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Some Researches on Real Piecewise Algebraic Curves

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Abstract The piecewise algebraic curve, defined by a bivariate spline, is a generalization of the classical algebraic curve. In this paper, we present some researches on real piecewise algebraic curves using elementary algebra. A real piecewise algebraic curve is studied according to the fact that a real spline for the curve is indefinite, definite or semidefinite (nondefinite). Moreover, the isolated points of a real piecewise algebraic curve is also discussed.

Keywords bivariate splines; piecewise algebraic curves; piecewise algebraic varieties; algebraic geometry.

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1. Introduction

Let K denote either $\mathbb{R}$ or $\mathbb{C}$ and $K^n := \{(\alpha_1,\dots,\alpha_n)|\alpha_i \in K, i = 1,\dots,n\}$ be an n-dimensional affine space. Using finite number of hypersurfaces in K^n, we partition a simply connected region $D \subseteq K^n$ into finite number of simply connected regions, which are called the partition cells. Denote the partition of the region D by Δ, which is the union of all partition cells $\delta_1,\dots,\delta_T$ and their edges $S_1,\dots,S_E$, where $S_1,\dots,S_E$ are algebraic hypersurfaces or algebraic families of dimension$\leq n$ that are called the partition net surfaces. If the net surface S_i is a common edge of some two cells and does not belong to the boundary of D, then S_i is called an interior net surface, otherwise S_i is called a boundary net surface.

Denote by $P(\Delta)$ the piecewise polynomial ring with respect to the partition Δ,

$$P(\Delta) := \{f|f_i = f|_{\delta_i} \in K[x_1,\dots,x_n],\ i = 1,2,\dots,T\}.$$

Let $P_d(\Delta)$ be the subset of $P(\Delta)$ with total degree d. For integers $d > \mu \geq 0$, let

$$S^\mu(\Delta) := \{f|f \in C^\mu(\Delta)\cap P(\Delta)\},$$

$$S^\mu_d(\Delta) := \{f|f \in C^\mu(\Delta)\cap P_d(\Delta)\},$$

where $S^\mu(\Delta)$ is called the C^μ spline ring, and $S^\mu_d(\Delta)$ is called the C^μ spline space with total degree d. In fact, $S^\mu(\Delta)$ is a Nöther ring[14,15].

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If $F \subset S^\mu(\Delta)$, then the zero set of F is defined as:

$$Z_K(F) := \{x \in D | f(x) = 0, \forall f \in F\}.$$

Clearly if I is the ideal of $S^\mu(\Delta)$ generated by F, then $Z_K(F) = Z_K(I)$. Furthermore, since $S^\mu(\Delta)$ is a Nöther ring, any ideal I has a finite set of generators $f_1, \dots, f_l$. Thus $Z_K(F)$ can be expressed as the common zeros of the finite set of splines $f_1, \dots, f_l$. It is called a C^μ piecewise algebraic variety[9,14–16,21,25].

If $n = 2$, then the curve $Z_K(f)$ is called a C^μ piecewise algebraic curve, where $f \in S^\mu(\Delta)$. It is obvious that the piecewise algebraic curve is a generalization of the classical algebraic curve[14,15]. The following theorem showed a basic result[15].

Theorem 1.1 *A set* $\mathtt{A} = \{z_i\}_{i=1}^k$ *of points, where* $k = \dim S_d^\mu(\Delta)$*, is a Lagrange interpolation set for* $S_d^\mu(\Delta)$ *if and only if there is no spline* $g(x, y) \in S_d^\mu(\Delta) \setminus \{0\}$ *such that* $\mathtt{A}$ *lies on the piecewise algebraic curve* $Z_K(g)$*.*

The researches on piecewise algebraic curve are very important for bivariate spline interpolation. Zhu and Wang presented a new method for constructing Lagrange interpolation sets for $S_d^0(\Delta)$ in [24]. Furthermore, the literatures show that the piecewise algebraic curve is not only very important for the interpolation by the bivariate splines but also a useful tool for studying traditional algebraic curve and other subjects[5,12,14,15,17]. Because of the possibility

$$\{(x, y) \in D | f|_{\delta_i} = f_i(x, y) = 0\} \cap \overline{\delta_i} = \phi,$$

it is very difficult to study the piecewise algebraic curves. For the recent researches on the piecewise algebraic curves, we refer to [9]–[11], [14]–[25].

One can plot real piecewise algebraic curves with the help of a computer. Indeed, these computer plots are somewhat unreliable. For instance, one cannot be totally sure that a curve is empty, based on the fact that the plot on the screen looks empty. Also, plots are not very precise near singular points of piecewise algebraic curves, just as shown in Fig.1(c). These facts make it necessary to have some theoretical results about real piecewise algebraic curves at hand.

Let $D = \mathbb{R}^2$. Denote by Δ_l the partition consisting of only a line l which divides $\mathbb{R}^2$ into two parts δ_1 and δ_2. Without loss of generality, we assume that $l : x = 0$. Then we have $\delta_1 = \{(x, y) \in \mathbb{R}^2 | x \le 0\}$ and $\delta_2 = \{(x, y) \in \mathbb{R}^2 | x \ge 0\}$.

What are the qualitative differences between the following piecewise algebraic curves of $\mathbb{R}^2$:

$$C_i = Z_{\mathbb{R}}(f_i),$$

where splines $f_i(x, y) \in S^1(\Delta_l), i = 1, 2, 3, 5, 6, f_4(x, y) \in S^0(\Delta_l)$, are defined as follows,

$$f_1(x, y) = \begin{cases} (x+1)^2 + y^2 - 1, (x, y) \in \delta_1 \\ (x+1)^2 + y^2 - 1 - 2x^3, (x, y) \in \delta_2 \end{cases}, \tag{1}$$

$$f_2(x, y) = \begin{cases} x^2 + (y+1)^2 - 1, (x, y) \in \delta_1 \\ 3x^2 + (y+1)^2 - 1, (x, y) \in \delta_2 \end{cases}, \tag{2}$$

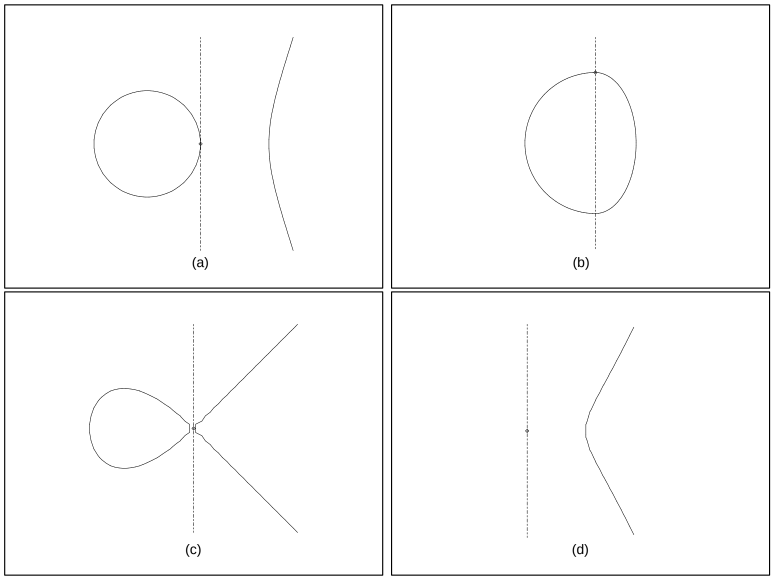

Figure 1: Four piecewise algebraic curves.

$$f_3(x,y) = \begin{cases} -x^2 + y^2 - x^3, (x,y) \in \delta_1 \\ -x^2 + y^2, (x,y) \in \delta_2 \end{cases}, \tag{3}$$

$$f_4(x,y) = \begin{cases} x^2 + y^2 - x^3, (x,y) \in \delta_1 \\ x^2 + y^2 - x^3 + 0.1x, (x,y) \in \delta_2 \end{cases}, \tag{4}$$

$$f_5(x,y) = \begin{cases} x^2(x+1)^2 + y^2, (x,y) \in \delta_1 \\ x^2(x+1)^2 + y^2 - 4x^3, (x,y) \in \delta_2 \end{cases}, \tag{5}$$

$$f_6(x,y) = \begin{cases} -x^2 - y^2 - 1, (x,y) \in \delta_1 \\ -y^2 - 1, (x,y) \in \delta_2 \end{cases} ? \tag{6}$$

This paper contains results and techniques to prove some properties of these piecewise algebraic curves, such as

• The sets C_1, C_2, C_3 and C_4 are infinite, C_5 consists of three points and C_6 is empty.

• The point $(0,0)$ belongs to C_1, C_2, C_3, C_4 and C_5; it is isolated in C_4 and C_5 but not so in C_1, C_2 and C_3 .

• The sets C_2 and C_5 are bounded, while C_1, C_3 and C_4 are not.

See Fig.1 (a)–(d), for C_1, C_2, C_3 and C_4. These and other topological and geometrical properties of the piecewise algebraic curves C_j partially depend on properties of the splines f_j and, very particularly, on whether f_j has a constant sign when evaluated at points of $\mathbb{R}^2$. In this paper, we study the real piecewise algebraic curves like the C_j above.

The remainder of the paper is organized as follows. The second section starts with some properties of piecewise algebraic curves. Next, we define the character of real bivariate spline and, using elementary methods in algebra and algebraic geometry, we study the spline and the piecewise algebraic curve it defines. At the end of this section, we present a discussion about isolated points of piecewise algebraic curves. Some results of the paper can be found in [22, 23].

2. Some properties of piecewise algebraic curves

Definition 2.1[15] *A C^μ piecewise algebraic variety in D is a set of the form*

$$Z_K(f_1,\dots,f_s)=\{(x_1,\dots,x_n)\in D|f_i(x_1,\dots,x_n)=0, i=1,\dots,s\},$$

where $s\in\mathbb{N}$ and $f_1,\dots,f_s\in S^\mu(\Delta)$.

Definition 2.2[15] *Let $X\subset D$ be a nonempty C^μ piecewise algebraic variety. If X can be expressed as the union of two nonempty proper C^μ piecewise algebraic varieties, then X is called reducible in D, otherwise X is called irreducible in D.*

Clearly, every C^μ piecewise algebraic variety can be expressed as the union of finite number of irreducible C^μ piecewise algebraic varieties. For the recent researches of piecewise algebraic varieties, one can refer to [9, 11, 15, 21].

Let $D=K^2$ and Δ be the partition on D. A C^μ piecewise algebraic curve $Z_K(f)$ is, by definition, the zero set in K^2 of a nonconstant spline $f\in S^\mu(\Delta)$. A C^μ piecewise algebraic curve $Z_K(f)$ is irreducible if it is irreducible as a C^μ piecewise algebraic variety. Then we can have the following result directly.

Theorem 2.1 *Every C^μ piecewise algebraic curve of $S^\mu_{\mu+1}(\Delta)$ is irreducible.*

Definition 2.3 *Let $f\in S^\mu(\Delta)$. If there exists $i\in\{1,\dots,T\}$ such that $f|_{\delta_i}$ is reducible, then f is called reducible locally. If for every $i\in\{1,\dots,T\}$ such that $f|_{\delta_i}$ is reducible, then f is called reducible. Otherwise, f is called irreducible.*

Definition 2.4 *Let $f\in S^{\mu_1}(\Delta), g\in S^{\mu_2}(\Delta)$. If there exists $i\in\{1,\cdots,T\}$ such that $f|_{\delta_i}$ divides $g|_{\delta_i}$, then we say that f divides g locally. If for every $i\in\{1,\dots,T\}$, $f|_{\delta_i}$ divides $g|_{\delta_i}$, then we say that f divides g.*

Definition 2.5 *Let $f\in S^{\mu_1}(\Delta), g\in S^{\mu_2}(\Delta)$. If two polynomials $f_i=f|_{\delta_i}$ and $g_i=g|_{\delta_i}$ have a non-constant common factor $r_i\in K[x,y]$, we say that they have a local common factor. Moreover, if f_i and g_i are coprime for all $i\in\{1,\dots,T\}$, we say that they are coprime.*

Lemma 2.1 *Let $f\in S^{\mu_1}(\Delta)$ and $g\in S^{\mu_2}(\Delta)$ be coprime. Then there exist $d,l,h\in P(\Delta)$ with $d_i=d|_{\delta_i}\in K[x]\setminus\{0\}, i=1,\dots,T$, such that $d=lf+hg$.*

Proof We can think of the elements of $K(x)[y]$ as polynomials in y having coefficients in $K(x)$, where the elements of $K(x)$ are rational expressions in x. Assume that $f_i=f|_{\delta_i}$ and $g_i=g|_{\delta_i}$ are coprime in $K[x,y]$. It follows from the Gauss lemma that f_i and g_i remain coprime in $K(x)[y]$, so that there exist $l_i',h_i'\in K(x)[y]$ with $1=l_i'f_i+h_i'g_i$. Removing denominators, we get $d_i\in K[x]\setminus\{0\}$ such that $d_i=l_if_i+h_ig_i$, where $l_i=d_il_i'$ and $h_i=d_ih_i'\in K[x,y]$. □

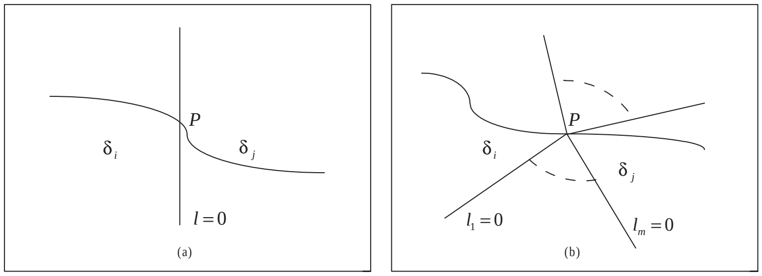

Figure 2: The point P lies on $\delta_i \cap \delta_j$.

Theorem 2.2 *Let $f \in S^{\mu_1}(\Delta)$ and $g \in S^{\mu_2}(\Delta)$ be coprime. Then $Z_K(f) \cap Z_K(g)$ is finite, possibly empty.*

Proof By Lemma 2.1, we get $d, l, h \in P(\Delta)$ such that $d = lf + hg$ with $d_i = d|_{\delta_i} \in K[x] \setminus \{0\}, i = 1, \ldots, T$. If (x_0, y_0) lies in $Z_K(f) \cap Z_K(g)$, then x_0 is a root of the piecewise polynomial d, so that only finitely many x_0 qualify. Now reproducing the argument on $K(y)[x]$, we prove that only finitely many y_0 qualify. Altogether, only finitely many points may lie in $Z_K(f) \cap Z_K(g)$.□

Theorem 2.3 *Except for $\mathbb{R}^2$ itself, every C^μ piecewise algebraic variety in $\mathbb{R}^2$ is a real C^μ piecewise algebraic curve.*

Proof From the Definition 2.1, we have

$$Z_{\mathbb{R}}(f_1, \ldots, f_s) = Z_{\mathbb{R}}(f_1) \cap \cdots \cap Z_{\mathbb{R}}(f_s) = Z_{\mathbb{R}}(f),$$

where $f = f_1^2 + \cdots + f_s^2$ and $f_1, \ldots, f_s \in S^\mu(\Delta)$. Then the result is proved. □

From [6], we know that for every finite set V in $\mathbb{R}^2$ there exists a polynomial $f \in \mathbb{R}[x, y]$, such that $V = Z_{\mathbb{R}}(f)$. Therefore, for arbitrary $\mu \in \mathbb{N}$ and partition Δ on $\mathbb{R}^2$, every finite set in $\mathbb{R}^2$ is a C^μ real piecewise algebraic curve.

Lemma 2.2 *Let $f \in S^\mu(\Delta)$, $\mu \geq 1$. If the point $P \in Z_K(f)$ and $P \in \delta_i \cap \delta_j, i, j \in \{1, \ldots, T\}$, then P is the regular (singular) point of the algebraic curve $Z_K(f_i) = Z_K(f|_{\delta_i})$ if and only if P is the regular (singular) point of the algebraic curve $Z_K(f_j) = Z_K(f|_{\delta_j})$.*

Proof If δ_i and δ_j are adjacent in Δ (as shown in Fig.2), from [13, 15], then we have

$$f_j - f_i = q l^{\mu+1}. \tag{7}$$

If δ_i and δ_j are not adjacent in Δ, then P is a vertex of Δ (as shown in Fig.2). From [13, 15], we get

$$f_j - f_i = q_1 l_1^{\mu+1} + \cdots + q_m l_m^{\mu+1}. \tag{8}$$

Taking first partial derivatives of x and y in (7), (8) respectively, by $\mu \geq 1$, we have

$$\frac{\partial f_j}{\partial x}|_P = \frac{\partial f_i}{\partial x}|_P, \tag{9}$$

$$\frac{\partial f_j}{\partial y}|_P = \frac{\partial f_i}{\partial y}|_P. \tag{10}$$

Then the result is proved. □

Remark 2.1 If the spline $f \in S^0(\Delta)$ and the point P satisfy the conditions of Lemma 2.2, then P is a regular (singular) point of $Z_K(f_i)$ while P may be a singular (regular) point of $Z_K(f_j)$. Just as shown in Fig.2 (d), $(0,0)$ is a singular point of $Z_{\mathbb{R}}(f_4|_{\delta_1})$ while $(0,0)$ is a regular point of $Z_{\mathbb{R}}(f_4|_{\delta_2})$.

Definition 2.6 *Let $f \in S^\mu(\Delta)$ and the point $P \in Z_K(f)$. If $P \in \overline{\delta_i}, i \in \{1,\ldots,T\}$, and P is a singular point of algebraic curve $Z_K(f_i) = Z_K(f|_{\delta_i})$, then P is called a singular point of C^μ piecewise algebraic curve $Z_K(f)$. Otherwise, P is called a regular point of piecewise algebraic curve $Z_K(f)$.*

Using the result of classical algebraic curves[3,5,8,12], we can get the following result easily.

Theorem 2.4 *Every C^μ piecewise algebraic curve has finitely many singular points.*

3. Character of real bivariate spline

Let $D = \mathbb{R}^2$ and Δ be the partition on D. Now we introduce the character of a real spline.

Definition 3.1 *The spline $f \in S^\mu(\Delta)$ is indefinite if there exist $a, b \in \mathbb{R}^2$ such that $f(a) < 0 < f(b)$. Otherwise, f is positive definite if $f(a) > 0$, for every $a \in \mathbb{R}^2$, negative definite if $f(a) < 0$, for every $a \in \mathbb{R}^2$, positive semi-definite if $f(a) \geq 0$, for every $a \in \mathbb{R}^2$ or negative semi-definite if $f(a) \leq 0$, for every $a \in \mathbb{R}^2$.*

By the character of a spline we mean one of the following three mutually exclusive conditions:

- indefinite,
- definite,
- semi–definite (non-definite).

Note that the character of a spline does not change by an affine/projective change of coordinates. First, we will study the piecewise algebraic curves defined by the indefinite splines.

Theorem 3.1 *If $f \in S^\mu(\Delta)$ is indefinite, then after an affine change of coordinates, there exists an open interval $\phi \neq I \subseteq \mathbb{R}$ such that, for each $u \in I$ there exists $b_u \in \mathbb{R}$ with $(u, b_u) \in Z_{\mathbb{R}}(f)$.*

Proof An affine change of coordinates allows us to assume that there exist a and $b_1 < b_2$ in $\mathbb{R}$ such that $f(a, b_1) < 0 < f(a, b_2)$. By continuity, there exists $\varepsilon > 0$ such that $f(u, b_1) < 0 < f(u, b_2)$, for every u in the open interval $I_\varepsilon := (a - \varepsilon, a + \varepsilon)$. By the intermediate value theorem, for each $u \in I_\varepsilon$ there exists b_u with $b_1 < b_u < b_2$ such that $f(u, b_u) = 0$, i.e., $(u, b_u) \in Z_{\mathbb{R}}(f)$. □

Corollary 3.1 *If $f \in S^\mu(\Delta)$ is indefinite, then $Z_{\mathbb{R}}(f)$ is infinite.*

Denote by Δ_T the partition on $\mathbb{R}^2$ containing only $T-1$ lines: $x - x_i = 0, i = 1, \ldots, T-1$.

Theorem 3.2 *Let $f \in S^\mu(\Delta_T)$. Then f is indefinite if the degree of one of $\{f|_{\delta_i}\}_{i=1}^T$ is odd with respect to y.*

Proof Let $\deg_y(f|_{\delta_i}) = d$ be odd and $f_i = f|_{\delta_i} = a_d(x)y^d + \cdots + a_0(x)$. There must exist a line $x - x' = 0$ on the cell δ_i such that $a_d(x') \neq 0$ and $a_0(x') \neq 0$. Let $g_i(y) := f_i(x', y)$. Then g_i is a univariate polynomial of odd degree. Now, it is well known that there exist $y', y'' \in \mathbb{R}$ such that $g_i(y') < 0 < g_i(y'')$. Therefore, f is indefinite. □

Theorem 3.3 *Let $f \in S^\mu(\Delta)$, $\mu \geq 1$. If there exists $P \in Z_{\mathbb{R}}(f)$ such that the gradient of f at P does not vanish, then f is indefinite.*

Proof Suppose that $P = (p_1, p_2)$ and $\frac{\partial f}{\partial x} \neq 0$. Then the univariate $C^\mu(\mu \geq 1)$ piecewise function

$$f(x, p_2) : \mathbb{R} \to \mathbb{R}$$

is strictly monotonous on the neighborhood of p_1 by continuity. Since this function vanishes at p_1, it must change signs on such a neighborhood. Then the result is proved. □

Remark 3.1 If $\mu = 0$, Theorem 3.3 may not hold. For example, $f \in S^0(\Delta_l)$ defined as

$$f|_{\delta_1} = (x-1)^2 + y^2 - 1, f|_{\delta_2} = (x+1)^2 + y^2 - 1.$$

It is clear that the gradient of f at $(0,0)$ does not vanish while $f(a) > 0$ for every $a \in \mathbb{R}^{\nvDash}\backslash\{(\nvdash,\nvdash)\}$. In fact, f is positive semi-definite and $Z_{\mathbb{R}}(f) = \{(0,0)\}$.

If we assume that $Z_{\mathbb{R}}(f)$ is infinite, then we conclude that f is indefinite by applying the previous theorem. From the Real Study's lemma of real algebraic curves[6], we can get the following two results directly.

Theorem 3.4 (Real Study's lemma) *Let $f \in S^{\mu_1}(\Delta), g \in S^{\mu_2}(\Delta)$. If f is irreducible, indefinite and $Z_{\mathbb{R}}(f) \subseteq Z_{\mathbb{R}}(h)$, then f divides h locally. Moreover, if every $f|_{\delta_i}$ is indefinite in the cell δ_i, then f divides h.*

Theorem 3.5 *Let $f \in S^{\mu_1}(\Delta), g \in S^{\mu_2}(\Delta)$. If f is irreducible and indefinite. Assume that after a certain affine change of coordinates, there exists an open interval $\phi \neq I \subseteq \mathbb{R}$ such that, for each $u \in I$ there exists $b_u \in \mathbb{R}$ with $(u, b_u) \in Z_{\mathbb{R}}(f) \cap Z_{\mathbb{R}}(h)$. Then f divides h locally.*

Now, we discuss the relationship between the irreducibility of a real spline and the irreducibility of the piecewise algebraic curve it defines.

Theorem 3.6 *Let $f \in S^\mu(\Delta)$. If f is irreducible and, for every $i \in \{1, \ldots, T\}$, $f|_{\delta_i}$ is indefinite in the cell δ_i, then $Z_{\mathbb{R}}(f)$ is irreducible.*

Proof Suppose that $Z_{\mathbb{R}}(f) = Z_1 \cup Z_2$, where $Z_1 = Z_{\mathbb{R}}(h_1, \ldots, h_s)$ and $Z_2 = Z_{\mathbb{R}}(g_1, \ldots, g_t)$ for some $s, t \in \mathbb{N}$ and some $h_1, \ldots, h_s, g_1, \ldots, g_t \in S^\mu(\Delta)$. Thus we have $Z_1 = Z_{\mathbb{R}}(h)$ and $Z_2 = Z_{\mathbb{R}}(g)$, where $h = h_1^2 + \cdots + h_s^2$ and $g = g_1^2 + \cdots + g_t^2$. Then $Z_{\mathbb{R}}(f) = Z_{\mathbb{R}}(h) \cup Z_{\mathbb{R}}(g) = Z_{\mathbb{R}}(hg)$. Using Theorem 3.4, we have $f|hg$. By the irreducibility of f, either $f|h$ or $f|g$ holds. Thus either $Z_{\mathbb{R}}(f) \subseteq Z_{\mathbb{R}}(h)$ or $Z_{\mathbb{R}}(f) \subseteq Z_{\mathbb{R}}(g)$ holds, so that $Z_{\mathbb{R}}(f) = Z_1$ or $Z_{\mathbb{R}}(f) = Z_2$, from which follows the result. □

Following [4,6], we consider the function

$$\alpha(d)=\max\{\frac{d^2}{4},\frac{(d-1)(d-2)}{2}\},$$

where $d\in\mathbb{N}$.

Denote by $S(n,\Delta)$ the bound of the number of singular points of spline with degree n on the partition Δ. Clearly, if Δ is a partition of $\mathbb{C}^2$ and f remains irreducible in $S^\mu(\Delta)$, then

$$S(n,\Delta)\leq\sum_{i=1}^{T}\frac{(n-1)(n-2)}{2}.$$

Suppose that $f\in S^\mu(\Delta), m=\deg(f)$, and $d_i=\deg(f|_{\delta_i})$. Let

$$\alpha(f)=\min\{\sum_{i=1}^{T}\alpha(d_i),S(m,\Delta)\}. \tag{11}$$

By using the character of real polynomia[6] and Theorem 3.3, we can have following results easily.

Theorem 3.7 *Let $f\in S^\mu(\Delta)$, $\mu\geq 1$. If f is semi-definite irreducible, then $|Z_{\mathbb{R}}(f)|\leq\alpha(f)$.*

Corollary 3.2 *For any semi-definite spline $f\in S^\mu(\Delta)$ with $\mu\geq 1$, the following are equivalent:*

- *$|Z_{\mathbb{R}}(f)|>\alpha(f)$,*
- *$Z_{\mathbb{R}}(f)$ is infinite.*

Remark 3.2 The upper bound of $Z_{\mathbb{R}}(f)$ (f is a semi-definite irreducible spline) as shown in (11) can be obtained by using the number of singular points of $Z_{\mathbb{R}}(f)$ and the Bezout number of piecewise algebraic curves. The estimation of Bezout number of piecewise algebraic curves was studied in [10, 17]. Unfortunately, the estimation of the number of singular points of a piecewise algebraic curve has not been studied yet. It is a tough work for the authors in the future.

The proof of the following result is trivial.

Corollary 3.3 *If $f\in S^\mu(\Delta)$ with $\mu\geq 1$ without multiple irreducible factors (in particular, if f is irreducible), then*

- *$Z_{\mathbb{R}}(f)$ is infinite if and only if f is indefinite,*
- *$Z_{\mathbb{R}}(f)$ is finite nonempty if and only if f is semi–definite non-definite,*
- *$Z_{\mathbb{R}}(f)$ is empty if and only if f is definite.*

Moreover, in case that f is semi-definite (non-definite), then $|Z_{\mathbb{R}}(f)|\leq\alpha(f)$.

4. Isolated points of piecewise algebraic curves

Let $\phi\neq C\subset\mathbb{R}^2$ be a C^μ piecewise algebraic curve with $\mu\geq 1$ and consider a point $P\in C$. If P is nonsingular, then P is nonisolated in C by Theorem 3.3. If P is singular, then P may or may not be isolated in C. For example, for the splines defined in (3) and (5), $(0,0)$ is singular and nonisolated in $Z_{\mathbb{R}}(f_3)$ while singular and isolated in $Z_{\mathbb{R}}(f_5)$.

Let $C = Z_{\mathbb{R}}(f)$ for some nonconstant spline $f \in S^{\mu}(\Delta)$ with $\mu \geq 1$. Consider a singular point $P \in C$ and suppose that $C \setminus \{P\}$ is nonempty. The point P is isolated in C if and only if the minimum of the set

$$\{\mathrm{dist}(P, Q) | Q \in C \setminus \{P\}\}$$

exists and is positive, where the distance considered is Euclidean. Without loss of generality, we may assume $P = (0, 0)$. Note that if $Q \in C \setminus \{P\}$ is a local extreme for the function $\mathrm{dist}(P, \cdot)$, then the point Q must belong to the piecewise algebraic curve $Z_{\mathbb{R}}(p(f))$, where $p(f)$ is defined as $y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$. This is because either the vector $\mathrm{grad}_Q(f)$ is zero or it is nonzero and proportional to the vector from $P = (0, 0)$ to Q. Therefore, we may consider the minimum of the set

$$\{\mathrm{dist}(P, Q) | Q \in C \cap Z_{\mathbb{R}}(p(f)) \setminus \{P\}\}.$$

By $\mu \geq 1$, we have $p(f) \in S^{\mu-1}(\Delta)$ and $Z_{\mathbb{R}}(p(f))$ is a $C^{\mu-1}$ piecewise algebraic curve.

Lemma 4.1 *If $f \in S^{\mu}(\Delta)$ and $\min_i\{\deg(f|_{\delta_i})\} \geq 2$ for every $i \in \{1, \ldots, T\}$, then $p(f)$ is reducible. Moreover, if f is irreducible, then f and $p(f)$ are coprime.*

Proof If $f|_{\delta_i}$ is a real polynomial with order greater than 1, by [6], then $p(f|_{\delta_i})$ is reducible. Therefore $p(f)$ is reducible. Using the irreducibility of f and the fact that $\deg(p(f)) \leq \deg(f)$, we have f and $p(f)$ are coprime. □

We consider $f \in S^{\mu}(\Delta)$ is irreducible. If $\min_i\{\deg(f|_{\delta_i})\} \geq 2$ for every $i \in \{1, \ldots, T\}$, then f and $p(f)$ are coprime, by Lemma 4.1. By Bezout's theorem of piecewise algebraic curves[10,15,17], the set $Z_{\mathbb{R}}(f) \cap Z_{\mathbb{R}}(p(f))$ is finite, say $\{Q_1, \ldots, Q_s\}$, for some $s \in \mathbb{N}$. Then P is isolated in C if and only if the function $\mathrm{dist}(P, \cdot)$ attains a local minimum at Q_j, for some $1 \leq j \leq s$.

By the compactness of algebraic curves[3,6,8], we present a simple characterization of bounded piecewise algebraic curves.

Theorem 4.1 *Let $C \in \mathbb{R}^2$ be a C^{μ} piecewise algebraic curve and $\overline{C} \in P^2(\mathbb{R})$ be the projective closure of C. Then C is compact if and only if each point in $\overline{C} \setminus C$ is isolated in $\overline{C}$.*

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