. Quart. J. Math., Oxford Ser. (2), 1950, **1**: 318–320.
- [8] BAILEY W N. On the simplification of some identities of the Rogers-Ramanujan type [J]. Proc. London Math. Soc., 1951, **3**(1): 217–221.
- [9] BAXTER R J. *Exactly Solved Models in Statistical Mechanics* [M]. Academic Press, New York, 1982.
- [10] BERNDT B C. *Ramanujan's Notebooks Part I-V* [M]. Springer-Verlag, New York, 1985–1998.
- [11] BHARGAVA S. A simple proof of the quintuple product identity [J]. J. Indian Math. Soc. (N.S.), 1995, **61**(3-4): 226–228.
- [12] CARLITZ L. Some inverse relations [J]. Duke Math. J., 1973, **40**: 893–901.
- [13] CARLITZ L, SUBBARAO M V. A simple proof of the quintuple product identity [J]. Proc. Amer. Math. Soc., 1972, **32**: 42–44.
- [14] CHEN W Y C, CHU Wenchang, GU N S S. Finite form of the quintuple product identity [J]. J. Combin. Theory Ser. A, 2006, **113**(1): 185–187.
- [15] CHU Wenchang. A reverse derivation for Gould-Hsu inversions and some comments [J]. J. Dalian Univ. Tech., 1990, **30**(3): 345–349. (in Chinese)
- [16] CHU Wenchang. Gould-Hsu-Carlitz inversions and Rogers-Ramanujan identities [J]. Acta Math. Sinica, 1990, **33**(1): 7–12.
- [17] CHU Wenchang. The lattice path method and classical partition identities [J]. J. Systems Sci. Math. Sci., 1992, **12**(1): 52–57. (in Chinese)
- [18] CHU Wenchang. A trigonometric identity with its q -analogue [J]. Amer. Math. Monthly 99:5(1992), Problem 10226; *ibid* 103:2 (1996), **Solution**
- [19] CHU Wenchang. An Algebraic Identity and Its Application: The unification of some matrix inverse pairs [J]. Pure Math. Appl., 1993, **4**(2): 175–190.
- [20] CHU Wenchang. Inversion techniques and combinatorial identities [J]. Boll. Un. Mat. Ital. B (7), 1993, **7**(4): 737–760.
- [21] CHU Wenchang. Durfee rectangles and the Jacobi triple product identity [J]. Acta Math. Sinica (N.S.), 1993, **9**(1): 24–26.
- [22] CHU Wenchang. Inversion Techniques and Combinatorial Identities: Basic hypergeometric identities [J]. Publ. Math. Debrecen, 1994, **44**(3-4): 301–320.
- [23] CHU Wenchang. Inversion Techniques and Combinatorial Identities: Strange evaluations of basic hypergeometric series [J]. Compositio Math., 1994, **91**(2): 121–144.
- [24] CHU Wenchang. Partial fractions and bilateral summations [J]. J. Math. Phys., 1994, **35**(4): 2036–2042.
- [25] CHU Wenchang. Basic Hypergeometric Identities: An introductory revisiting through the Carlitz inversions [J]. Forum Math., 1995, **7**(1): 117–129.
- [26] CHU Wenchang. Inversion Techniques and Combinatorial Identities: Jackson's q -analogue of the Dougall-Dixon theorem and the dual formulae [J]. Compositio Math., 1995, **95**(1): 43–68.
- [27] CHU Wenchang. Symmetry on q -Pfaff-Saalschutz-Sheppard-series [J]. Pure Math. Appl., 1995, **6**(1): 31–39.
- [28] CHU Wenchang. The extended Cesàro theorem and hypergeometric asymptotics [J]. Analysis, 1996, **16**(4): 379–384.
- [29] CHU Wenchang. Basic almost poised Hypergeometric series [M]. Mem. Amer. Math. Soc., 1998, **135**(642): 99.
- [30] CHU Wenchang. Partial-fraction expansions and well-poised bilateral series [J]. Acta Sci. Math. (Szeged), 1998, **64**(3-4): 495–513.

- [31] CHU Wenchang. *The Saalschütz chain reactions and bilateral basic hypergeometric series* [J]. Constr. Approx., 2002, **18**(4): 579–597.
- [32] CHU Wenchang. *Theta function identities and Ramanujan’s congruences on the partition function* [J]. Q. J. Math., 2005, **56**(4): 491–506.
- [33] CHU Wenchang. *The Saalschütz Chain Reactions and Multiple q -Series Transformations* [M]. Theory and Applications of Special Functions, 99–121, Dev. Math., 13, Springer, New York, 2005.
- [34] CHU Wenchang. *Bailey’s very well-poised ${}_6\psi_6$ -series identity* [J]. J. Combin. Theory Ser. A, 2006, **113**(6): 966–979.
- [35] CHU Wenchang, DICLAUDIO L. *Classical Partition Identities and Basic Hypergeometric Series* [M]. Quaderno n. 6 anno 2004 del Dipartimento di Matematica “Ennio De Giorgi”, Università degli Studi di Lecce; Edizioni del Grifo (2004), pp. viii+153.
- [36] COOPER S. *The quintuple product identity* [J]. Int. J. Number Theory, 2006, **2**(1): 115–161.
- [37] FINE N J. *Basic Hypergeometric Series and Applications* [M]. American Mathematical Society, Providence, RI, 1988.
- [38] GASPER G. *Summation formulas for basic hypergeometric series* [J]. SIAM J. Math. Anal., 1981, **12**(2): 196–200.
- [39] GASPER G, RAHMAN M. *Basic Hypergeometric Series* [M]. Cambridge University Press, Cambridge, 2004.
- [40] GESSEL I M. *Some generalized Durfee square identities* [J]. Discrete Math., 1984, **49**(1): 41–44.
- [41] GOULD H W, HSU L C. *Some new inverse series relations* [J]. Duke Math. J., 1973, **40**: 885–891.
- [42] GRAHAM R L, KNUTH D E, PATASHNIK O. *Concrete Mathematics* [M]. Addison-Wesley Publ. Company, Reading, Massachusetts, 1989.
- [43] HARDY G H, WRIGHT E M. *An Introduction to the Theory of Numbers* [M]. 4th ed. Oxford University Press, London, 1960.
- [44] JACKSON F H. *Summation of q -hypergeometric series* [J]. Messenger of Math., 1921, **50**: 101–112.
- [45] JACKSON F H. *Certain q -identities* [J]. Quart. J. Math., Oxford Ser. 12, 1941, 167–172.
- [46] JOUHET F, SCHLOSSER M. *Another proof of Bailey’s ${}_6\psi_6$ summation* [J]. Aequationes Math., 2005, **70**(1–2): 43–50.
- [47] KAC V G. *Infinite Dimensional Lie Algebras* [M]. Cambridge University Press, Cambridge, 1985.
- [48] KARLSSON P W. *Hypergeometric formulas with integral parameter differences* [J]. J. Mathematical Phys., 1971, **12**(2): 270–271.
- [49] LEPOWSKY J, MILNE S. *Lie algebraic approaches to classical partition identities* [J]. Adv. in Math., 1978, **29**(1): 15–59.
- [50] MILNE S C. *Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions* [J]. Ramanujan J., 2002, **6**(1): 7–149.
- [51] MINTON B M. *Generalized hypergeometric function of unit argument* [J]. J. Mathematical Phys., 1970, **11**: 1375–1376.
- [52] RAMANUJAN S. *On certain arithmetical functions* [J]. Trans. Cambridge Philos. Soc., 1916, **22**(9): 159–184.
- [53] SCHLOSSER M. *A simple proof of Bailey’s very-well-poised ${}_6\psi_6$ summation* [J]. Proc. Amer. Math. Soc., 2002, **130**(4): 1113–1123.
- [54] SEARS D B. *On the transformation theory of basic hypergeometric functions* [J]. Proc. London Math. Soc. (2), 1951, **53**: 158–180.
- [55] SLATER L J. *A note on equivalent product theorems* [J]. Math. Gazette, 1954, **38**: 127–128.
- [56] SLATER L J. *Some new results on equivalent products* [J]. Proc. Cambridge Philos. Soc., 1954, **50**: 394–403.
- [57] SLATER L J. *Generalized Hypergeometric Functions* [M]. Cambridge University Press, Cambridge, 1966.
- [58] WATSON G N. *A new proof of the Rogers-Ramanujan identities* [J]. J. London Math. Soc., 1929, **4**: 4–9.
- [59] WHITTAKER E T, WATSON G N. *A Course of Modern Analysis* [Fourth Edition] [M]. Cambridge University Press, Cambridge, 1952.
- [60] WINQUIST L. *An elementary proof of $p(11m + 6) \equiv 0 \pmod{11}$* [J]. J. Combinatorial Theory, 1969, **6**: 56–59.