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Boundedness of Some Operators and Commutators in Morrey–Herz Spaces on Non-Homogeneous Spaces

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Abstract The authors introduce the homogeneous Morrey-Herz spaces and the weak homogeneous Morrey-Herz spaces on non-homogeneous spaces and establish the boundedness in homogeneous Morrey-Herz spaces for a class of sublinear operators including Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators. Furthermore, some weak estimate of these operators in weak homogeneous Morrey-Herz spaces are also obtained. Moreover, the authors discuss the boundedness in homogeneous Morrey-Herz spaces of the maximal commutators associated with Hardy-Littlewood maximal operators and multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with RBMO(μ) functions.

Keywords Hardy-Littlewood maximal operators; Calderón-Zygmund operators; fractional integral operators; RBMO(μ) functions; multilinear commutators; homogeneous Morrey-Herz spaces; weak homogeneous Morrey-Herz spaces; non-homogeneous spaces.

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1. Preliminaries

It is well known that the doubling condition on the underlying measure is a key assumption in the harmonic analysis on Euclidean spaces or more general spaces of homogeneous type. We recall that the measure μ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \operatorname{supp}\mu$ and $r > 0$, where we denote by $B(x, r)$ the open ball centered at x and having the radius r. However, some recent research has revealed that the most results of classical Calderón-Zygmund operator theory are still true with the condition that the underlying measure μ does not satisfy the doubling condition[1]−[3]. In this case the measure μ only satisfies the following growth condition, namely, there exists a constant $C_0 > 0$ such that

$$\mu(B(x, r)) \le C_0 r^n \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where n is a fixed number and $0 < n \le d$. We call the Euclidean space, which is endowed with the usual Euclidean distance and a non-negative Radon measure

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μ only satisfying the above growth condition (1.1), a non-homogeneous space. The analysis on non-homogeneous spaces was proved to play a striking role in solving the long open Painlevé's problem by Tolsa[4].

Sawano and Tanaka[5] introduced the Morrey spaces on the non-homogeneous spaces and proved the boundedness in Morrey spaces of Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators. On the base of the above results, later Yang and Meng[6] considered the boundedness in Morrey spaces of the commutators generated by Calderón-Zygmund operators or fractional integral operators with RBMO(μ) functions . Motivated by these results, in this paper, we will introduce the homogeneous Morrey-Herz spaces and the weak homogeneous Morrey-Herz spaces on non-homogeneous spaces and establish the boundedness in these spaces for a class of sublinear operators including Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators. We also discuss the boundedness in homogeneous Morrey-Herz spaces of the commutators generated by Hardy-Littlewood maximal operators or Calderón-Zygmund operators or fractional integral operators with RBMO(μ) functions. We should point out that the Morrey spaces introduced by Sawano and Tanaka are the subspaces of the homogeneous Morrey-Herz spaces when some special indexes are taken. So in a sense our results extend the results of Sawano and Tanaka and Yang to more extensive situation. Otherwise, when μ is the $d-$dimensional Lebesgue measure, Xu[7],[8] systematically studied the boundedness of singular integral operators with rough kernel and the boundedness of the commutators generated by some sublinear operators with rough kernel with BMO($\mathbb{R}^d$) functions in the classical homogeneous Morrey-Herz spaces. Zhao, Jiang and Cao[9] proved the boundedness in classical homogeneous Morrey-Herz spaces of the maximal commutators associated with Hardy-Littlewood maximal operators and commutators generated by Calderón-Zygmund operators with BMO($\mathbb{R}^d$) functions. Our results in this paper can be regarded as a natural extension of the classical results with Lebesgue measure on non-homogeneous spaces.

In 2 Section, we study the boundedness of Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators in homogeneous Morrey-Herz spaces; In 3 Section, we establish some weak type estimate of the above operators in weak homogeneous Morrey-Herz spaces on non-homogeneous spaces; In 4 Section, we discuss the boundedness in homogeneous Morrey-Herz spaces of the multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with RBMO(μ) functions and maximal commutators associated with Hardy-Littlewood maximal operators. What should be pointed out is that one can formally define the non-homogeneous Morrey-Herz spaces and weak non-homogeneous Morrey-Herz spaces on non-homogeneous spaces. However, it is a pity that one still cannot obtain the boundedness in non-homogeneous Morrey-Herz spaces for some important operators such as Calderón-Zygmund operators, fractional integral operators and Hardy-Littlewood maximal operators and the weak type estimate in weak non-homogeneous Morrey-Herz spaces. So we only discuss the case of homogeneous Morrey-Herz spaces and weak homogeneous Morrey-Herz spaces.

Before stating the main results, we first give some necessary notations. In the following, unless otherwise stated, any cube is a closed cube in $\mathbb{R}^d$ with sides parallel to the axes and centered at some point of $\operatorname{supp}(\mu)$. For any cube $Q \subset \mathbb{R}^d$, we denote its side length by $l(Q)$. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that some cube $Q \in \mathbb{R}^d$ is a (α, β)-doubling cube if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ denotes the cube concentric with Q and having side length $\alpha l(Q)$. If α and β are not specified, all doubling cubes in this paper are $(2, 2^{d+1})$-doubling cubes. Given two cubes $Q_1 \subset Q_2$, we set

$$K_{Q_1, Q_2} = 1 + \sum_{k=1}^{N_{Q_1, Q_2}} \frac{\mu(2^k Q_1)}{l(2^k Q_1)^n},$$

where N_{Q_1, Q_2} is the first positive integer k such that $l(2^k Q_1) \ge l(Q_2)$. Some basic properties of K_{Q_1, Q_2} can be found in [10].

For any $k \in \mathbb{Z}$, denote $B_k = \{x \in \mathbb{R}^d : \ |x| \le 2^k\}$ and $A_k = B_k \backslash B_{k-1}$. The notation $\chi_k(x) = \chi_{A_k}(x)$ is the characteristic function of the set A_k. In addition, for a function $f \in L^1_{\mathrm{loc}}(\mu)$, denote $f_k = f\chi_k$.

In what follows, $C > 0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

2. The boundedness in homogenous Morrey-Herz spaces of sublinear operators

In this section, the homogenous Morrey-Herz spaces will be introduced and the boundedness in homogeneous Morrey-Herz spaces for a class of sublinear operators including Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators will be discussed.

Definition 2.1 *Let $-\infty < \alpha < \infty$, $0 \le \lambda < \infty$, $0 < p \le \infty$ and $0 < q < \infty$. The homogeneous Morrey-Herz space $M\dot{K}^{\alpha, \lambda}_{p, q}(\mu)$ is defined by*

$$M\dot{K}^{\alpha, \lambda}_{p, q}(\mu) = \left\{ f \in L^q_{\mathrm{loc}}\left(\mathbb{R}^d \backslash \{0\}\right) : \ \|f\|_{M\dot{K}^{\alpha, \lambda}_{p, q}(\mu)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha, \lambda}_{p, q}(\mu)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mu)} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

From the Difinition 2.1, $M\dot{K}^{\alpha, 0}_{p, q}(\mu)$ is just the homogeneous Herz space defined in the reference [11] when $\lambda = 0$. In this case, we have established the boundedness in Herz space of the Hardy-Littlewood maximal operators, Calderón-Zygmund operators and fractional integral operators. Thus, for all the associate results in this section, we only consider the case of $\lambda > 0$.

Theorem 2.1 *Let $0 < \lambda < \infty$, $0 < p < \infty$ and $-\infty < \beta_1, \beta_2 < \infty$. If for all $\beta \in (\beta_1, \beta_2)$ and some $1 < q < \infty$ satisfying $\beta_1/q + \lambda < \alpha < \beta_2/q + \lambda$, the sublinear operator L is bounded on*

$L^q(\mathbb{R}^d,\,|x|^\beta\,\mathrm{d}\mu(x))$, *then the operator* L *is also bounded on* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$.

Proof For all $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$, write $f(x)=\sum_{j=-\infty}^{\infty}f_j(x)$. Then

$$\begin{aligned}\|L(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}\le& C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=-\infty}^{k+1}\|\chi_kL(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p}+\\ &C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=k+2}^{\infty}\|\chi_kL(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p}\\ :=&\mathrm{D}_1+\mathrm{D}_2.\end{aligned}$$

We first estimate D_1. Choose α_2 such that $\alpha-\lambda<\alpha_2/q<\beta_2/q$. Then from the boundedness on $L^q(\mathbb{R}^d,\,|x|^{\alpha_2}\,d\mu(x))$ of L, we have

$$\begin{aligned}\mathrm{D}_1&\le C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k(\alpha-\alpha_2/q)p}\left[\sum_{j=-\infty}^{k+1}\|\chi_kL(f_j)\|_{L^q(\mathbb{R}^d,\,|x|^{\alpha_2}\,\mathrm{d}\mu(x))}\right]^p\right\}^{1/p}\\ &\le C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k(\alpha-\alpha_2/q)p}\left[\sum_{j=-\infty}^{k+1}\|f_j\|_{L^q(\mathbb{R}^d,\,|x|^{\alpha_2}\,\mathrm{d}\mu(x))}\right]^p\right\}^{1/p}\\ &\le C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}\left[\sum_{j=-\infty}^{k+1}2^{j\alpha_2/q+k(\alpha-\alpha_2/q)}\|f_j\|_{L^q(\mu)}\right]^p\right\}^{1/p}\\ &\le C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}\left[\sum_{j=-\infty}^{k+1}2^{(j-k)(\alpha_2/q-\alpha)+j\lambda}2^{-j\lambda}\left(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_l\|^p_{L^q(\mu)}\right)^{1/p}\right]^p\right\}^{1/p}\\ &\le C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\lambda p}\left[\sum_{j=-\infty}^{k+1}2^{(j-k)(\alpha_2/q-\alpha+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}\right]^p\right\}^{1/p}\\ &\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.\end{aligned}$$

Now we turn to estimate D_2. We choose α_1 such that $\beta_1/q<\alpha_1/q<\alpha-\lambda$. For the boundedness on $L^q(\mathbb{R}^d,\,|x|^{\alpha_1}\,\mathrm{d}\mu(x))$ of L, using a similar argument to the estimate for D_1, we get

$$\begin{aligned}\mathrm{D}_2\le& C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k(\alpha-\alpha_1/q)p}\left[\sum_{j=k+2}^{\infty}\|\chi_kL(f_j)\|_{L^q(\mathbb{R}^d,\,|x|^{\alpha_1}\,\mathrm{d}\mu(x))}\right]^p\right\}^{1/p}\\ \le& C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k(\alpha-\alpha_1/q)p}\left[\sum_{j=k+2}^{\infty}\|f_j\|_{L^q(\mathbb{R}^d,\,|x|^{\alpha_1}\,\mathrm{d}\mu(x))}\right]^p\right\}^{1/p}\\ \le& C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}\left[\sum_{j=k+2}^{\infty}2^{j\alpha_1/q+k(\alpha-\alpha_1/q)}\|f_j\|_{L^q(\mu)}\right]^p\right\}^{1/p}\end{aligned}$$

$$\le C\sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\lambda p}\left[\sum_{j=k+2}^{\infty} 2^{(j-k)(\alpha_1/q-\alpha+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}\right]^p\right\}^{1/p}$$
$$\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.$$

Combining the estimate for D_1 and D_2, we obtain $\|L(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}$. Thus we complete the proof of Theorem 2.1.

For $f \in L^1_{\mathrm{loc}}(\mu)$, define the Hardy-Littlewood maximal operators M by

$$Mf(x) = \sup_{Q\ni x}\frac{1}{l(Q)^n}\int_Q |f(y)|\,\mathrm{d}\mu(y). \tag{2.1}$$

Then from Lemma 3.1 in the reference [11] and Proposition 7.1 in the reference [1], the operator M is bounded on $L^q(\mathbb{R}^d,\, |x|^\beta\,\mathrm{d}\mu(x))$, where $1<q<\infty$ and $-n<\beta<n(q-1)$. As an application of Theorem 2.1, we can get the boundedness on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ of the Hardy-Littlewood maximal operators M as follows.

Corollary 2.1 *Let $0<\lambda<\infty$, $0<p<\infty$, $1<q<\infty$ and $-n/q+\lambda<\alpha<n/q'+\lambda$. Then the Hardy-Littlewood maximal operators M defined by (2.1) is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$.*

Now we discuss the boundedness in homogeneous Morrey-Herz spaces of Calderón-Zygmund operator and fractional integral operator.

Let $K(x,\,y)$ be a function on $\mathbb{R}^d\times\mathbb{R}^d\backslash\{(x,\,y):\,x=y\}$ and satisfy that

$$|K(x,\,y)| \le C|x-y|^{-n} \tag{2.2}$$

for $x\ne y$, and if $|x-y|\ge 2|x-x'|$,

$$|K(x,\,y)-K(x',\,y)|+|K(y,\,x)-K(y,\,x')| \le C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}},$$

where $\delta\in(0,\,1]$ and $C>0$ is a positive constant. The Calderón-Zygmund operator associated to the above kernel K and the measure μ is formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,\,y)f(y)\mathrm{d}\mu(y). \tag{2.3}$$

This integral may be not convergent for many functions. Thus we consider the truncated operator T_ε for $\epsilon>0$ defined by

$$T_\epsilon(f)(x) = \int_{|x-y|>\epsilon} K(x,\,y)f(y)\mathrm{d}\mu(y).$$

We say that T is bounded on $L^p(\mu)$ if the operators T_ϵ are bounded on $L^p(\mu)$ uniformly on $\epsilon>0$, where $1<p<\infty$; see the reference [10]. Similarly, the boundedness on other function spaces of T also means the boundedness on these spaces uniformly on $\epsilon>0$ of the truncated operators T_ϵ.

Using Corollary 2.1, we can establish the boundedness in $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ of Calderón-Zygmund operator as follows.

Theorem 2.2 *Let $0<\lambda<\infty$, $0<p<\infty$, $1<q<\infty$ and $-n/q+\lambda<\alpha<n/q'+\lambda$. If the operator T defined by (2.3) is bounded on $L^2(\mu)$, then T is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$.*

Proof We only need to prove the truncated operators T_ϵ are bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ uniformly on $\epsilon > 0$. Write $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Then we have

$$\begin{aligned}\|T_\epsilon(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \le& C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}\|\chi_k T_\epsilon(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} + \\ & C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=k-2}^{k+2}\|\chi_k T_\epsilon(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} + \\ & C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=k+3}^{\infty}\|\chi_k T_\epsilon(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} \\ :=& \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.\end{aligned}$$

We first estimate E_1. Note that $x \in A_k$, $y \in A_j$ and $j \le k-3$, then from (2.2) we get

$$|T_\epsilon(f_j)(x)| \le CM(f_j)(x).$$

Thus using an argument similar to the estimate for D_1 in Theorem 2.1 and combining Corollary 2.1, we obtain

$$\mathrm{E}_1 \le C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}\|\chi_k M(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.$$

For E_2, from the boundedness on $L^q(\mu)\,(1 < q < \infty)$ of T, we have

$$\mathrm{E}_2 \le C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=k-2}^{k+2}\|f_j\|_{L^q(\mu)}\right]^p\right\}^{1/p} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.$$

Finally, we estimate E_3. Note that $x \in A_k$, $y \in A_j$ and $j \ge k+3$. Then from (2.2) and the Hölder inequality we conclude that $|T_\epsilon(f_j)(x)| \le C2^{-jn/q}\|f_j\|_{L^q(\mu)}$. Therefore an argument similar to the estimate for D_2 in Theorem 2.1 leads to

$$\begin{aligned}\mathrm{E}_3 \le& C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=k+2}^{\infty} 2^{(k-j)n/q}\|f_j\|_{L^q(\mu)}\right]\right\}^{1/p} \\ \le& C \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\lambda p}\left[\sum_{j=k+2}^{\infty} 2^{(k-j)(n/q+\alpha-\lambda)}\right]^p\right\}^{1/p} \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \\ \le& C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.\end{aligned}$$

The estimates for E_1, E_2 and E_3 tell us that there exists a constant $C > 0$ independent of ϵ satisfying

$$\|T_\epsilon(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}$$

for all $\epsilon > 0$. Thus the proof of Theorem 2.2 is completed.

Given $0 < l < n$, the fractional integral operator I_l of order l is defined by

$$I_l(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-l}} \, \mathrm{d}\mu(y). \tag{2.4}$$

Then from the boundedness on $(L^p(\mu), L^q(\mu))$ of the fractional integral operator I_l, where $1 < p < n/l$ and $1/q = 1/p - l/n$[1], and an completely similar argument to Theorem 2.2, we can also prove the boundedness in homogeneous Morrey-Herz spaces of the fractional integral operator as follows.

Theorem 2.3 *Let $0 < \lambda < \infty$, $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $0 < p_1 \le p_2 < \infty$ and $-n/q_1 + l + \lambda < \alpha < n/q_1' + \lambda$. Suppose the fractional integral operator I_l is as in (2.4). Then I_l is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mu)$ into $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mu)$.*

Theorem 2.4 *Let $0 < \lambda < \infty$, $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1(1 - lp_1/n)$, $0 < p_2 < \infty$, $0 < p_1 \le \min\{q_1, p_2\}$, $-n/q_1 + l + \lambda < \alpha_1 < n/q_1' + \lambda$ and $\alpha_2 = \alpha_1 + l(p_1/q_1 - 1)$. Suppose the fractional integral operator I_l is as in (2.4). Then I_l is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p_1,q_1}(\mu)$ into $M\dot{K}^{\alpha_2,\lambda}_{p_2,q_2}(\mu)$.*

3. The boundedness in weak homogeneous Morrey-Herz spaces of sublinear operators

In order to consider the boundedness of Calderón-Zygmund operators and fractional integral operators at the endpoint case of the homogeneous Morrey-Herz spaces, Xu[7] introduced the weak homogeneous Morrey-Herz spaces. In this section, motivated by the results of Xu, we will introduce the weak homogeneous Morrey-Herz spaces on non-homogeneous spaces and establish the weak type estimate in weak homogeneous Morrey-Herz spaces of the Hardy-Littlewood radial maximal function, Calderón-Zygmund operators and fractional integral operators.

Definition 3.1 *Let $-\infty < \alpha < \infty$, $0 \le \lambda < \infty$, $0 < p \le \infty$ and $0 < q < \infty$. The weak homogeneous Morrey-Herz space $WM\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ is defined by*

$$WM\dot{K}^{\alpha,\lambda}_{p,q}(\mu) = \left\{ f : \ f \text{ is measurable on } \mathbb{R}^d \text{ and } \|f\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} < \infty \right\},$$

where

$$\|f\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} = \sup_{\gamma>0} \gamma \sup_{K\in\mathbb{Z}} 2^{-K\lambda} \left(\sum_{k=-\infty}^{K} 2^{k\alpha p} \mu(\{x \in A_k : |f(x)| > \gamma\})^{p/q} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

For the weak type estimate in the weak homogeneous spaces of Hardy-Littlewood maximal function, we have the following result.

Theorem 3.1 *Let $0 \le \lambda < \infty$, $0 < p < \infty$ and $-n + \lambda < \alpha < \lambda$. Suppose the Hardy-Littlewood radial maximal function M is as in (2.1). Then M is bounded from $M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)$ into $WM\dot{K}^{\alpha,\lambda}_{p,1}(\mu)$.*

Proof In order to prove Theorem 3.1, we only need to find a constant $C > 0$ such that

$$\gamma \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\mu(\{x\in A_k : |Mf(x)|>\gamma\})^p\right\}^{1/p} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)} \tag{3.1}$$

for all $\gamma > 0$.

Write

$$f(x)=\sum_{j=-\infty}^{\infty} f_j(x),$$

then

$$\begin{aligned}
&\gamma \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\mu(\{x\in A_k : |Mf(x)|>\gamma\})^p\right\}^{1/p}\\
&\le C\gamma \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\mu(\{x\in A_k : |M(\sum_{j=-\infty}^{k+1} f_j)(x)|>\gamma\})^p\right\}^{1/p}+\\
&\quad C\gamma \sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\mu(\{x\in A_k : |M(\sum_{j=k+2}^{\infty} f_j)(x)|>\gamma\})^p\right\}^{1/p}\\
&:= \mathrm{F}_1+\mathrm{F}_2.
\end{aligned}$$

On the one hand, applying the fact that M is bounded from $L^1(\mu)$ into weak $L^1(\mu)$, we get the estimate for F_1

$$\begin{aligned}
\mathrm{F}_1 &\le C\sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\alpha p}\left[\sum_{j=-\infty}^{k+1}\|f_j\|_{L^1(\mu)}\right]^p\right\}^{1/p}\\
&\le C\sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}\left[\sum_{j=-\infty}^{k+1} 2^{(k-j)\alpha+j\lambda}2^{-j\lambda}\left(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^1(\mu)}\right)^{1/p}\right]^p\right\}^{1/p}\\
&\le C\sup_{K\in\mathbb{Z}} 2^{-K\lambda}\left\{\sum_{k=-\infty}^{K} 2^{k\lambda p}\left[\sum_{j=-\infty}^{k+1} 2^{(j-k)(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)}\right]^p\right\}^{1/p}\\
&\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)}.
\end{aligned}$$

On the other hand, for F_2, we first have

$$\left\|\chi_k M\left(\sum_{j=k+2}^{\infty} f_j\right)\right\|_{L^1(\mu)} \le C\sum_{j=k+2}^{\infty} 2^{-jn}\|f_j\|_{L^1(\mu)}\mu(A_k)\le C\sum_{j=k+2}^{\infty}2^{(k-j)n}\|f_j\|_{L^1(\mu)}.$$

Then using an estimate similar to F_1, we obtain $\mathrm{F}_2 \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)}$.

The estimates for F_1 and F_2 yield (3.1). This completes the proof of Theorem 3.1.

Note that the Calderón-Zygmund operator T is bounded from $L^1(\mu)$ into weak $L^1(\mu)$ and the fractional integral operator I_l is bounded from $L^1(\mu)$ into weak $L^{n/n-l}(\mu)$, where $0 < l < n$[13]. Applying an argument completely similar to Theorem 3.1, we prove the weak type estimates of

Calderón-Zygmund operator and the fractional integral operator in the following respectively.

Theorem 3.2 *Let $0 \le \lambda < \infty$, $0 < p < \infty$ and $-n+\lambda < \alpha < \lambda$. Suppose the Calderón-Zygmund operator T is as in (2.3). Then T is bounded from $M\dot{K}^{\alpha,\lambda}_{p,1}(\mu)$ into $WM\dot{K}^{\alpha,\lambda}_{p,1}(\mu)$.*

Theorem 3.3 *Let $0 \le \lambda < \infty$, $0 < l < n$, $1/q = 1-l/n$, $0 < p_1 \le p_2 < \infty$ and $-n+l+\lambda < \alpha < \lambda$. Suppose the fractional integral operator I_l is as in (2.4). Then I_l is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,1}(\mu)$ into $WM\dot{K}^{\alpha,\lambda}_{p_2,q}(\mu)$.*

4. The boundedness in homogeneous Morrey-Herz spaces of commutators

In this section, we will establish the boundedness in homogeneous Morrey-Herz spaces of the maximal commutators associated with the Hardy-Littlewood radial maximal function and the multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with RBMO(μ) functions.

Therefore, we first recall the space RBMO(μ) with the nondoubling measure μ which was introduced by Tolsa[10].

Definition 4.1 *Let $\rho > 1$ be some fixed constant. We say that a function $b \in L^1_{\rm loc}(\mu)$ is in* RBMO(μ) *if there exists some constant $B > 0$ such that for any cube Q centered at some point of* $\operatorname{supp}(\mu)$,

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q |b(x) - m_{\widetilde{Q}}(b)|\,\mathrm{d}\mu(x) \le B < \infty$$

and for any two doubling cubes $Q_1 \subset Q_2$, $|m_{Q_1}(b) - m_{Q_2}(b)| \le BK_{Q_1,Q_2}$, where $\widetilde{Q}$ denotes the smallest doubling cube which is like $2^kQ(k \in \mathbb{N} \cup \{0\})$ and $m_{\widetilde{Q}}(b)$ denotes the mean of b over the cube $\widetilde{Q}$, that is,

$$m_{\widetilde{Q}}(b) = \frac{1}{\mu(\widetilde{Q})} \int_{\widetilde{Q}} b(x)\,\mathrm{d}\mu(x).$$

The minimal constant B as above is the RBMO(μ) *norm of b and is denoted by $\|b\|_*$.*

Theorem 4.1 *Let $b \in$ RBMO(μ), $0 < \lambda < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $-n/q + \lambda < \alpha < n/q' + \lambda$. The maximal commutator M_b is defined by*

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{l(Q)^n} \int_Q |b(x) - b(y)||f(y)|\,\mathrm{d}\mu(y). \tag{4.1}$$

Then M_b is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$.

Proof By the homogeneous property we can suppose $\|b\|_* = 1$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f_j(x),$$

then

$$\begin{aligned}\|M_b(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \le & C\sup_{K\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}\|\chi_k M_b(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} + \\ & C\sup_{K\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=k-2}^{k+2}\|\chi_k M_b(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} + \\ & C\sup_{K\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=k+3}^{\infty}\|\chi_k M_b(f_j)\|_{L^q(\mu)}\right]^p\right\}^{1/p} \\ :=& \mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3.\end{aligned}$$

We first estimate G_1. Note that $k-j\ge 3$ and $x\in A_k$, then from (4.1) and some simple geometric computation we obtain

$$M_b(f_j)(x)\le\frac{C}{2^{kn}}\int_{\mathbb{R}^d}|b(x)-b(y)||f_j(y)|\,\mathrm{d}\mu(y). \tag{4.2}$$

Denote by Q_j the smallest cube centered at the origin and containing A_j. Furthermore, we write $b_j=m_{\widetilde{Q_j}}(b)$. Then from (4.2), the Hölder inequality and Corollary 3.5 in the reference [10], we have

$$\begin{aligned}\|\chi_k M_b(f_j)\|_{L^q(\mu)} \le & C2^{-kn}\left\{\int_{A_k}\left[\int_{A_j}|b(x)-b(y)|\,|f(y)|\,\mathrm{d}\mu(y)\right]^q\mathrm{d}\mu(x)\right\}^{1/q} \\ \le & C2^{-kn}\|f_j\|_{L^1(\mu)}\left[\int_{A_k}|b(x)-b_j|^q\,\mathrm{d}\mu(x)\right]^{1/q} + \\ & C2^{kn(1/q-1)}\|f_j\|_{L^q(\mu)}\left[\int_{A_j}|b(y)-b_j|^{q'}\,\mathrm{d}\mu(y)\right]^{1/q'} \\ \le & C\|b\|_*(k-j)2^{(j-k)n(1-1/q)}\|f_j\|_{L^q(\mu)},\end{aligned} \tag{4.3}$$

where we used the fact that $K_{\widetilde{Q_j},\widetilde{Q_k}}\le C(k-j)$. It follows from (4.3) and $\alpha<n/q'+\lambda$ that

$$\begin{aligned}\mathrm{G}_1 \le & C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\alpha p}\left[\sum_{j=-\infty}^{k-3}(k-j)2^{(j-k)n/q'}\|f_j\|_{L^q(\mu)}\right]^p\right\}^{1/p} \\ \le & C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\lambda p}\left[\sum_{j=-\infty}^{k-3}(k-j)2^{(j-k)(n/q'-\alpha+\lambda)}2^{-j\lambda}\left(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_j\|^p_{L^q(\mu)}\right)^{1/p}\right]^p\right\}^{1/p} \\ \le & C\sup_{K\in\mathbb{Z}}2^{-K\lambda}\left\{\sum_{k=-\infty}^{K}2^{k\lambda p}\left[\sum_{j=-\infty}^{k-3}(k-j)2^{(j-k)(n/q'-\alpha+\lambda)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}\right]^p\right\}^{1/p} \\ \le & C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.\end{aligned}$$

Next we estimate G_2. Noting that M_b is bounded on $L^q(\mu)$, where $1 < q < \infty$[14], we get

$$\begin{aligned}\mathrm{G}_2 &\le C \sup_{K\in\mathbb{Z}} 2^{-K\lambda} \left\{ \sum_{k=-\infty}^{K} 2^{k\alpha p} \left[\sum_{j=k-2}^{k+2} \|f_j\|_{L^q(\mu)} \right]^p \right\}^{1/p} \\ &\le C \sup_{K\in\mathbb{Z}} 2^{-K\lambda} \left\{ \sum_{j=-\infty}^{K} 2^{j\alpha p} \|f_j\|^p_{L^q(\mu)} \right\}^{1/p} \\ &\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.\end{aligned}$$

Finally, we estimate G_3. Similar to the estimate for (4.3) we easily obtain

$$\|\chi_k M_b(f_j)\|_{L^q(\mu)} \le C 2^{(k-j)n/q}(j-k)\|f_j\|_{L^q(\mu)}.$$

And note that $\alpha > -n + l + \lambda$, hence

$$\begin{aligned}\mathrm{G}_3 &\le C \sup_{K\in\mathbb{Z}} 2^{-K\lambda} \left\{ \sum_{k=-\infty}^{K} 2^{k\lambda p} \left[\sum_{j=k+3}^{\infty} (j-k) 2^{(k-j)(n/q+\alpha-\lambda)} 2^{-j\lambda} 2^{j\alpha} \|f_j\|_{L^q(\mu)} \right]^p \right\}^{1/p} \\ &\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.\end{aligned}$$

The estimates for G_1, G_2 and G_3 indicate that $\|M_b(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}$. And we complete the proof of Theorem 4.1. □

Applying the similar method in Theorem 4.1, we prove the boundedness in homogeneous Morrey-Herz spaces of the multilinear commutators generated by Calderón-Zygmund operators with RBMO(μ) functions.

Let $m \in \mathbb{N}$, $b_i \in \mathrm{RBMO}(\mu)$, for $i = 1, 2, \ldots, m$. Write $\vec{b} = (b_1, b_2, \ldots, b_m)$. The multilinear commutator $T_{\vec{b}}$ generated by Calderón-Zygmund operators with RBMO(μ) functions is defined by

$$T_{\vec{b}}(f)(x) = [b_m, [b_{m-1}, \ldots, [b_1, T]\cdots]](f)(x), \tag{4.4}$$

where

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

And T stands for a weak limit as $\epsilon \to 0$ of some subsequence of uniformly bounded operators T_ϵ on $L^2(\mu)$[10]. It can be verified that T is still bounded on $L^2(\mu)$ and for some function $f \in L^2(\mu)$ with compact support,

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \mathrm{d}\mu(y), \quad \mu\text{-a. e. } x \in \mathbb{R}^d \setminus \operatorname{supp} f,$$

where the kernel function K is as in (2.3).

Theorem 4.2 *Let $0 < \lambda < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $-n/q + \lambda < \alpha < n/q' + \lambda$. Suppose the multilinear commutator $T_{\vec{b}}$ is as in (4.4). Then $T_{\vec{b}}$ is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$.*

Accordingly, we can also prove the boundedness in homogeneous Morrey-Herz spaces of the multilinear commutators generated by fractional integral operators with RBMO(μ) functions.

Theorem 4.3 *Let $m \in \mathbb{N}$, $b_i \in \mathrm{RBMO}(\mu)$ for $i = 1, 2, \dots, m$. The multilinear commutator $I_{l;\vec{b}}$ is defined by*

$$I_{l;\vec{b}}(f)(x) = \int_{\mathbb{R}^d} \prod_{i=1}^{m} [b_i(x) - b_i(y)] \frac{f(y)}{|x-y|^{n-l}} \, \mathrm{d}\mu(y). \tag{4.5}$$

Then $I_{l;\vec{b}}$ is bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mu)$ into $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mu)$, where $0 < \lambda < \infty$, $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $0 < p_1 \le p_2 < \infty$ and $-n/q_1 + l + \lambda < \alpha < n/q_1' + \lambda$.

Theorem 4.4 *Let $0 < \lambda < \infty$, $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1(1 - lp_1/n)$, $0 < p_2 < \infty$, $0 < p_1 \le \min\{q_1, p_2\}$, $-n/q_1 + l + \lambda < \alpha_1 < n/q_1' + \lambda$ and $\alpha_2 = \alpha_1 + l(p_1/q_1 - 1)$. Suppose the multilinear commutator $I_{l;\vec{b}}$ is as in (4.5). Then $I_{l;\vec{b}}$ is bounded from $M\dot{K}^{\alpha_1,\lambda}_{p_1,q_1}(\mu)$ to $M\dot{K}^{\alpha_2,\lambda}_{p_2,q_2}(\mu)$.*

Remark Since the established results in the reference [11] contain the boundedness in homogeneous Herz spaces of the maximal commutators associated with the Hardy-Littlewood radial maximal function and the commutators generated by Calderón-Zygmund operators with RBMO(μ) functions, we only consider the case $\lambda > 0$ in this section.

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References

[1] GARCÍA-CUERVA J, MARTELL J M. *Two-weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces* [J]. Indiana Univ. Math. J., 2001, **50**(3): 1241–1280.

[2] NAZAROV F, TREIL S, VOLBERG A. *Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces* [J]. Internat. Math. Res. Notices, 1997, **15**: 703–726.

[3] OROBITG J, PÉREZ C. *A_p weights for nondoubling measures in R^n and applications* [J]. Trans. Amer. Math. Soc., 2002, **354**(5): 2013–2033.

[4] TOLSA X. *Painlevé's problem and the semiadditivity of analytic capacity* [J]. Acta Math., 2003, **190**(1): 105–149.

[5] SAWANO Y, TANAKA H. *Morrey spaces for non-doubling measures* [J]. Acta Math. Sin. (Engl. Ser.), 2005, **21**(6): 1535–1544.

[6] YANG Dongyong, MENG Yan. *Boundedness of commutators in Morrey-Herz spaces on nonhomogeneous spaces* [J]. Beijing Shifan Daxue Xuebao, 2005, **41**(5): 449–454.(in Chinese).

[7] Xu Lifang. *Boundedness of singular integrals and Marcinkiewicz integrals* [D]. Thesis for Ma. D. at Beijing Normal University, 2003. (in Chinese)

[8] XU Lifang. *Boundedness of commutators on the homogeneous Morrey-Herz spaces* [J]. Beijing Shifan Daxue Xuebao, 2004, **40**(3): 297–303. (in Chinese)

[9] ZHAO Xiangqing, JIANG Yinsheng, CAO Yonghui. *Boundedness of commutators on Herz-Morrey spaces* [J]. J. Lanzhou Univ. Nat. Sci., 2006, **42**(5): 116–123.

[10] TOLSA X. *BMO, H^1 and Calderón-Zygmund operators for non doubling measures* [J]. Math. Ann., 2001, **319**(1): 89–149.

[11] GUO Yan, MENG Yan. *Boundedness of some operators and commutators in Herz spaces with non-doubling measures* [J]. Beijing Shifan Daxue Xuebao, 2004, **40**(6): 725–731. (in Chinese)

[12] TOLSA X. *A proof of the weak $(1,1)$ inequality for singular integrals with non doubling measures based on a Calderón-Zygmund decomposition* [J]. Publ. Mat., 2001, **45**(1): 163–174.

[13] GARCÍA-CUERVA J, GATTO A E. *Boundedness properties of fractional integral operators associated to non-doubling measures* [J]. Studia Math., 2004, **162**(3): 245–261.

[14] MENG Yan. *New proof for boundedness of maximal commutators with non-doubling measures* [J]. Beijing Shifan Daxue Xuebao, 2004, **40**(3): 309–314. (in Chinese)