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On the Gibbs Phenomenon of Fourier Series of a Classical Function

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Abstract In this paper, we point out that the Fourier series of a classical function $\sum_{k=1}^{\infty}\frac{\sin kx}{k}$ has the Gibbs phenomenon in the neighborhood of zero. Furthermore, we estimate the upper bound of its partial sum and get:

$$\sup_{n\geq 1}\|\sum_{k=1}^{n}\frac{\sin kx}{k}\|=\int_0^{\pi}\frac{\sin x}{x}\mathrm{d}x\doteq 1.85194,$$

which is better than that in [1].

Keywords Fourier series; partial sum; upper bound.

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1. Introduction

Fourier analysis is a classical and useful field in both mathematics and applications. The Gibbs phenomenon in Fourier convergence is very interesting. Many results have been established on it[2]. The Gibbs phenomenon means that in the convergence process of a Fourier series, for every scalar in a certain segment, there exists a sequence of points whose partial sum sequence converge to this very scalar. A typical signal function with the period 2π is frequently mentioned in Fourier analysis, which is:

$$f(x)=\begin{cases}\dfrac{\pi-x}{2}, & x\in(0,2\pi)\\ 0, & x=0,2\pi.\end{cases}$$

Its Fourier series is:

$$f(x)\sim\sum_{k=1}^{\infty}\frac{\sin kx}{k}.$$

Denote its partial sum as:

$$S_n(x)=\sum_{k=1}^{n}\frac{\sin kx}{k}. \tag{1}$$

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From the smoothness of $f(x)$, we know $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ is pointwise convergent in $(0, 2\pi)$. Furthermore, with the help of Abel transformation, we can get the uniform convergence of (1) in any internal closed interval of $(0, \pi)$. While it is not right in $(0, 2\pi)$. This conclusion can also be got by the T.W.Chaundy and A.E.Jolliffe's result on necessary and sufficient condition about Fourier convergence in [1]. Although, these series are not uniformly convergent in R^1, they are boundedly convergent. The result in [1], [4] is that: for any integer $n > 0$,

$$\sup_{n\geq 1} \| \sum_{k=1}^{n} \frac{\sin kx}{k} \| \leq 3\sqrt{\pi}. \tag{2}$$

With some trifling modification in their proof, one can obtain a new upper bound $2\sqrt{\pi}$.

In this paper, we will gain an upper bound of (2), which is also the best one. Moreover, we will point out that the left of (2) increases to 1.85 around, which is much smaller than $3\sqrt{\pi}$. At last, we will study the Gibbs phenomenon of this series in the neighborhood of zero, and show that (1) reaches its maximum there. This maximum is 0.2 times higher than the maximum of $f(x)$.

2. Extreme points and zero points of $S_n(x)$

In this part, we will discuss the distribution of the extreme points and zero points of $S_n(x)$, together with its maximum. Because $S_n(0) = S_n(\pi) = 0$ and $S_n(x)$ is an odd function, we only need to consider the problem in $[0, \pi]$. First, we have:

Theorem 1 *When n is odd, $S_n(x)$ has n extreme points in $(0, \pi)$; when n is even, $S_n(x)$ has n-1 extreme points in $(0, \pi)$. They are:*

$$x_1 = \frac{\pi}{n+1}, \quad x_2 = \frac{2\pi}{n}, \quad x_3 = \frac{3\pi}{n+1}, \quad x_4 = \frac{4\pi}{n}, \ldots.$$

Furthermore, the maximal points and the minimal points are alternating, with both the first and last are maxima points.

Proof Since $S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$, we can easily get

$$S_n'(x) = \sum_{k=1}^{n} \cos kx.$$

Adding $\frac{1}{2}$ to both sides, we have

$$S_n'(x) + \frac{1}{2} = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin \frac{(2n+1)x}{2}}{2 \sin \frac{x}{2}},$$

$$S_n'(x) = \frac{\cos \frac{(n+1)x}{2} \sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$

From this formula, we obtain the extreme points of $S_n(x)$ in $(0, \pi)$ as follows:

$$x_k' = \frac{2k\pi}{n}, \ 2k < n; \quad x_k'' = \frac{(2k+1)\pi}{n+1}, \ 2k < n.$$

The expressions of x'_k, x''_k tell us: when n is odd, there are n extreme points; when n is even, there are n-1 extreme points. They are given explicitly as

$$x_1 = \frac{\pi}{n+1}, \quad x_2 = \frac{2\pi}{n}, \quad x_3 = \frac{3\pi}{n+1}, \quad x_4 = \frac{4\pi}{n}, \ldots.$$

By analyzing $S'_n(x)$, we find that $x_{2k} = \frac{2k\pi}{n}$ are the minimal points of $S_n(x)$, while $x_{2k-1} = \frac{(2k-1)\pi}{n+1}$ are the maximal points of it. When n is even, the last extreme point is $x_{n-1} = \frac{(n-1)\pi}{n+1}$; when n is odd, it is $x_n = \frac{n\pi}{n+1}$. Therefore, the last extreme point of $S_n(x)$ in $(0, \pi)$ is always a maximal point. Theorem 1 is proved. □

Secondly, we find that $S_n(x)$ in $(0, \pi)$ is nonnegative. To obtain this conclusion, we only need to show the minimal value of $S_n(x)$ is nonnegative. In fact, we get some more profound statements as follows:

Theorem 2 *For $k = 1, 2, \ldots, [\frac{n}{2}]$, $S_n(x_{2k})$ have k positive and negative alternating parts. Among these k parts, the following is always correct: the absolute value of the former one is larger than that of the latter one. What's more, the maximum of $S_n(x)$ is obtained at $x_1 = \frac{\pi}{n+1}$.*

Proof The first statement of this theorem is quite clear. In the second part of this theorem, $S_n(\frac{\pi}{n+1}) > 0$ can also be obtained easily. To prove the remainder, we begin with $S_n(\frac{2\pi}{n})$. For even n, we have

$$\begin{aligned} S_n(\frac{2\pi}{n}) &= \sin\frac{2\pi}{n} + \frac{\sin\frac{4\pi}{n}}{2} + \cdots + \frac{\sin\frac{2(n-1)\pi}{n}}{n-1} + \frac{\sin\frac{2n\pi}{n}}{n} \\ &= \sin\frac{2\pi}{n} + \frac{\sin\frac{4\pi}{n}}{2} + \cdots + \frac{\sin\frac{2\pi}{n}\frac{n}{2}}{\frac{n}{2}} - \\ &\quad \left\{\frac{\sin\frac{2\pi}{n}}{\frac{n}{2}+1} + \frac{\sin\frac{4\pi}{n}}{\frac{n}{2}+2} + \cdots + \frac{\sin\frac{2\pi(n-1)}{n}}{n}\right\} \\ &> 0. \end{aligned}$$

While if n is odd, there are also few difficulties in noticing $S_n(\frac{2\pi}{n}) > 0$.

Generally, $S_n(\frac{2j\pi}{n})$ has $2j$ parts. Denote $n_0 = [\frac{n}{2j}]$ and $\Delta = \frac{n}{2j} - n_0$. Now we intend to prove the absolute value of the m-th part is larger than that of the $(m+1)$-th. Assume the former $m+1$ parts have $(m+1)n_0$ items, namely, every part has n_0 items. Denote the absolute value of the m-th part and the $(m+1)$-th as I_m and I_{m+1}, respectively.

$$I_m = \frac{\sin\alpha}{(m-1)n_0+1} + \frac{\sin(\alpha + \frac{2j\pi}{n})}{(m-1)n_0+2} + \cdots + \frac{\sin(\alpha + \frac{2n_0 j\pi}{n})}{mn_0},$$

$$I_{m+1} = \frac{\sin\beta}{mn_0+1} + \frac{\sin(\beta + \dfrac{2j\pi}{n})}{mn_0+2} + \cdots + \frac{\sin(\beta + \frac{2n_0 j\pi}{n})}{(m+1)n_0},$$

here $\alpha = [(m-1)n_0+1]\frac{2j\pi}{n} - (m-1)\pi$, $\beta = (mn_0+1)\frac{2j\pi}{n} - m\pi$ and $\alpha > \beta$.

From the monotonicity of $\sin x$ for $x \in (0, \frac{\pi}{2})$, we know the former half part of I_m is larger than that of I_{m+1}.

In what follows, we will prove: for the latter half parts, this is also right. The reciprocal k-th items of I_m and I_{m+1} are

$$\frac{\sin(m\Delta\frac{2j\pi}{n}+\frac{2kj\pi}{n})}{mn_0-k},\quad \frac{\sin[(m+1)\Delta\frac{2j\pi}{n}+\frac{2kj\pi}{n}]}{(m+1)n_0-k},$$

respectively. Using the following inequality

$$\sin\gamma x>\gamma\sin x,\ \ x\in(0,\pi),\ \ 0<\gamma<1 \tag{3}$$

we can gain

$$\frac{\sin(m\Delta\frac{2j\pi}{n}+\frac{2kj\pi}{n})}{mn_0-k}>\frac{\sin[(m+1)\Delta\frac{2j\pi}{n}+\frac{2kj\pi}{n}]}{(m+1)n_0-k}.$$

Up to now, we have $I_m>I_{m+1}$ on above assumption. If the assumption is not right, this method is also effective. We will not repeat the work again.

On the other hand, from the Inequality (3), we can infer this conclusion: for any integer $k,\ l,\ kl\neq 1$, the following holds

$$\sum_{i=(l-1)k+1}^{lk}\frac{\sin ix}{i}>\frac{\sin lkx}{l},\quad lkx\in(0,\pi).$$

From the above inequality, we can get

$$\sin x_1'+\frac{\sin 2x_1'}{2}+\cdots+\frac{\sin nx_1'}{n}>\sin x_k'+\frac{\sin 2x_k'}{2}+\cdots+\frac{\sin ix_k'}{i}, \tag{4}$$

here x_k' is a maximal point, $ix_k'<\pi$. Combining (4) with the former part of Theorem 2, we can infer that $S_n(x)$ reaches its maximum at $x_1=\frac{\pi}{n+1}$. Theorem 2 is proved. □

3. Upper bound and its approximation properties

As we know, for $x\in[0,2\pi]$,

$$\lim_{n\to\infty}S_n(x)=f(x). \tag{5}$$

And for any $\delta>0$, (5) holds uniformly in $[\delta,2\pi-\delta]$. However, this is not the case in the neighborhoods of the two extreme points. In this part, we give the approximation speed of $S_n(x)$ in the neighborhood of π; while in the neighborhood of zero, we will focus on its approximation properties.

First, in the neighborhood of π, we have

Theorem 3 *For $x\in(\pi-\dfrac{\pi}{n},\pi]$, we have $S_n(x)=O(\frac{1}{n})$.*

Proof Theorem 2 shows that the last extreme point of $S_n(x)$ in $(0,\pi)$ is a maximal point. When n is even, it is $x_{n-1}=\frac{(n-1)\pi}{n+1}$ and the maximal value is

$$S_n(\frac{(n-1)\pi}{n+1})=\sin\frac{2\pi}{n+1}-\frac{\sin\frac{4\pi}{n+1}}{2}+\cdots.$$

Then we have:

$$\sin\frac{2\pi}{n+1}-\frac{\sin\frac{4\pi}{n+1}}{2}<S_n(\frac{(n-1)\pi}{n+1})<\sin\frac{2\pi}{n+1}\leq\frac{2\pi}{n+1}. \tag{6}$$

Using (3) on the left side of (6), we obtain

$$\frac{16}{(n+1)^3} \le S_n(\frac{(n-1)\pi}{n+1}) \le \frac{2\pi}{n+1}.$$

When n is odd, we can get a similar inequality in the same way,

$$\frac{4}{(n+1)^3} \le S_n(\frac{n\pi}{n+1}) \le \frac{\pi}{n+1}.$$

So that $S_n(x) = O(\frac{1}{n})$, $x \in (\pi - \frac{\pi}{n}, \pi]$. The proof of Theorem 3 is completed. □

At last, we consider the approximation properties of $S_n(x)$ in the neighborhood of zero. For any $\delta > 0$, it converges to $\dfrac{\pi - x}{2}$ uniformly in $[\delta, \pi]$. So when n is large enough, the value of $S_n(x)$ in $[\delta, \pi]$ is less than $\frac{\pi}{2}$. Many results have been obtained in neighborhood of zero, such as the Gibbs phenomenon of $S_n(x)$ and its boundness. The usual result is $\|S_n\| \le 3\sqrt{\pi}$, which can be improved to $\|S_n\| \le 2\sqrt{\pi}$ conveniently. However, based on Theorem 2, we will improve such a result in essence.

Theorem 4 *Denote* $S_n = \max_{x\in[0,2\pi]} |S_n(x)|$. *Then* S_n *is increasing. Moreover,*

$$\lim_{n\to\infty} S_n = \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x.$$

Proof From Theorem 2 and the periodicity of $\sin x$, we obtain

$$S_n = S_n(\frac{\pi}{n+1}) = \sum_{k=1}^{n} \frac{\sin\frac{k\pi}{n+1}}{k} = \frac{\pi}{n+1} \sum_{k=1}^{n} \frac{\sin\frac{k\pi}{n+1}}{\frac{k\pi}{n+1}}.$$

As $\frac{\sin x}{x}$ is decreasing, S_n is its lower Riemann sum. Whereas, [3] and [5] inform us that the lower Riemann sum of a function which is either convex or concave increases with n. Thus S_n is increasing, and

$$\lim_{n\to\infty} S_n = \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x.$$

Up to now, we successfully improve the Inequality (2) to

$$\sup_{n\ge 1} \|\sum_{k=1}^{n} \frac{\sin kx}{k}\| = \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x \doteq 1.85194. \tag{7}$$

From the values of S_1 to S_{20} as well as the graphs of $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_4(x)$, $S_5(x)$, $S_{10}(x)$, $S_{15}(x)$ and $S_{20}(x)$, we can notice the change of S_n and $S_n(x)$ clearly.

n	S_n	n	S_n	n	S_n	n	S_n	n	S_n
1	1.	2	1.29904	3	1.44281	4	1.52728	5	1.58285
6	1.62219	7	1.65149	8	1.67417	9	1.69224	10	1.70697
11	1.71922	12	1.72956	13	1.73840	14	1.74605	15	1.75274
16	1.75863	17	1.76386	18	1.76854	19	1.77274	20	1.77654

Table 1 Values of S_1 to S_{20}

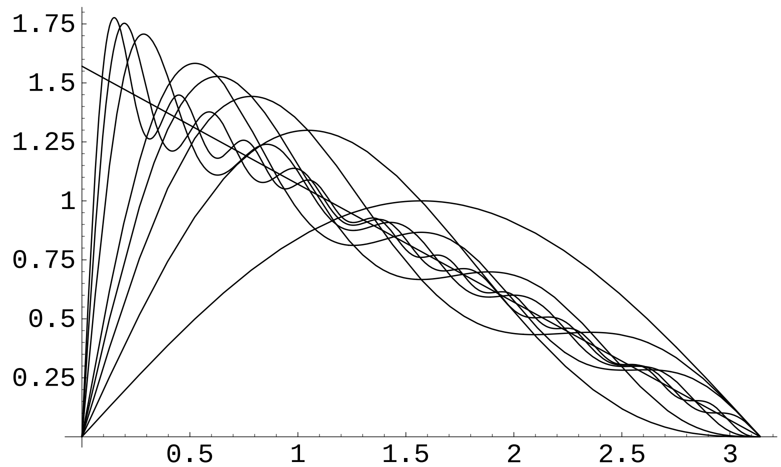

Figure 1 Graphs of $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_4(x)$, $S_5(x)$, $S_{10}(x)$, $S_{15}(x)$ and $S_{20}(x)$

References

[1] XIE Tingfan, ZHOU Songping. *Approximation of Real Function* [M]. Hangzhou: Hangzhou University Press, 1998, 71–82. (in Chinese)

[2] PAN Wenjie. *Fourier Analysis and It's Application* [M]. Beijing: Beijing University Press, 2002, 45–48 (in Chinese)

[3] BENNETT G, JAMESON G. *Monotonic averages of convex functions* [J]. J. Math. Anal. Appl., 2000, **252**(1): 410–430.

[4] LE Ruijun, ZHOU Songping. *A new condition and applications in Fourier analysis. II.* [J]. Adv. Math. (China), 2005, **34**(2): 249–252.

[5] LIN Meijuan. *Convegence rate for a kind of Reimann sum of convex function* [J]. Jinhua College Transaction, 2005, **5**: 50–52. (in Chinese)