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Some Applications on the Method of Eigenvalue Interlacing for Graphs

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Abstract The Method of Eigenvalue Interlacing for Graphs is used to investigate some problems on graphs, such as the lower bounds for the spectral radius of graphs. In this paper, two new sharp lower bounds on the spectral radius of graphs are obtained, and a relation between the Laplacian spectral radius of a graph and the number of quadrangles in the graph is deduced.

Keywords eigenvalues interlacing; adjacency matrix; Laplace matrix; quotient matrix.

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1. Introduction

In this paper, we consider only finite graphs without multiple edges and loops[1]. Let G be a graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\ldots,e_m\}$. Its adjacency matrix $A(G)=(a_{ij})$ is defined to be the $n\times n$ matrix (a_{ij}), where

$$a_{ij}=\begin{cases}1, & (v_i,v_j)\in E(G),\\ 0, & \text{otherwise}.\end{cases}$$

The characteristic polynomial of G is just $\det(\lambda I-A(G))$, which is denoted by $P(G;\lambda)$. Since $A(G)$ is a real symmetric matrix, all of its eigenvalues are real. We assume, without loss of generality, that they are ordered in decreasing order, i.e., $\lambda_1(G)\geq\lambda_2(G)\geq\lambda_3(G)\geq\cdots\geq\lambda_n(G)$, and call them the eigenvalues of G. Particularly, $\lambda_1(G)$ is called the spectral radius of G. The Laplace matrix $L(G)=(l_{ij})$ of G is defined by

$$l_{ij}=\begin{cases}d_{v_i}, & i=j,\\ -1, & (v_i,v_j)\in E(G),\\ 0, & \text{otherwise}.\end{cases}$$

The Laplace matrix $L(G)$ of a graph G is singular and positive semidefinite with eigenvalues $\theta_1\geq\theta_2\geq\cdots\geq\theta_n=0$, say Laplace eigenvalues of G. Particularly, θ_1 is called the Laplace spectral radius of G.

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The eigenvalues and Laplace eigenvalues of a graph G have physical interpretations in the quantum chemical theory, so they have long been attracting researcher's attention. It is significant and necessary to investigate the relations between the graph-theoretic properties of G and its eigenvalues (or Laplace eigenvalues). There have been a lot of research papers published continually on this topic[3−6].

2. Preliminaries

Let G be a simple undirected graph. For $v \in V(G)$, the degree and the neighbors of v are denoted by $d(v)$ and $N(v)$, respectively. A pendant vertex is a vertex of degree 1 in G and a pendant edge is an edge incident with a pendant vertex. We denote by K_n, S_n and C_n the complete graph, the star and the cycle , respectively, each on n vertices. Two edges of a graph are said to be independent if they are not adjacent. An m-matching M of G is a set of m mutually independent edges. A vertex v is said to be M-saturated, if some edge of M is incident with v; otherwise, v is M-unsaturated. If every vertex of G is M-saturated, the matching M is perfect.

Consider two sequences of real numbers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ with $(m < n)$. The second sequence is said to interlace the first one whenever

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Lemma 1[3] *If B is a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

Suppose that the rows and columns of the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$$

are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$ with characteristic matrix $\tilde{S}$, that is,

$$(\tilde{S})_{ij} = \begin{cases} 1, & i \in X_j, \\ 0, & \text{otherwise.} \end{cases}$$

The quotient matrix is a matrix $\tilde{B}$ whose entries are the average row sums of the blocks of A. More precisely,

$$(\tilde{B})_{ij} = \frac{1}{|X_i|}(\bar{S}^{\mathrm{T}} A \bar{S})_{ij}, \quad i, j = 1, 2, \ldots, m.$$

Lemma 2[3] *Suppose $\tilde{B}$ is the quotient matrix of a symmetric partitioned matrix A. Then the eigenvalues of $\tilde{B}$ interlace the eigenvalues of A.*

Let G be a graph with the adjacency matrix $A(G)$. Suppose that the rows and columns of $A(G)$ are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $V(G)$. Then A_{ii} ($i = 1, 2, \ldots, m$) is a adjacency matrix of the graph induced by X_i ($i = 1, 2, \ldots, m$), and

$$(\tilde{B})_{ij} = \begin{cases} \frac{2e(X_i)}{|X_i|}, & i = j, \\ \frac{2e(X_i, X_j)}{|X_i|}, & \text{otherwise,} \end{cases}$$

where $e(X_i)$ is the number of edges in the graph induced by X_i, and $e(X_i, X_j)$ is the number of edges joining a vertex in X_i to that of X_j $(i, j = 1, 2, \ldots, m)$.

3. Main results

Theorem 1 *Let G be a connected graph of order n $(G \neq K_2)$ with k $(k > 0)$ pendant vertices. Then*

$$\lambda_1 \geq \frac{((n+c-1)-k)}{n-k} + \frac{\sqrt{(n+c-1)^2 - nk - 2k(c-1)}}{n-k},$$

where c is the cyclomatic number of G.

Proof Since G is a connected graph with order n and the cyclomatic number c, the number of edges in G is $\mid E(G) \mid = n + c - 1$. Let S be the set of all pendant vertices in G $(G \neq K_2)$. Then S is an independent set of G, and $\mid S \mid = k$. Partitioning the rows and columns of $A(G)$ according to a partitioning S, $V(G)/S$ of $V(G)$, induces a corresponding quotient matrix $\tilde{B}$ of 2×2

$$\tilde{B} = \begin{pmatrix} 0 & 1 \\ \frac{k}{n-k} & \frac{2(n+c-1)-2k}{n-k} \end{pmatrix}.$$

It is easy to calculate the eigenvalues of $\tilde{B}$

$$\mu_1 = \frac{((n+c-1)-k)}{n-k} + \frac{\sqrt{(n+c-1)^2 - nk - 2k(c-1)}}{n-k},$$

$$\mu_2 = \frac{((n+c-1)-k)}{n-k} - \frac{\sqrt{(n+c-1)^2 - nk - 2k(c-1)}}{n-k}.$$

By Lemma 2, the eigenvalues of $\tilde{B}$ interlace the eigenvalues of $A(G)$. We have

$$\lambda_1 \geq \mu_1 = \frac{((n+c-1)-k)}{n-k} + \frac{\sqrt{(n+c-1)^2 - nk - 2k(c-1)}}{n-k}. \qquad \square$$

In Theorem 1, when $c = 0$, G $(G \neq K_2)$ is a tree with k $(k \geq 2)$ pendant vertices. Then we have

Corollary 1.1 *Let T $(T \neq K_2)$ be a tree with k pendant vertices. Then*

$$\lambda_1 \geq \frac{n-k-1}{n-k} + \frac{\sqrt{(n-2)(n-k)+1}}{n-k}.$$

In Corollary 1.1, when $k = n-1$, $T = S_n$. Then we have $\lambda_1 \geq \sqrt{n-1}$. Since $\lambda_1(S_n) = \sqrt{n-1}$, the lower bound in Corollary 1.1 is sharp.

In Theorem 1, when $c = 1$, G is a unicyclic graph with k $(k \geq 0)$ pendant vertices. We have

Corollary 1.2 *Let G be a unicyclic graph with k $(k > 0)$ pendant vertices. Then*

$$\lambda_1 \geq 1 + \frac{\sqrt{n}}{\sqrt{n-k}}.$$

Let G' be a graph obtained from C_m by attaching the same number t pendant vertices to every vertex of C_m. Then the order of G' is $n = mt + m$, $k = mt$. It is easy to calculate that

$\lambda_1(G') = 1+\sqrt{t+1}$. In Corollary 1.2, when $G = G'$, i.e., $k = mt$, $n = mt + m$, we have $\lambda_1(G') \geq 1+\sqrt{t+1}$. So the lower bound in the Corollary 1.2 is sharp.

By Corollary 1.1 and Corollary 1.2, we know that the lower bound in Theorem 1 is sharp.

Theorem 2 *Let $G = (X, Y)$ be a bipartite graph of order n, and $e(X, Y) = m$. Then*

$$\lambda_1 \geq \frac{m}{\sqrt{|X|(n-|X|)}}.$$

Proof Partitioning the rows and columns of $A(G)$ according to the bipartition X, Y of $V(G)$ induces a corresponding quotient matrix

$$\tilde{B} = \begin{pmatrix} 0 & \frac{m}{|X|} \\ \frac{m}{n-|X|} & 0 \end{pmatrix}.$$

It is easy to calculate the eigenvalues of $\tilde{B}$:

$$\mu_1 = \frac{m}{\sqrt{|X|(n-|X|)}}, \quad \mu_2 = -\frac{m}{\sqrt{|X|(n-|X|)}}.$$

By Lemma 2, the eigenvalues of $\tilde{B}$ interlace the eigenvalues of $A(G)$. Then we have

$$\lambda_1 \geq \mu_1 = \frac{m}{\sqrt{|X|(n-|X|)}}. \qquad \square$$

When G is a $k-$regular bipartite graph, then $|X| = \frac{n}{2}$, $m = \frac{nk}{2}$. By Theorem 2, we have $\lambda_1 \geq k$. Since $\lambda_1(G) = k$, the bound of Theorem 2 is sharp.

Notice that for any partitioning $V(G) = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_m$ of a graph G, the Laplace matrix of the graph induced by X_i $(i = 1, 2, \ldots, n)$ is not a submatrix of the Laplace matrix of G. So the eigenvalue interlacing techniques of previous section do not work in such a straightforward manner here. We need to do some transformations to the Laplace matrix $L(G)$ of G. We define the matrix $M(G) = (M_{ij}) = L(G) + ee^{\mathrm{T}}$ (e denotes the n-dimensional column vector whose entries are all ones), where

$$M_{ij} = \begin{cases} d_{v_i} + 1, & i = j, \\ 0, & (v_i, v_j) \in E(G), \\ 1, & \text{otherwise}. \end{cases}$$

Obviously, the sum of each row of $M(G)$ is n.

Lemma 3[7] *If G is a connected graph of order n with Laplace eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n = 0$, then the eigenvalues of $M(G)$ are $n \geq \theta_1 \geq \cdots \geq \theta_{n-2} \geq \theta_{n-1}$.*

We construct the following matrix

$$A' = \begin{pmatrix} 0 & M(G) \\ M(G) & 0 \end{pmatrix},$$

which is a $2n \times 2n$ real symmetric matrix with eigenvalues denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{2n}$. By Lemma 3 and knowledge of matrix theory, we can find $\mu_i = -\mu_{2n-i+1}$ $(i = 1, 2, \ldots, n)$, and the eigenvalues of A' are equal to the square roots of the eigenvalues of $M^T(G)M(G)$. Thus we have

$$\mu_1 = -\mu_{2n} = n,$$

$$\mu_2 = -\mu_{2n-1} = \theta_1,$$

$$\cdots \quad \cdots$$

$$\mu_n = -\mu_{n+1} = \theta_{n-1}.$$

Lemma 4 *Let X and Y be disjoint sets of vertices of connected graph G, such that for $\forall x \in X, \forall y \in Y$, there is always one edge xy between X and Y. Then*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \frac{\theta_1^2}{n^2}.$$

Proof Let $A' = \begin{pmatrix} 0 & M(G) \\ M(G) & 0 \end{pmatrix}$. Partitioning the first n rows and columns of A' according to a partitioning Y, $V(G)/Y$ of $V(G)$, and Partitioning the other n rows and columns of A' according to a partitioning $V(G)/X$, X of $V(G)$ induces a corresponding quotient matrix

$$\tilde{B} = \begin{pmatrix} 0 & 0 & \frac{n|Y|}{|Y|} & 0 \\ 0 & 0 & n - \frac{n|Y|}{n-|Y|} & \frac{n|X|}{n-|Y|} \\ \frac{n|Y|}{n-|X|} & n - \frac{n|Y|}{n-|X|} & 0 & 0 \\ 0 & \frac{n|X|}{|X|} & 0 & 0 \end{pmatrix}.$$

Let the eigenvalues of $\tilde{B}$ be $\beta_1 \geq \beta_2 \geq \beta_3 \geq \beta_4$. It is easy to get $\beta_1 = n$, $\beta_4 = -n$, and

$$\det(\tilde{B}) = n^4 \frac{|X||Y|}{(n-|X|)(n-|Y|)} = -n^2\beta_2\beta_3,$$

so

$$n^2 \frac{|X||Y|}{(n-|X|)(n-|Y|)} = -\beta_2\beta_3.$$

By Lemma 2, the eigenvalues of $\tilde{B}$ interlace the eigenvalues of A'. We have $\mu_2 \geq \beta_2$, $\mu_{2n-4+3} \leq \beta_3$. Then $-\mu_{2n-1} \geq -\beta_3$, and $\mu_2 = -\mu_{2n-1} = \theta_1$. So we have

$$\theta_1^2 = \mu_2(-\mu_{2n-1}) \geq -\beta_2\beta_3 = n^2 \frac{|X||Y|}{(n-|X|)(n-|Y|)}.$$

That is

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \frac{\theta_1^2}{n^2}.$$

□

Theorem 3 *Let G $(G \neq K_n)$ be a graph of order n, and u, v be two vertices of G such that $d(u,v) = 2$. τ_4 denotes the number of quadrangles in G through the vertex u, v. Then*

$$\left(\frac{n^2}{\theta_1^2(n-2)} + \frac{1}{2}\right)\sqrt{1+8\tau_4} \leq \left(n - \frac{1}{2} - \frac{n^2}{\theta_1^2(n-2)}\right).$$

Proof The number of quadrangles in G though the vertices u, v is

$$\tau_4 = \begin{pmatrix} |N(u)\bigcap N(v)| \\ 2 \end{pmatrix}.$$

Then $|\mathrm{N(u)}\bigcap \mathrm{N(v)}|=\frac{1+\sqrt{1+8\tau_4}}{2}$. In Lemma 4, let $X = \{u, v\}$ and $Y = N(u)\bigcap N(v)$. Since $|N(u)\bigcap N(v)|=\frac{1+\sqrt{1+8\tau_4}}{2}$, we have

$$2\frac{\frac{1+\sqrt{1+8\tau_4}}{2}}{(n-2)(n-\frac{1+\sqrt{1+8\tau_4}}{2})} \leq \frac{\theta_1^2}{n^2}.$$

Then

$$(\frac{n^2}{\theta_1^2(n-2)}+\frac{1}{2})\sqrt{1+8\tau_4} \leq (n-\frac{1}{2}-\frac{n^2}{\theta_1^2(n-2)}). \qquad \square$$

Let K'_{2n} be a graph obtained form K_{2n} by deleting a perfect matching M. Then K'_{2n} is a $(2n-2)-$rugular graph. It is not difficult to get that $\theta_1 = n$, $\tau_4 = \begin{pmatrix} n-2 \\ 2 \end{pmatrix}$. One can easily check that the equality holds in Theorem 3.

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