Journal of Mathematical Research & Exposition
May, 2008, Vol. 28, No. 2, pp. 366–370
DOI:10.3770/j.issn:1000-341X.2008.02.017
Http://jmre.dlut.edu.cn

Dual Toeplitz Algebra on the Polydisk

CHENG Guo Zheng,   *YU Tao*
(Department of Mathematics, Zhejiang Normal University, Zhejiang 321004, China)
(E-mail: chgzh04@yahoo.com.cn)

Abstract In this paper, we prove the dual Toeplitz algebra $\mathcal{I}\ (C(\overline{D^n}))$ contains the ideal $\mathcal{K}$ of compact operators as its semicommutator ideal, and study its algebraic structure. We also get some results about spectrum of dual Toeplitz operators.

Keywords Bergman space; dual Toeplitz operator ; semicommutator ideal.

Document code A
MR(2000) Subject Classification 47B35
Chinese Library Classification O177.1

1. Introduction

Let $D^n(n \geq 2, n \in Z)$ be the unit polydisk with normalized Lebesgue measure dA. The Bergman space $L_a^2(D^n)$ is defined by $L_a^2(D^n) = L^2(D^n) \bigcap H(D^n)$, where $H(D^n)$ denotes the class of all the analytic functions on D^n. For $w \in D^n$, the Möbius transformation is given by $\varphi_w(z) = (\varphi_{w_1}(z_1), \ldots, \varphi_{w_n}(z_n))$, where $\varphi_{w_i}(z_i) = \frac{w_i - z_i}{1-\bar{w}_i z_i}$, $z_i \in D$, $1 \leq i \leq n$. For $f, g \in L^2(D^n)$, we define the one rank operator $f \otimes g$ as

$$(f \otimes g)h = \langle h, g\rangle f, \quad h \in L^2(D^n).$$

Let P denote the orthogonal projection from $L^2(D^n)$ to $L_a^2(D^n)$, which can be represented by

$$(Ph)(z) = \int_{D^n} h(w)\overline{K_z(w)}\mathrm{d}A(w), \quad h \in L^2(D^n), w \in D^n,$$

where $K_z(w) = \prod_{i=1}^n \frac{1}{(1-w_i\bar{z}_i)^2}$ are the reproducing kernels for $L_a^2(D^n)$. The functions $k_z(w) = \prod_{i=1}^n \frac{1-|z_i|^2}{(1-w_i\bar{z}_i)^2}$ are the normalized reproducing kernels.

For $f \in L^\infty(D^n)$, Toeplitz operator $T_f : L_a^2(D^n) \to L_a^2(D^n)$ is defined by

$$T_f(g) = P(fg), \quad g \in L_a^2(D^n).$$

Let $Q = I - P$ and $L_a^2(D^n)^\perp$ be the orthogonal complement of $L_a^2(D^n)$ in $L^2(D^n)$. Then two other operators are given as follows:

Hankel operator $H_f : L_a^2(D^n) \to L_a^2(D^n)^\perp$ is defined by

$$H_f(h) = Q(fh), \quad h \in L_a^2(D^n).$$

Received date: 2006-03-01; **Accepted date**: 2006-07-02
Foundation item: the National Natural Science Foundation of China (No. 10771064).

Dual Toeplitz operator $S_f : L_a^2(D^n)^\perp \to L_a^2(D^n)^\perp$ is defined by

$$S_f(u) = Q(fu), \quad u \in L_a^2(D^n)^\perp.$$

Based on the definitions of these operators as above, we can decompose the multiplication operator $M_f(f \in L^\infty(D^n))$ on $L^2(D^n)$ as

$$M_f = \begin{pmatrix} T_f & H_{\bar{f}}^* \\ H_f & S_f \end{pmatrix}.$$

The identity $M_{fg} = M_f M_g$ implies

$$T_{fg} = T_f T_g + H_{\bar{f}}^* H_g, \tag{1.1}$$

$$S_{fg} = S_f S_g + H_f H_{\bar{g}}^*, \tag{1.2}$$

$$H_{fg} = H_f T_g + S_f H_g. \tag{1.3}$$

If f is analytic in (1.3), then we have

$$H_{fg} = S_f H_g = H_g T_f. \tag{1.4}$$

These identities show the tight relationships between dual Toeplitz operators and Toeplitz operators, and Hankel operators. There have been many results about Toeplitz algebra. In 1969, R.Douglas[1] obtained the short exact sequence:

$$(0) \longrightarrow \mathrm{com}\mathcal{I}(L^\infty(T)) \longrightarrow \mathcal{I}(L^\infty(T)) \longrightarrow L^\infty(T) \longrightarrow (0),$$

where $\mathcal{I}(L^\infty(T))$ is the Toeplitz algebra on Hardy space $H^2(T)$ generated by $\{T_f : f \in L^\infty(T)\}$, and $\mathrm{com}\mathcal{I}(L^\infty(T))$ is the commutator ideal in $\mathcal{I}(L^\infty(T))$. In [2], C.Yan and Sh.Sun completely characterized the automorphism group of the Toeplitz algebra generated by continuous symbols. G.McDonald and C.Sunberg[3] discussed the Toeplitz algebra with the symbols in $C(\mathcal{M})$, where $\mathcal{M}$ is the maximal ideal space of $H^\infty(D)$.

For the dual Toeplitz operators, K.Stroethoff and D.Zheng[4] has proved the following short exact sequence:

$$(0) \longrightarrow \mathrm{semi}\mathcal{I}(L^\infty(D)) \longrightarrow \mathcal{I}(L^\infty(D)) \longrightarrow L^\infty(D) \longrightarrow (0),$$

where $\mathrm{semi}\mathcal{I}(L^\infty(D))$ is the semicommutator ideal in $\mathcal{I}(L^\infty(D))$. [4] also poved that $\mathrm{semi}\mathcal{I}(L^\infty(D))$ is the class of all compact operators. This paper will extend these results to the polydisk.

2. Main resultes

The symbols $\mathcal{I}$, *semi* are the same as in introduction.

Theorem 1 *The sequence*

$$(0) \longrightarrow \mathrm{semi}\mathcal{I}(L^\infty(D^n)) \longrightarrow \mathcal{I}(L^\infty(D^n)) \longrightarrow L^\infty(D^n) \longrightarrow (0)$$

is short exact.

When $n=1$, Theorem 1 has been proved in [4] by using the functions

$$g_{w,s}(z)=\overline{(z-w)}\chi_{w+sD}(z),$$

where $w\in D$, $0<s<1-|w|$. In the case of polydisk, we can use the functions $G_{w,s}(z)=\prod_{i=1}^{n}\overline{(z_i-w_i)}\chi_{w_i+sD}(z_i)$, $w\in D^n, 0<s<\min\{1-|w_i|:1\le i\le n\}$. Since the proof is similar to that in [4], we omit it here.

Theorem 2 $\mathcal{K}\subseteq \mathrm{semi}\mathcal{I}(L^\infty(D^n))$, *where* $\mathcal{K}$ *is the class of all the compact operators on* $L_a^2(D^n)^\perp$.

Before proving this theorem, we give some notations and lemmas. For any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, where α_k $(1\le k\le n)$ are the nonnegative integers, we write $|\alpha|=\alpha_1+\dots+\alpha_n$. For $z=(z_1,\dots,z_n)\in D^n$, z^α denotes $z_1^{\alpha_1}\cdots z_n^{\alpha_n}$.

Lemma 3 *For* $w\in D^n$, *the identity*

$$k_w\otimes k_w=\prod_{i=1}^{n}(I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}}) \tag{2.1}$$

holds on $L_a^2(D^n)$.

Proof By the proposition 4.1 in [5], for $w\in D$, we have that

$$k_w\otimes k_w=I-2T_{\varphi_w}T_{\bar\varphi_w}+T^2_{\varphi_w}T^2_{\bar\varphi_w}$$

on $L_a^2(D)$. Suppose $f=z^\alpha\in L_a^2(D^n)$, where α is a multi-index. Then

$$(k_w\otimes k_w)z^\alpha=\prod_{i=1}^{n}\langle z_i^{\alpha_i},k_{w_i}\rangle k_{w_i}=\prod_{i=1}^{n}(I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}})z_i^{\alpha_i}.$$

But

$$\begin{aligned}T_{\bar\varphi_{w_i}}(z^\alpha)&=\int_{D^n}\overline{\varphi_{w_i}(\lambda_i)}\lambda^\alpha\overline{K_z(\lambda)}\mathrm{d}A(\lambda)\\&=\int_D\overline{\varphi_{w_i}(\lambda_i)}\lambda_i^{\alpha_i}\overline{K_{z_i}(\lambda_i)}\mathrm{d}A(\lambda_i)\prod_{k\ne i}^{n}\int_D\lambda_k^{\alpha_k}\overline{K_{z_k}(\lambda_k)}\mathrm{d}A(\lambda_k)\\&=(T_{\bar\varphi_{w_i}}(z_i^{\alpha_i}))\prod_{k\ne i}^{n}z_k^{\alpha_k}.\end{aligned}$$

Similarly, we have

$$T^2_{\bar\varphi_{w_i}}(z^\alpha)=(T^2_{\bar\varphi_{w_i}}(z_i^{\alpha_i}))\prod_{k\ne i}^{n}z_k^{\alpha_k}.$$

So

$$(I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}})z^\alpha=((I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}})z_i^{\alpha_i})\prod_{k\ne i}^{n}z_k^{\alpha_k}.$$

At last we have

$$\prod_{i=1}^{n}(I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}})z_i^{\alpha_i}=(\prod_{i=1}^{n}(I-2T_{\varphi_{w_i}}T_{\bar\varphi_{w_i}}+T^2_{\varphi_{w_i}}T^2_{\bar\varphi_{w_i}}))z^\alpha.$$

Since $k_\omega\otimes k_\omega$ is linear, (2.1) holds for any $f\in L_a^2(D^n)$.

In the following proof, we let $\mathcal{S}$ denote semi$\mathcal{I}(L^\infty(D^n))$ briefly.

Lemma 4 *For $1 \le i, j \le n$, $\bar{z}_i \otimes \bar{z}_j \in$ semi$\mathcal{I}(L^\infty(D^n))$.*

Proof We only prove the case $n = 2$, and the general case can be proved similarly. The following identities are obvious:

$$\bar{z}_1 \otimes \bar{z}_2 = (H_{\bar{z}_1}1) \otimes (H_{\bar{z}_2}1) = H_{\bar{z}_1}(1 \otimes 1)H^*_{\bar{z}_2}.$$

From Lemma 3 we have

$$\begin{aligned} 1 \otimes 1 =& k_0 \otimes k_0 \\ =& \prod_{i=1}^{2}(I - 2T_{z_i}T_{\bar{z}_i} + T^2_{z_i}T^2_{\bar{z}_i}) \\ =& I - 2(T_{z_1}T_{\bar{z}_1} + T_{z_2}T_{\bar{z}_2}) + (T^2_{z_1}T^2_{\bar{z}_1} + T^2_{z_2}T^2_{\bar{z}_2} + 4T_{z_1}T_{\bar{z}_1}T_{z_2}T_{\bar{z}_2}) - \\ & 2(T_{z_1}T_{\bar{z}_1}T^2_{z_2}T^2_{\bar{z}_2} + T_{z_2}T_{\bar{z}_2}T^2_{z_1}T^2_{\bar{z}_1}) + T^2_{z_1}T^2_{\bar{z}_1}T^2_{z_2}T^2_{\bar{z}_2}. \end{aligned}$$

By (1.4), the identity $H_{\bar{z}_i}T_{z_j} = S_{z_j}H_{\bar{z}_i} (1 \le i, j \le 2)$ holds. So

$$\begin{aligned} \bar{z}_1 \otimes \bar{z}_2 =& H_{\bar{z}_1}H^*_{\bar{z}_2} - 2S_{z_1}H_{\bar{z}_1}H^*_{\bar{z}_2}S_{\bar{z}_1} - 2S_{z_2}H_{\bar{z}_1}H^*_{\bar{z}_2}S_{\bar{z}_2} + \\ & S^2_{z_1}H_{\bar{z}_1}H^*_{\bar{z}_2}S^2_{\bar{z}_1}S^2_{z_2}H_{\bar{z}_1}H^*_{\bar{z}_2}S^2_{\bar{z}_2} + 4S_{z_1}H_{\bar{z}_1}T_{\bar{z}_1}T_{z_2}H^*_{\bar{z}_2}S_{\bar{z}_2} - 2S_{z_1}H_{\bar{z}_1}T_{\bar{z}_1}T^2_{z_2}H^*_{\bar{z}_2}S^2_{\bar{z}_2} - \\ & 2S_{z_2}H_{\bar{z}_1}T_{\bar{z}_2}T^2_{z_1}H^*_{\bar{z}_2}S^2_{\bar{z}_1} + S^2_{z_1}H_{\bar{z}_1}T^2_{\bar{z}_1}T^2_{z_2}H^*_{\bar{z}_2}S^2_{\bar{z}_2}. \end{aligned} \tag{2.2}$$

By (1.2) we have

$$H_{\bar{z}_1}H^*_{\bar{z}_2} = S_{\bar{z}_1 z_2} - S_{\bar{z}_1}S_{z_2} \in \mathcal{S}. \tag{2.3}$$

So first five items in (2.2) are in $\mathcal{S}$. Since

$$T_{\bar{z}_1}T_{z_2} = T_{\bar{z}_1 z_2} = T_{z_2\bar{z}_1} = T_{z_2}T_{\bar{z}_1} + H^*_{\bar{z}_2}H_{\bar{z}_1},$$

we know that

$$4S_{z_1}H_{\bar{z}_1}T_{\bar{z}_1}T_{z_2}H^*_{\bar{z}_2}S_{\bar{z}_2} = 4S_{z_1}H_{\bar{z}_1}T_{z_2}T_{\bar{z}_1}H^*_{\bar{z}_2}S_{\bar{z}_2} + 4S_{z_1}H_{\bar{z}_1}H^*_{\bar{z}_2}H_{\bar{z}_1}H^*_{\bar{z}_2}S_{\bar{z}_2}.$$

The above expression is in $\mathcal{S}$ by (2.3), in other words, the sixth item in (2.2) belongs to $\mathcal{S}$. Similarly, the other items in (2.2) are in $\mathcal{S}$. So $\bar{z}_1 \otimes \bar{z}_2 \in \mathcal{S}$. Similarly, we can also get $\bar{z}_1 \otimes \bar{z}_1$, $\bar{z}_2 \otimes \bar{z}_1$ and $\bar{z}_2 \otimes \bar{z}_2$ are in $\mathcal{S}$ respectively.

Proof of Theorem 2 By Lemma 4 and Theorem 5.39 in [1], it suffices to prove that $\mathcal{S}$ is irreducible in $\mathcal{B}(L^2_a(D^n)^\perp)$, where $\mathcal{B}(L^2_a(D^n)^\perp)$ denotes all the bounded linear operators on $L^2_a(D^n)^\perp$. Suppose $\mathcal{N}$ is a closed linear subspace of $L^2_a(D^n)^\perp$ which reduces $\mathcal{S}$, we have to show that $\mathcal{N} = L^2_a(D^n)^\perp$. Firstly, we will prove that $\bar{z}_i \in \mathcal{N}$ $(1 \le i \le n)$.

Since $\mathcal{N}$ is nonzero, it contains a nonzero function φ. Since the linear combinations of the functions in $\{z^\alpha\bar{z}^\beta : \alpha, \beta$ are multi-indices$\}$ are dense in $L^2(D^n)$, and thus there exist α, β such that $\langle\varphi, z^\alpha\bar{z}^\beta\rangle \ne 0$. Since $\varphi \in L^2_a(D^n)^\perp$, there exists $\beta_i > 0$, in other words, $|\beta| > 0$. Now we have

$$\langle\varphi, z^\alpha\bar{z}^\beta\rangle\bar{z}_i = \langle\bar{z}^\alpha z_i^{\beta_i - 1}\prod_{j\ne i} z_j^{\beta_j}(\varphi), \bar{z}_i\rangle\bar{z}_i = \langle S_{\bar{z}^\alpha z_i^{\beta_i-1}\prod_{j\ne i} z_j^{\beta_j}}(\varphi), \bar{z}_i\rangle\bar{z}_i$$

$$= (\bar{z}_i \otimes \bar{z}_i) S_{\bar{z}^\alpha z_i^{\beta_i - 1} \prod_{j \neq i} z_j^{\beta_j}}(\varphi).$$

Following Lemma 4, $(\bar{z}_i \otimes \bar{z}_i) S_{\bar{z}^\alpha z_i^{\beta_i - 1} \prod_{j \neq i} z_j^{\beta_j}} \in \mathcal{S}$. Since $\mathcal{N}$ reduces $\mathcal{S}$, $\langle \varphi, z^\alpha \bar{z}^\beta \rangle \bar{z}_i \in \mathcal{N}$. Because $\langle \varphi, z^\alpha \bar{z}^\beta \rangle \neq 0$, we conclude that $\bar{z}_i \in \mathcal{N}$.

For $j \neq i$, since $\bar{z}_j \otimes \bar{z}_i \in \mathcal{S}$ and $(\bar{z}_j \otimes \bar{z}_i)\bar{z}_i = \bar{z}_j$, we have $\bar{z}_j \in \mathcal{N}$.

Now suppose $\psi \in L_a^2(D^n)^\perp$ and $\psi \perp \mathcal{N}$. For any multi-indices α, β, if $|\beta| = 0$, it is obvious that $\langle \psi, z^\alpha \bar{z}^\beta \rangle = 0$. If not, then there exists $\beta_i > 0$ such that

$$\langle \psi, z^\alpha \bar{z}^\beta \rangle \bar{z}_i = (\bar{z}_i \otimes \bar{z}_i) S_{\bar{z}^\alpha z_i^{\beta_i - 1} \prod_{j \neq i} z_j^{\beta_j}}(\psi).$$

Since $\mathcal{N}$ reduces $\mathcal{S}$ and $\bar{z}_i \in \mathcal{N}$, we have $\langle \psi, z^\alpha \bar{z}^\beta \rangle = 0$. Thus ψ is zero almost everywhere on D^n, in other words, $\mathcal{N} = L_a^2(D^n)^\perp$. So we conclude that $\mathcal{S}$ is irreducible in $\mathcal{B}(L_a^2(D^n)^\perp)$.

Theorem 3 $\mathcal{K} = \mathrm{semi}\mathcal{I}(C(\overline{D^n}))$.

Proof For $f, g \in C(\overline{D^n})$, then H_f and H_g are compact. By (1.2) we have

$$S_{fg} - S_f S_g = H_f H_{\bar{g}}^*.$$

Thus $\mathrm{semi}\mathcal{I}(C(\overline{D^n})) \subseteq \mathcal{K}$. Combining Theorem 2, we conclude that $\mathcal{K} = \mathrm{semi}\mathcal{I}(C(\overline{D^n}))$.

Following Theorems 1 and 3, we have the short exact sequence

$$(0) \longrightarrow \mathcal{K} \longrightarrow \mathcal{I}(C(\overline{D^n})) \longrightarrow C(\overline{D^n}) \longrightarrow (0).$$

If f is a measurable function on D^n, then we denote by $\mathcal{R}(f)$ the essential rang of f, and by $\sigma(S_f)$ and $\sigma_e(S_f)$ respectively the spectrum and essential spectrum of S_f. For a subset E of the complex plane, let $h(E)$ denote the closed convex hull of E.

Since the proofs of the following theorems are similar to those in [4], we omit them here.

Theorem 4 *If $f \in L^\infty(D^n)$, then $\mathcal{R}(f) \subset \sigma(S_f) \subset h(\mathcal{R}(f))$.*

Theorem 5 *If $f \in L^\infty(D^n)$, and H_f and $H_{\bar{f}}$ are compact, then $\sigma_e(S_f) = \mathcal{R}(f)$.*

References

[1] DOUGLAS R G. *Banach Algebra Techniques in Operator Theory* [M]. Academic Press, New York-London, 1972.

[2] YAN Congquan, SUN Shunhua. *The automorphism group of the Toeplitz algebra on $H^2(T^2)$* [J]. Sci. China Ser. A, 2000, **43**(3): 225–233.

[3] MCDONALD G, SUNDBERG C. *Toeplitz operators on the disc* [J]. Indiana Univ. Math. J., 1979, **28**(4): 595–611.

[4] STROETHOFF K, ZHENG Dechao. *Algebraic and spectral properties of dual Toeplitz operators* [J]. Trans. Amer. Math. Soc., 2002, **354**(6): 2495–2520.

[5] STROETHOFF K, ZHENG Dechao. *Products of Hankel and Toeplitz operators on the Bergman space* [J]. J. Funct. Anal., 1999, **169**(1): 289–313.