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A Note to Hutton's Theorem

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Abstract It is proved in this paper that there is a bijection φ from $\mathcal{J} \cup \mathcal{J}'$ to $\delta \cup \delta'$ which satisfies: (1) $\varphi|\mathcal{J} : (\mathcal{J}, \subset) \longrightarrow (\delta, \leq)$ is a frame isomorphism; (2) $\varphi|\mathcal{J}' : (\mathcal{J}', \subset) \longrightarrow (\delta', \leq)$ is a coframe isomorphism, where $\mathcal{J}$ is the ordinary topology on $[0, 1]$, δ is the ordinary L-topology on L-unit interval $I(L)$, and L is a frame with an order-reversing involution. This result improves Theorem 3 in Hutton's paper.

Keywords L-unit interval; ordinary topology; meet-preserving mapping; join-preserving mapping.

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1. Introduction and preliminaries

In this paper, L is supposed to be a frame with the least element 0 and the greatest element 1, and equipped with an order-reversing involution $'$ satisfying $a \vee a' = 1$ for every $a \in L$. Again let L^X be the set of all mappings (called L-subsets) from a set X to L, R the real line, $I = [0, 1]$ the ordinary unit interval, $\mathcal{J}$ the ordinary topology on $[0, 1]$, $H(I)$ the set of all monotonic decreasing mappings $\lambda : R \longrightarrow L$ satisfying $\lambda(t) = 1$ for $t \in (-\infty, 0)$ and $\lambda(t) = 0$ for $t \in [1, +\infty)$. For any two elements λ_1 and λ_2 in $H(I)$, define $\lambda_1 \sim \lambda_2 \iff \lambda_1(t+) = \lambda_2(t+)$ and $\lambda_1(t-) = \lambda_2(t-)$ $(\forall t \in R)$, where $\lambda(t+) = \bigvee_{s>t} \lambda(s)$ and $\lambda(t-) = \bigwedge_{s<t} \lambda(s)$ $(\lambda \in H(I), t \in R)$. It can be verified that $\sim$ is an equivalence relationship on $H(I)$. The equivalence class containing element λ is written as $[\lambda]$, and the ordinary L-topology δ on $I(L) = H(I)/\!\sim$ is generated by the subbase $\{R_t, L_t \mid t \in R\}$, where $R_t([\lambda]) = \lambda(t+)$ and $L_t([\lambda]) = (\lambda(t-))'$ $(\forall [\lambda] \in I(L))$. The L-topological space $(I(L), \delta)$ is called L-unit interval which was first defined by B. Hutton[1]. Hutton also proved the following:

Theorem 1[1] *If L is a completely distributive complete lattice with an order-reversing involution $'$ satisfying $a \vee a' = 1$ $(\forall a \in L)$, then there exists a frame isomorphism (i.e. a bijection or one-one correspondence which preserves both arbitrary joins and finite meets) $\varphi : (\mathcal{J}, \subset) \longrightarrow (\delta, \leq)$.*

In this paper we will generalize the above theorem.

2. Main results and their proofs

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For each $W \in \mathcal{J}$, W is the union of a family $\{(a_i, b_i)\}_{i<\alpha}$ of disjoint nonempty intervals (called constitution intervals of W), where α is a cardinal number not more than $\aleph_0$ (the cardinal number of the set of all natural numbers), $a_i \in \{-\infty\} \cup [0,1), b_i \in \{+\infty\} \cup (0,1]$, and we make the convention that $(-\infty, b) = (x, b) = [0, b)$ $(x < 0), (a, +\infty) = (a, y) = (a, 1]$ $(y > 1)$. We use the equality $W = \bigsqcup_{i<\alpha}(a_i, b_i)$ to denote this fact, and define a mapping $\varphi : \mathcal{J} \cup \mathcal{J}' \longrightarrow \delta \cup \delta'$ as follows (where $\mathcal{J}' = \{[0,1] - V \mid V \in \mathcal{J}\}$, and $\delta' = \{A' \mid A \in \delta\}$):

$$\forall[\lambda] \in I(L), \quad \varphi(W)([\lambda]) = \begin{cases} \bigvee_i(\lambda(a_i+) \wedge \lambda(b_i-)'), & \text{if } W = \bigsqcup_i(a_i, b_i) \in \mathcal{J}, \\ \bigwedge_i(\lambda(a_i+)' \vee \lambda(b_i-)), & \text{if } W = [0,1] - \bigsqcup_i(a_i, b_i) \in \mathcal{J}', \end{cases}$$

that is,

$$\forall[\lambda] \in I(L), \quad \varphi(W) = \begin{cases} \bigvee_i(R_{a_i} \wedge L_{b_i}), & \text{if } W = \bigsqcup_i(a_i, b_i) \in \mathcal{J}, \\ \bigwedge_i(R'_{a_i} \vee L'_{b_i}), & \text{if } W = [0,1] - \bigsqcup_i(a_i, b_i) \in \mathcal{J}'. \end{cases}$$

This mapping has the following properties:

Theorem 2 *(1) φ is a bijection which preserves the order-reversing involution.*

(2) φ preserves the partial order.

(3) If $\mathcal{A} \subset \mathcal{J}$, then $\varphi(\bigcup \mathcal{A}) = \bigvee_{A \in \mathcal{A}} \varphi(A)$.

(4) If $\mathcal{A} \subset \mathcal{J}'$ then $\varphi(\bigcap \mathcal{A}) = \bigwedge_{A \in \mathcal{A}} \varphi(A)$.

(5) If $A, B \in \mathcal{J} \cup \mathcal{J}'$ and $A \cap B \in \mathcal{J} \cup \mathcal{J}'$, then $\varphi(A \cap B) = \varphi(A) \wedge \varphi(B)$.

(6) If $A, B \in \mathcal{J} \cup \mathcal{J}'$ and $A \cup B \in \mathcal{J} \cup \mathcal{J}'$, then $\varphi(A \cup B) = \varphi(A) \vee \varphi(B)$.

Proof **Step 1.** φ preserves the order-reversing involution. For any $W \in \mathcal{J}' \cup \mathcal{J}$, if $W \in \mathcal{J}$, say $W = \bigsqcup_i(a_i, b_i)$, then $\varphi(W) = \bigvee_i(R_{a_i} \wedge L_{b_i})$, and thus $\varphi(W') = \varphi([0,1] - \bigsqcup_i(a_i, b_i)) = \bigwedge_i(R'_{a_i} \vee L'_{b_i}) = [\bigvee_i(R_{a_i} \wedge L_{b_i})]' = [\varphi(W)]'$. Similarly, $\varphi(W') = [\varphi(W)]'$ if $W \in \mathcal{J}'$.

Step 2. For any $A, B \in \mathcal{J}, \varphi(A \cap B) = \varphi(A) \wedge \varphi(B)$. Suppose $A, B \in \mathcal{J}$, say $A = \bigsqcup_i(a_i, b_i)$ and $B = \bigsqcup_j(c_j, d_j)$. Then

$$A \cap B = [\bigsqcup_i(a_i, b_i)] \cap [\bigsqcup_j(c_j, d_j)] = \bigcup_{i,j}[(a_i, b_i) \cap (c_j, d_j)].$$

By the join infinite distributive law, for every $[\lambda] \in I(L)$, we have

$$\begin{aligned} &[\varphi(A) \wedge \varphi(B)]([\lambda]) = \varphi(A)([\lambda]) \wedge \varphi(B)([\lambda]) \\ &\quad = [\bigvee_i(\lambda(a_i+) \wedge \lambda(b_i-)')] \wedge [\bigvee_j(\lambda(c_j+) \wedge \lambda(d_j-)')] \\ &\quad = \bigvee_{i,j}[\lambda(\max\{a_i, c_j\}+) \wedge \lambda(\min\{b_i, d_j\}-)']. \end{aligned}$$

Notice that if $(a, b) \cap (c, d) = \emptyset$, then $\min\{b, d\} \leq \max\{a, c\}$, $\lambda(\min\{b, d\}-)' \leq \lambda(\max\{a, c\}+)'$, $\lambda(\max\{a, c\}+) \wedge \lambda(\min\{b, d\}-)' = 0$, and thus

$$[\varphi(A) \wedge \varphi(B)]([\lambda]) = \bigvee_{i,j;(a_i,b_i)\cap(c_j,d_j)\neq\emptyset}(\lambda(\max\{a_i, c_j\}+) \wedge \lambda(\min\{b_i, d_j\}-)') \quad (\forall[\lambda] \in I(L)).$$

Furthermore, if $(a_i, b_i) \cap (c_j, d_j) \neq \emptyset$, then $(a_i, b_i) \cap (c_j, d_j) = (\max\{a_i, c_j\}, \min\{b_i, d_j\})$, and

$$A \cap B = \bigcup_{i,j} [(a_i, b_i) \cap (c_j, d_j)] = \bigcup_{i,j;(a_i,b_i)\cap(c_j,d_j)\neq\emptyset} (\max\{a_i, c_j\}, \min\{b_i, d_j\}).$$

As $(\max\{a_{i_1}, c_{j_1}\}, \min\{b_{i_1}, d_{j_1}\}) \cap (\max\{a_{i_2}, c_{j_2}\}, \min\{b_{i_2}, d_{j_2}\}) \subset [(a_{i_1}, b_{i_1}) \cap (a_{i_2}, b_{i_2})] \cap [(c_{j_1}, d_{j_1}) \cap (c_{j_2}, d_{j_2})] = \emptyset$ if $(i_1, j_1) \neq (i_2, j_2)$, $\{(\max\{a_i, c_j\}, \min(b_i, d_j) \mid (a_i, b_i) \cap (c_j, d_j) \neq \emptyset\}$ are the constitution intervals of $A \cap B$. Thus $A \cap B = \bigsqcup\{(\max\{a_i, c_j\}, \min\{b_i, d_j\}) \mid i, j, (a_i, b_i) \cap (c_j, d_j) \neq \emptyset\}$. By definition of φ we have

$$\varphi(A \cap B)([\lambda]) = \bigvee_{i,j;(a_i,b_i)\cap(c_j,d_j)\neq\emptyset} (\lambda(\max\{a_i, c_j\}+) \wedge \lambda(\min\{b_i, d_j\}-)') \quad (\forall[\lambda] \in I(L)).$$

Therefore, $\varphi(A) \wedge \varphi(B) = \varphi(A \cap B)$.

Step 3. (proof of (3)) As $\varphi(\bigvee \mathcal{A}) = \varphi(\emptyset) = 0_{I(L)} = \bigvee_{A\in\emptyset} \varphi(A)$ for $\mathcal{A} = \emptyset$, we only need to show the case $\mathcal{A} \neq \emptyset$.

First we consider the special case $W = (a, b)$ and $V = (c, d)$. By definition of φ, when $W \cap V = \emptyset$, $\varphi(W \cup V) = \varphi(W) \vee \varphi(V)$, thus we only need to show the case $a < c < b < d$. Take an $e \in (c, b)$, then by the distributive law and the property of the order-reversing involution of L, we have

$$\begin{aligned}
[\varphi(W) \vee \varphi(V)]([\lambda]) &= [\varphi(W)([\lambda])] \vee [\varphi(V)([\lambda])] = [\lambda(a+) \wedge \lambda(b-)'] \vee [\lambda(c+) \wedge \lambda(d-)'] \\
&\geq [\lambda(a+) \wedge \lambda(e)'] \vee [\lambda(e) \wedge (\lambda(d-))'] \\
&= \lambda(a+) \wedge \lambda(d-)' = \varphi((a, d))([\lambda]) = \varphi(W \cup V)([\lambda]) \quad (\forall \lambda \in I(L)),
\end{aligned}$$

i.e., $\varphi(W \cup V) \leq \varphi(W) \vee \varphi(V)$. By Step 3, for any $A, B \in \mathcal{J}$ with $A \subset B$, we have $\varphi(A) \leq \varphi(B)$. It follows that $\varphi(W) \leq \varphi(W \cup V)$, $\varphi(V) \leq \varphi(W \cup V)$, $\varphi(W) \vee \varphi(V) \leq \varphi(W \cup V)$, and $\varphi(W \cup V) = \varphi(W) \vee \varphi(V)$. This implies that $\varphi(\bigcup_{i=1}^n (a_i, b_i)) = \bigvee_{i=1}^n \varphi((a_i, b_i))$.

Now we consider the general case. Let $\mathcal{A} = \{A_i\}_{i\in I} \subset \mathcal{J}, A_i = \bigcup_{j\in J_i} (a_j^i, b_j^i)$, $\bigcup_{i\in I} A_i = A = \bigsqcup \mathcal{B}$, where $\mathcal{B}$ is a collection of disjoint nonempty open intervals. Take $(a, b) \in \mathcal{B}$ and $[c, d] \subset (a, b)$, then $[c, d] \subset \bigcup_{i,j} (a_j^i, b_j^i)$. As $[c, d]$ is compact, there exists a finite subcollection $\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\} (n \in N)$ of $\{(a_j^i, b_j^i)\}_{i,j}$ such that $[c, d] \subset \bigcup_{k=1}^n (a_k, b_k)$. We have

$$\varphi((c, d)) \leq \varphi(\bigcup_{k=1}^n (a_k, b_k)) = \bigvee_{k=1}^n \varphi((a_k, b_k)) \leq \bigvee_{k=1}^n \varphi(A_{i_k}) \leq \bigvee_{i\in I} \varphi(A_i).$$

By the join infinite distributive law, we have

$$\begin{aligned}
\lim_{c\to a+} \varphi((c, d))([\lambda]) &= \bigvee_{c,d} \varphi((c, d))([\lambda]) = \bigvee_{c>a} (\lambda(c+) \wedge \lambda(d-)') \\
&= (\bigvee_{c>a} \lambda(c+)) \wedge \lambda(d-)' = \lambda(a+) \wedge \lambda(d-)' = \varphi((a, d))([\lambda]).
\end{aligned}$$

Furthermore, let $c \longrightarrow a+$, and $d \longrightarrow b-$. Then we have

$$\varphi((c, d))([\lambda]) = \lambda(c+) \wedge \lambda(d-)' \longrightarrow \lambda(a+) \wedge \lambda(b-)' = \varphi((a, b))([\lambda]) (\forall[\lambda] \in I(L)).$$

This means that $\varphi((a, b)) \leq \bigvee_{i\in I} \varphi(A_i)$. Therefore, $\varphi(\bigcup A_i) = \varphi(\bigsqcup \mathcal{B}) = \bigvee_{(a,b)\in\mathcal{B}} \varphi((a, b)) \leq$

$\bigvee_{i\in I}\varphi(A_i)$. By Step 3, we have $\varphi(A)\geq\bigvee_{i\in I}\varphi(A_i)$, hence $\varphi(\bigcup_{i\in I}A_i)=\bigvee_{i\in I}\varphi(A_i)$.

Step 4. (proof of (4)) Let $\mathcal{A}\subset\mathcal{J}'$. By (1) and (3), $\varphi(\bigcap\mathcal{A})=\varphi([0,1]-\bigcup_{(a,b)\in\emptyset}(a,b))=\bigwedge_{(a,b)\in\emptyset}(R'_{a_i}\vee L'_{b_i})=1_{I(L)}=\bigwedge_{A\in\mathcal{A}}\varphi(A)$ if $\mathcal{A}=\emptyset$. If $\mathcal{A}\neq\emptyset$, then $\varphi(\bigcap\mathcal{A})=\varphi((\bigcup_{A\in\mathcal{A}}A')')=[\bigvee_{A\in\mathcal{A}}(\varphi(A'))]'=\bigwedge_{A\in\mathcal{A}}\varphi(A)''=\bigwedge_{A\in\mathcal{A}}\varphi(A)$.

Step 5. φ is a bijection. Firstly, for any $W,V\in\mathcal{J}$ with $W-V\neq\emptyset$, let $x\in W-V$ and $[\lambda]=\chi_{(-\infty,x]}$. Then $[\lambda]\in I(L)$ and $\varphi(W)([\lambda])=1\neq 0=\varphi(V)([\lambda])$ by definition of φ. By Step 1, φ is an injection. Secondly, as L is a frame with an order-reversing involution and $\{(a,b)\mid a,b\in[0,1]\cup\{-\infty,+\infty\},a<b\}$ is a base of $\mathcal{J}$, $\{\varphi((a,b))\mid a,b\in[0,1]\cup\{-\infty,+\infty\}$ and $a<b\}$ is a base of δ by Step 1 and Step 3, which means φ is a surjection.

Step 6. (proof of (2)) Suppose $A,B\in\mathcal{J}\cup\mathcal{J}'$ with $A\subset B$. We consider the following four cases:

Case 1 $A,B\in\mathcal{J}$. By Step 3, $\varphi(A)\leq\varphi(B)$.

Case 2 $A,B\in\mathcal{J}'$. By (1) and the result of Case 1, we have $\varphi(B)'\leq\varphi(A)'$, i.e., $\varphi(A)\leq\varphi(B)$.

Case 3 $A\in\mathcal{J}$ and $B\in\mathcal{J}'$. As φ is a bijection, both $\varphi|\mathcal{J}'$ and $\varphi^{-1}|\mathcal{J}'$ are order-preserving, and $\varphi|\mathcal{J}'$ is meet-preserves, we have $\varphi(A_0^-)=\varphi(\bigcap\{B_0\mid B_0\in\mathcal{J}',A_0\subset B_0\})=\bigwedge\{\varphi(B_0)\mid B_0\in\mathcal{J}',A_0\subset B_0\}=\bigwedge\{D\mid D\in\delta',\varphi(A_0)\leq D\}=\varphi(A_0)^-$ for every $A_0\in\mathcal{J}\cup\mathcal{J}'$. Since $B\in\mathcal{J}'$, $A^-\subset B$, by the proof of Case 2, we have $\varphi(A)\leq\varphi(A)^-=\varphi(A^-)\leq\varphi(B)$.

Case 4 $A\in\mathcal{J}'$ and $B\in\mathcal{J}$. As $A\subset B$, we have $A\cap B'=\emptyset$, and thus $\varphi(A\cap B')=\varphi(A)\wedge\varphi(B')=\varphi(A)\wedge\varphi(B)'=0_X$, i.e., $\varphi(A)(x)\wedge\varphi(B)(x)'=0$ $(\forall x\in I(L))$. Since $a\vee a'=1$ $(\forall a\in L)$, $\varphi(A)(x)=\varphi(A)(x)\wedge 1=\varphi(A)(x)\wedge[\varphi(B)(x)\vee\varphi(B)(x)']=[\varphi(A)(x)\wedge\varphi(B)(x)]\vee[\varphi(A)(x)\wedge\varphi(B)(x)']=[\varphi(A)(x)\wedge\varphi(B)(x)]\vee 0=\varphi(A)(x)\wedge\varphi(B)(x)$, $\varphi(A)(x)\leq\varphi(B)(x)$ $(\forall x\in I(L))$, i.e., $\varphi(A)\leq\varphi(B)$.

Step 7 (proof of (5))

Case 1 $A,B\in\mathcal{J}\cup\mathcal{J}',A\cap B\in\mathcal{J}$. If $A,B\in\mathcal{J}'$, then $\varphi(A\cap B)=\varphi(A)\wedge\varphi(B)$ by (4). If $A,B\in\mathcal{J}$, then $\varphi(A\cap B)=\varphi(A)\wedge\varphi(B)$ by Step 2. If $A\in\mathcal{J}$ and $B\in\mathcal{J}'$, then $(A\cap B)^o=A^o\cap B^o=A\cap B^o=A\cap B$ because $A\cap B\in\mathcal{J}$, and thus $A\subset B'\cup B^o$. Since $B\in\mathcal{J}'$, $B',B^o\in\mathcal{J}$, by Step 4, $\varphi(A)\leq\varphi(B'\cup B^o)=\varphi(B')\vee\varphi(B^o)$. It follows that $\varphi(A)\wedge\varphi(B)\leq(\varphi(B')\vee\varphi(B^o))\wedge\varphi(B)=(\varphi(B')\wedge\varphi(B))\vee(\varphi(B^o)\wedge\varphi(B))=(\varphi(B)'\wedge\varphi(B))\vee(\varphi(B^o)\wedge\varphi(B))=\varphi(B^o)$, and that $\varphi(A)\wedge\varphi(B)\leq\varphi(A)\wedge\varphi(B^o)=\varphi(A\cap B^o)=\varphi(A\cap B)$. By Step 2, $\varphi(A\cap B)=\varphi(A\cap B^o)\leq\varphi(A)\wedge\varphi(B^o)\leq\varphi(A)\wedge\varphi(B)$. Therefore, $\varphi(A\cap B)=\varphi(A)\wedge\varphi(B)$. Similarly, $\varphi(A\cap B)=\varphi(A)\wedge\varphi(B)$ if $A\in\mathcal{J}'$ and $B\in\mathcal{J}$.

Case 2 $A,B\in\mathcal{J}\cup\mathcal{J}'$, $A\cap B\in\mathcal{J}'$. By Step 4, it suffices to show the case that $A\in\mathcal{J}$, $B\in\mathcal{J}'$ and $A\cap B\in\mathcal{J}'$. As φ preserves the partial order, $\varphi(A\cap B)\leq\varphi(A)\wedge\varphi(B)$. Thus we only need to show $\varphi(A\cap B)\geq\varphi(A)\wedge\varphi(B)$. Firstly, since $A\in\mathcal{J}$ and $A\cap B\in\mathcal{J}'$, we have $\varphi(A\cap B)\geq\varphi((A\cap$

$B)^{o-}) = \varphi((A \cap B^o)^-) = [\varphi(A \cap B^o)]^- = [\varphi(A) \wedge \varphi(B^o)]^- = \bigwedge\{C \in \delta' \mid \varphi(A) \wedge \varphi(B^o) \le C\}$. Secondly, for every $C \in \delta$ satisfying $\varphi(A) \wedge \varphi(B^o) \le C$ and every $D \le \varphi(A) \wedge \varphi(B)$, we have $D^o \le [\varphi(A) \wedge \varphi(B)]^o = \varphi(A) \wedge \varphi(B^o)$, $D^o \le C$, and $D \le D^- \le C$ since C is closed. Therefore, $\varphi(A) \wedge \varphi(B) \le C$. Since C is arbitrary, $\varphi(A) \wedge \varphi(B) \le \bigwedge\{C \in \delta' \mid \varphi(A) \wedge \varphi(B^o) \le C\} = \varphi(A \cap B)$.

Step 8 (proof of (6)) We only need to consider the following four cases:

Case 1 $A,\ B \in \mathcal{J}'$. By Step 2 and the fact that φ preserves the order-reversing involution, we have $\varphi(A \cup B) = \varphi(A) \vee \varphi(B)$.

Case 2 $A \in \mathcal{J}, B \in \mathcal{J}'$, and $A \cup B \in \mathcal{J}'$. By Case 1 of Step 7 and the fact that φ preserves the order-reversing involution, we have $\varphi(A \cup B) = \varphi(A) \vee \varphi(B)$.

Case 3 $A,\ B \in \mathcal{J}$. Then $\varphi(A \cup B) = \varphi(A) \vee \varphi(B)$ by Step 3.

Case 4 $A \in \mathcal{J}, B \in \mathcal{J}'$, and $A \cup B \in \mathcal{J}$. By Case 1 of Step 7 and the fact that φ preserves the order-reversing involution, we have $\varphi(A \cup B) = \varphi(A) \vee \varphi(B)$. This completes the proof of Theorem 2. $\square$

Let $A \in \delta$. As $R_a([\lambda]) \wedge L_b([\lambda]) = 0$ for all $[\lambda] \in I(L)$ and all $a, b \in [0,1]$ with $a \ge b$, we can write $A = \bigvee\{R_{a_s} \wedge L_{b_s} \mid s \in S_A\}$, where $a_s < b_s$ for all $s \in S_A$. The set $G_A = \{(a_s, b_s) \mid s \in S_A\}$ is called a collection of open intervals of $A^{[2]}$. Hence, for every $B \in \delta'$, B can be denoted by $B = \bigvee\{R_{a_t} \wedge L_{b_t} \mid t \in T_B\}' = \bigwedge\{R'_{a_t} \vee L'_{b_t} \mid t \in T_B\}$, where $a_t < b_t$, $a_t \le 1$, $b_t \ge 0$. Define a mapping $\psi : \delta \cup \delta' \to \mathcal{J} \cup \mathcal{J}'$ as follows:

$$\psi(A) = \begin{cases} \bigcup_{s \in S_A} (a_s, b_s), & \text{if } A = \bigvee\{R_{a_s} \wedge L_{b_s} \mid s \in S_A\} \in \delta, \\ \bigcap_{t \in T_A} [[0,1] - (a_t, b_t)], & \text{if } A = \bigwedge\{R'_{a_t} \vee L'_{b_t} \mid t \in T_A\} \in \delta'. \end{cases}$$

Then we have the following

Proposition 1 *$\psi : \delta \cup \delta' \to \mathcal{J} \cup \mathcal{J}'$ is the inverse mapping of $\varphi : \mathcal{J} \cup \mathcal{J}' \to \delta \cup \delta'$.*

Proof Firstly, $\psi \circ \varphi(W) = \psi(\varphi(W)) = \psi(\bigvee_i (R_{a_i} \wedge L_{b_i})) = \bigcup_i (a_i, b_i) = W$ if $W = \bigsqcup_i (a_i, b_i) \in \mathcal{J}$, and $\psi \circ \varphi(W) = \psi(\varphi(W)) = \psi(\bigwedge_i (R'_{a_i} \vee L'_{b_i})) = \bigcap_i [[0,1] - (a_i, b_i)] = W$ if $W = [0,1] - \bigsqcup_i (a_i, b_i) \in \mathcal{J}'$. Secondly, $\varphi \circ \psi(A) = \varphi(\psi(A)) = \varphi(\bigcup_{s \in S_A} (a_s, b_s)) = \bigvee_{s \in S_A} (R_{a_s} \wedge L_{b_s}) = A$ if $A = \bigvee\{R_{a_s} \wedge L_{b_s} \mid t \in S_A\} \in \delta$ by Step 3 in the proof of Theorem 2, and $\varphi \circ \psi(B) = \varphi(\psi(B)) = \varphi(\bigcap_{t \in T_B} ([0,1] - (a_t, b_t))) = \varphi([0,1] - \bigcup_{t \in T_B} (a_t, b_t)) = \varphi(\bigcup_{t \in T_B} (a_t, b_t))' = [\bigvee_{t \in T_B} (R_{a_t} \wedge L_{b_t})]' = B$ if $B = \bigwedge\{R'_{a_t} \vee L'_{b_t} \mid t \in T_B\} \in \delta'$. Therefore, $\varphi^{-1} = \psi$.

The mapping $\varphi^{-1} = \psi : \delta \cup \delta' \to \mathcal{J} \cup \mathcal{J}'$ has the following properties:

Theorem 3 *(1) φ^{-1} preserves the order-reversing involution.*

(2) Both $\varphi^{-1}|\delta$ and $\varphi^{-1}|\delta'$ preserve the partial order.

(3) $\varphi^{-1}|\delta$ preserves arbitrary joins, and $\varphi^{-1}|\delta'$ preserves arbitrary meets.

(4) $\varphi^{-1}|\delta$ preserves finite meets, and $\varphi^{-1}|\delta'$ preserves finite joins.

Proof (1) Take an $A \in \delta \cup \delta'$. If $A = \bigvee\{R_{a_s} \wedge L_{b_s} \mid s \in S_A\} \in \delta$ (where $a_s < b_s$, $a_s \le 1$, and $b_s \geq 0$), then $\varphi^{-1}(A) = \bigcup_{s\in S_A}(a_s, b_s)$, and thus $\varphi^{-1}(A') = \varphi^{-1}([\bigvee_{s\in S_A}(R_{a_s} \wedge L_{b_s})]') = \varphi^{-1}(\bigwedge_{s\in S_A}(R'_{a_s} \vee L'_{b_s})) = [0,1] - \bigsqcup_{s\in S_A}(a_s, b_s) = [\varphi^{-1}(A)]'$. Similarly, $\varphi^{-1}(A') = [\varphi^{-1}(A)]'$ if $A \in \mathcal{J}'$.

(2) Firstly, for every $A = \bigvee_{i\in S_A}(R_{a_i} \wedge L_{b_i}) \in \delta$ and $B = \bigvee_{j\in S_B}(R_{c_j} \wedge L_{d_j}) \in \delta$ with $A \leq B$, we have $\varphi^{-1}(B) = \varphi^{-1}(A \vee B) = \varphi^{-1}([\bigvee_{i\in S_A}(R_{a_i} \wedge L_{b_i})] \vee [\bigvee_{j\in S_B}(R_{c_j} \wedge L_{d_j})]) = [\bigcup_{i\in S_A}(a_i, b_i)] \cup [\bigcup_{j\in S_B}(c_j, d_j)] = \varphi^{-1}(A) \vee \varphi^{-1}(B)$. Thus $\varphi^{-1}(A) \leq \varphi^{-1}(B)$. Secondly, for $A,\ B \in \delta'$ with $A \leq B$, since φ^{-1} preserves the order-reversing involution, we have $A',\ B' \in \delta$ and $B' \leq A'$. Thus $\varphi^{-1}(B)' = \varphi^{-1}(B') \leq \varphi^{-1}(A') = \varphi^{-1}(A)'$, i.e., $\varphi^{-1}(A) \leq \varphi^{-1}(B)$.

(3) Suppose that $\mathcal{A} = \{A_i \mid i \in I\} = \{\bigvee_{j\in S_{A_i}}(R_{a_j^{(i)}} \wedge L_{b_j^{(i)}}) \mid i \in I\} \subset \delta$, where $0 \leq a_j^{(i)} < b_j^{(i)} \leq 1$ $(\forall i \in I, \forall j \in S_{A_i})$. By definition of φ^{-1}, $\varphi^{-1}(\bigvee \mathcal{A}) = \varphi^{-1}(\bigvee_{i\in I}\bigvee_{j\in S_{A_i}}(R_{a_j^{(i)}} \wedge L_{b_j^{(i)}})) = \bigcup_{i\in I}\bigcup_{j\in S_{A_i}}(a_j^{(i)}, b_j^{(i)}) = \bigcup_i \varphi^{-1}(A_i)$. Since φ^{-1} preserves the order-reversing involution, $\varphi^{-1}|\delta'$ preserves arbitrary meets.

(4) Suppose that $A = \bigvee_{i\in S_A}(R_{a_i} \wedge L_{b_i}) \in \delta$ and $B = \bigvee_{j\in S_B}(R_{c_j} \wedge L_{d_j}) \in \delta$. By the join infinite distributive law,

$$
\begin{aligned}
\varphi^{-1}(A \wedge B) &= \varphi^{-1}([\bigvee_{i\in S_A}(R_{a_i} \wedge L_{b_i})] \wedge [\bigvee_{j\in S_B}(R_{c_j} \wedge L_{d_j})]) \\
&= \varphi^{-1}(\bigvee_{i\in S_A, j}(R_{a_i} \wedge L_{b_i} \wedge R_{c_j} \wedge L_{d_j})) \\
&= \varphi^{-1}(\bigvee_{i\in S_A, j\in S_B}(R_{\max\{a_i,c_j\}} \wedge L_{\min\{b_i,d_j\}})) \\
&= \varphi^{-1}(\bigvee\{R_{\max\{a_i,c_j\}} \wedge L_{\min\{b_i,d_j\}} \mid \max\{a_i,c_j\} < \min\{b_i,d_j\}, i \in S_A, j \in S_B\}) \\
&= \bigcup\{(\max\{a_i,c_j\}, \min\{b_i,d_j\}) \mid \max\{a_i,c_j\} < \min\{b_i,d_j\}, i \in S_A, j \in S_B\} \\
&= \bigcup\{(a_i,b_i) \cap (c_j,d_j) \mid (a_i,b_i) \cap (c_j,d_j) \neq \emptyset, i \in S_A, j \in S_B\} \\
&= [\bigcup_{i\in S_A}(a_i,b_i)] \cap [\bigcup_{j\in S_B}(c_j,d_j)] = \varphi^{-1}(A) \cap \varphi^{-1}(B).
\end{aligned}
$$

Since φ^{-1} preserves the order-reversing involution, $\varphi^{-1}|\delta'$ preserves finite joins.

Proposition 2 *Mappings φ and φ^{-1} preserve existent joins and meets.*

Proof Suppose that $\{A_i\}_{i\in I} \subset \mathcal{J} \cup \mathcal{J}'$, and $\bigcup_{i\in I} A_i \in \mathcal{J} \cup \mathcal{J}'$. Since φ preserves the partial order, $\varphi(\bigcup_{i\in I} A_i) \geq \bigvee_{i\in I}\varphi(A_i)$. Next we prove $\varphi(\bigcup_{i\in I} A_i) \leq \bigvee_{i\in I}\varphi(A_i)$. For every $A \geq \bigvee_{i\in I}\varphi(A_i)$, we have $A \geq \varphi(A_i)$ $(\forall i \in I)$, since φ^{-1} preserves the partial order, then for every $i \in I$, $\varphi^{-1}(A) \geq \varphi^{-1} \circ \varphi(A_i) = A_i$. So $\varphi^{-1}(A) \geq \bigcup_{i\in I} A_i$. Since φ preserves the partial order, $A = \varphi \circ \varphi^{-1}(A) \geq \varphi(\bigcup_{i\in I} A_i)$. Therefore, $\varphi(\bigcup_{i\in I} A_i) \leq \bigvee_{i\in I}\varphi(A_i)$. So, $\varphi(\bigcup_{i\in I} A_i) = \bigvee_{i\in I}\varphi(A_i)$.

Similarly, φ preserves existent meets, φ^{-1} preserves existent joins and meets.

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