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On the Crossing Numbers of $K_5 \times S_n$

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Abstract By connecting the 5 vertices of K_5 to other n vertices, we obtain a special family of graph denoted by H_n. This paper proves that the crossing number of H_n is $Z(5,n)+2n+\lfloor \frac{n}{2} \rfloor+1$, and the crossing number of Cartesian products of K_5 with star S_n is $Z(5,n)+5n+\lfloor \frac{n}{2} \rfloor+1$.

Keywords graph; drawing; crossing number; star; Cartesian products.

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1. Introduction

For graph theory terminology not defined here we direct the reader to [1] and all the graphs are connected simple graphs. Let G be a graph with vertex set V_G and edge set E_G. For convenience, let E_v be the set of all the edges incident to v ($v \in V$) and for any edge set $E' \subseteq E$, let $\langle E' \rangle$ denote the edge-induced subgraph of G, and $G \backslash \{E'\}$ the subgraph of G obtained by deleting the edges of edge set E'. Specially, $G \backslash \{e\}$ denotes the subgraph of G obtained by deleting the edge e. If a vertex of graph G has degree k, then we call it a $k-vertex$. Formally, the Cartesian product $G \times H$ of two graphs G and H has vertex set $V(G \times H) = V(G) \times V(H)$ and edge set

$$E(G \times H) = \{(u_i, v_j)(u_h, v_k) : u_i = u_h \text{ and } v_j v_k \in E(H) \text{ or } v_j = v_k \text{ and } u_i u_h \in E(G); u_i, u_h \in V(G), v_j, v_k \in V(H)\}.$$

If one of the graphs is S_n, the star $K_{1,n}$, a less formal description is helpful: $G \times S_n$ is obtained by $n+1$ copies $G^0, G^1, \ldots, G^n$ of G and by joining the vertices of the copies $G^i, i = 1, 2, \ldots, n$, to G^0 correspondingly.

A drawing is called good, if for all arcs in A, no two with a common endpoint meet, no two meet in more than one point, and no three have a common point. A crossing in a good drawing is a point of intersection of two arcs in A. A good drawing is said to be optimal if it minimizes the number of crossings. The crossing number $\mathrm{cr}(G)$ of a graph G is the number of crossings in

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any optimal drawing of G in the plane. Let ϕ be a good drawing of graph G. We denote by $\mathrm{cr}_\phi(G)$ the number of crossings in ϕ.

Gray and Johnson have proved that in general the problem to determine the crossing numbers of graphs is NP-complete[2]. At present only a few families of graphs with arbitrarily large crossing numbers for the plane are known. Guy and Zarankiewicz presented two important conjectures on the determination of the crossing numbers of graphs.

On the crossing numbers of complete graphs K_n, Guy conjectured:

$$\mathrm{cr}(K_n) = \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor,$$

where $\lfloor x\rfloor$ denotes the maximum integer not greater than x. In 1993, Woodall proved that Guy's conjecture is correct when $n \leq 10$ in [3], but when $n \geq 11$, Guy's conjecture is undetermined.

On the crossing numbers of complete bipartite graphs $K_{m,n}$, Zarankiewicz conjectured:

$$\mathrm{cr}(K_{m,n}) = Z(m,n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor.$$

In 1970, Kleitman proved that when $\min(m,n) \leq 6$, Zarankiewicz's conjecture is true[4]; In [4], Kleitman proved that Zarankiewicz's conjecture is also true for $K_{7,7}$ and $K_{7,9}$. But for other m, n, Zarankiewicz's conjecture is only an upper bound.

On the crossing numbers of complete tripartite graphs, Kouhei Asano determined the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$[5].

On the crossing numbers of Cartesian product graphs, most are the Cartesian product of low order graphs with paths, cycles and stars[6−13]. Among them, Klešč determined the crossing numbers of $K_4 \times p_n$, $K_4 \times S_n$[8] and $K_5 \times p_n$[9]; Beineke and Ringeisen determined the crossing numbers of $K_4 \times C_n$[10]. But the crossing numbers of $K_5 \times C_n$ and $K_5 \times S_n$ are undetermined.

In this paper, for the convenience of determining the crossing numbers of $K_5 \times S_n$, we first construct the following graph H_n:

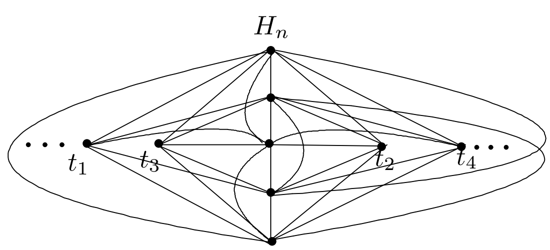

Figure 1 A good drawing of H_n

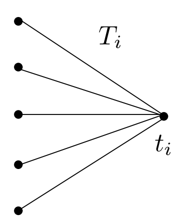

Figure 2 T_i

In the bipartite graph $K_{5,n}$ with bipartition $\{t_1, t_2, \ldots, t_n\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$, connecting the edges incident with any two vertices of set $\{v_1, v_2, v_3, v_4, v_5\}$ yields a new graph, denoted by H_n. Furthermore, H_n can also be obtained by joining all the 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$ of K_5 to n vertices t_i $(i = 1, 2, \ldots, n)$ (Figure 1). Let T_i $(i = 1, 2, \ldots, n)$ be the set of edges incident with vertex t_i $(i = 1, 2, \ldots, n)$ (Figure 2) and let E_0 be the set of edges of K_5. It is easy to obtain

$$E(H_n) = E_0 \cup (\bigcup_{i=1}^{n} T_i).$$

The following theorems are our two main results:

Theorem 1 $\mathrm{cr}(H_n) = Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1$.

Theorem 2 $\mathrm{cr}(K_5 \times S_n) = Z(5,n) + 5n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1$.

2. The proofs of Theorems

In order to prove the Theorems, first we introduce some lemmas.

Lemma 1 *If ϕ is a good drawing of a graph G, and E_1, E_2 and E_3 are three mutually disjoint edge subsets of G, then we have*

1) $\mathrm{cr}_\phi(E_1 \cup E_2) = \mathrm{cr}_\phi(E_1) + \mathrm{cr}_\phi(E_1, E_2) + \mathrm{cr}_\phi(E_2)$*;*

2) $\mathrm{cr}_\phi(E_1 \cup E_2, E_3) = \mathrm{cr}_\phi(E_1, E_3) + \mathrm{cr}_\phi(E_2, E_3)$.

Proof It is direct from the definitions. □

Lemma 2 *If ϕ is a good drawing of H_2 such that* $\mathrm{cr}_\phi(T_1, T_2) = 0$, *then* $\mathrm{cr}_\phi(E_0, T_1 \cup T_2) \geq 5$.

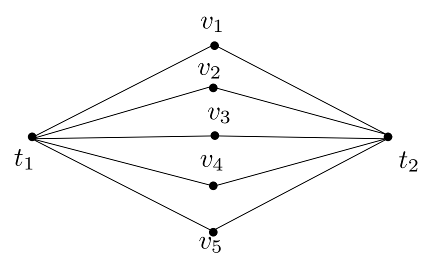

Figure 3 A good drawing ϕ' of $H$when $\mathrm{cr}_\phi(T_1, T_2) = 0$

Proof Let $H = \langle \{T^1 \cup T^2\} \rangle$ be the edge induced subgraph of H^2. Clearly, H is isomorphic to the complete bipartite graph $K_{2,5}$ with vertex partition $\{t^1, t^2\}$ and $V(K_5) := \{v_1, v_2, v_3, v_4, v_5\}$. Since $\mathrm{cr}_\phi(T^1, T^2) = 0$, the subdrawing ϕ' of H induced by ϕ must be isomorphic to Figure 3. From Figure 3, each edge of $E_0' := \{v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5\}$ has at least one crossing with the edge set $T_1 \cup T_2$, then $\mathrm{cr}_\phi(E_0, T_1 \cup T_2) \geq 5$. □

Lemma 3 $\mathrm{cr}(K_7 \backslash \{e\}) = 6$, $e \in E(K_7)$.

Proof See the [9]. □

Lemma 4 $\mathrm{cr}(H_1) = 3$, $\mathrm{cr}(H_2) = 6$.

Proof Since H_1 is isomorphic to K_6 and H_2 isomorphic to $K_7 \backslash \{e\}$, $e \in E(K_7)$, we have

$\mathrm{cr}(H_1) = \mathrm{cr}(K_6) = 3$; $\mathrm{cr}(H_2) = \mathrm{cr}(K_7 \backslash \{e\}) = 6$. □

Theorem 1 $\mathrm{cr}(H_n) = Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1$.

Proof First construct a good drawing θ of H_n as Figure 1. Clearly, in Figure 1, $H_n \backslash E_0$ is isomorphic to $K_{5,n}$ and the induced subdrawing by θ is an optimal drawing of $K_{5,n}$, so $\mathrm{cr}_\theta(H_n \backslash E_0) = \mathrm{cr}(K_{5,n}) = Z(5,n)$; $\mathrm{cr}_\theta(E_0, T_i) = 2$ (when i is odd), $\mathrm{cr}_\theta(E_0, T_i) = 3$ (when i is even); $\mathrm{cr}_\theta(E_0) = 1$. Combining (1) and Lemma 1, we have

$$\begin{aligned}\mathrm{cr}_\theta(H_n) &= \mathrm{cr}_\theta(H_n \backslash E_0) + \mathrm{cr}_\theta(E_0, \bigcup_{i=1}^{n} T_i) + \mathrm{cr}_\theta(E_0)\\ &= Z(5,n) + \sum_{i=1}^{n} \mathrm{cr}_\theta(E_0, T^i) + \mathrm{cr}_\theta(E_0)\\ &= Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.\end{aligned}$$

By the definition of crossing numbers, it is easy to know that $\mathrm{cr}(H_n) \leq \mathrm{cr}_\theta(H_n) = Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1$. In the following, we will prove that the theorem holds true by induction on n.

By Lemma 4, $\mathrm{cr}(H_1) = Z(5,1) + 2 \times 1 + \lfloor \frac{1}{2} \rfloor + 1 = 3$ and $\mathrm{cr}(H_2) = Z(5,2) + 2 \times 2 + \lfloor \frac{2}{2} \rfloor + 1 = 6$. So the Theorem holds when $n = 1, 2$. Now assume that $n \geq 3$ and when $\ell < n$, $\mathrm{cr}(H_\ell) = Z(5,\ell) + 2\ell + \lfloor \frac{\ell}{2} \rfloor + 1$. Let ϕ be a good drawing of H_n. Then we only need to prove that $\mathrm{cr}_\phi(H_n) \geq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$. We divide the problem into the following different cases:

Case 1 There exist two vertices t_i and t_j $(1 \leq i, j \leq n; i \neq j)$ such that $\mathrm{cr}_\phi(T_i, T_j) = 0$.

Without loss of generality, assume that $\mathrm{cr}_\phi(T_n, T_{n-1}) = 0$. When $1 \leq i \leq n-2$, as $\langle T_n \cup T_{n-1} \cup T_i \rangle$ is isomorphic to complete bipartite graph $K_{3,5}$ with $\mathrm{cr}_\phi(T_n, T_{n-1}) = 0$, we have

$$\mathrm{cr}_\phi(T_n \cup T_{n-1}, T_i) = \mathrm{cr}_\phi(K_{3,5}) - \mathrm{cr}_\phi(T_n \cup T_{n-1}) - \mathrm{cr}_\phi(T_i) \geq 4 - 0 - 0 = 4. \tag{2}$$

Furthermore, as $\langle E_0 \cup (\bigcup_{i=1}^{n-2} T_i) \rangle$ is isomorphic to H_{n-2}, combining Lemmas 1, 2 and Equations (1), (2), we have

$$\begin{aligned}\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(E_0 \cup T_n \cup T_{n-1} \cup \bigcup_{i=1}^{n-2} T_i)\\ &= \mathrm{cr}_\phi(T_n \cup T_{n-1}, \bigcup_{i=1}^{n-2} T_i) + \mathrm{cr}_\phi(T_n \cup T_{n-1}, E_0) + \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n-2} T_i)\\ &= \sum_{i=1}^{n-2} \mathrm{cr}_\phi(T_n \cup T_{n-1}, T_i) + \mathrm{cr}_\phi(T_n \cup T_{n-1}, E_0) + \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n-2} T_i)\\ &\geq 4(n-2) + 5 + Z(5, n-2) + 2(n-2) + \lfloor \frac{n-2}{2} \rfloor + 1\\ &= Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.\end{aligned}$$

Case 2 For every $1 \leq i < j \leq n$, there holds $\mathrm{cr}_\phi(T_i, T_j) \geq 1$.

Subcase 2.1 There exists a vertex t_i $(1 \leq i \leq n)$ such that $\mathrm{cr}_\phi(T_i, E_0) = 0$.

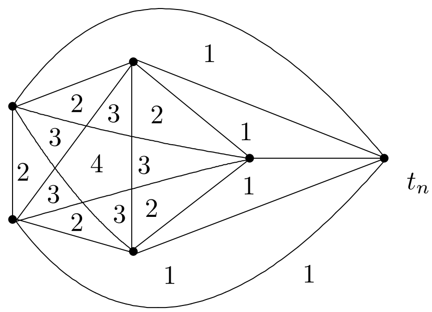

Figure 4 A good drawing of $\langle T_n \cup E_0 \rangle$ when $\mathrm{cr}_\phi(T_n, E_0) = 0$

Suppose $\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$. Without loss of generality, let $\mathrm{cr}_\phi(T_n, E_0) = 0$. Then consider the subdrawing ϕ_1 of $\langle T_n \cup E_0 \rangle$ induced by ϕ: as $\mathrm{cr}_\phi(T_n, E_0) = 0$, there is a disk C such that the vertices of K_5 are all located on the boundary of C, and the edges of K_5 are all located in the inner of C. Furthermore, as ϕ is a good drawing and the edges of K_5 can be presented by straight lines, vertex t_1 and the edges incident with t_1 are all located on the outside of C, see Figure 4. In Figure 4, we divide the regions of ϕ_1 into 2 classes:

1) On ϕ, when vertices t_i $(1 \le i \le n-1)$ are located in the region marked with 1, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 4$, and we can find that only if $\mathrm{cr}_\phi(T_i, E_0) = 0$, there is $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4$; if $\mathrm{cr}_\phi(T_i, E_0) > 0$, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$.

2) On ϕ, when vertices t_i $(1 \le i \le n-1)$ are located in the region marked with $2, 3, 4$, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$.

Let $A_1 := \{t_i | \mathrm{cr}_\phi(T_i, E_0) = 0, 1 \le i \le n-1\}$;

Let $A_2 := \{t_i | \mathrm{cr}_\phi(T_i, E_0) > 0, 1 \le i \le n-1\}$.

So A_1 presents set of some of the vertices located in region 1, and for any $t_i \in A_1$, $\mathrm{cr}_\phi(T_i, T_n) \ge 4$; also, $A_1 \cup A_2 = \{t_i, 1 \le i \le n-1\}$. As $\langle E_0 \cup (\bigcup_{i=1}^{n-1} T_i) \rangle$ is isomorphic to H_{n-1}, by Lemma 1, we get

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) \\
&= \mathrm{cr}_\phi(T_n, \bigcup_{i=1}^{n-1} T_i) + \mathrm{cr}_\phi(T_n, E_0) + \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n-1} T_i) \\
&= \mathrm{cr}_\phi(T_n, \bigcup_{t_i \in A_1} T_i) + \mathrm{cr}_\phi(T_n, \bigcup_{t_i \in A_2} T_i) + \mathrm{cr}_\phi(T_n, E_0) + \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n-1} T_i) \\
&= \sum_{t_i \in A_1} \mathrm{cr}_\phi(T_n, T_i) + \sum_{t_i \in A_2} \mathrm{cr}_\phi(T_n, T_i) + \mathrm{cr}_\phi(T_n, E_0) + \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n-1} T_i) \\
&\ge 4|A_1| + |A_2| + 0 + Z(5, n-1) + 2(n-1) + \lfloor \frac{n-1}{2} \rfloor + 1
\end{aligned}$$

$$= Z(5,n-1) + 2(n-1) + \lfloor \frac{n-1}{2} \rfloor + 1 + (n-1) + 3|A_1|$$
$$= Z(5,n-1) + 3(n-1) + \lfloor \frac{n-1}{2} \rfloor + 1 + 3|A_1|.$$

But at the beginning, we assume that $\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$. By a simple calculation, we have

$$|A_1| < \frac{1}{3}\{Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 3(n-1) + \lfloor \frac{n-1}{2} \rfloor + 1)\} \leq \frac{n+1}{3}. \qquad (3)$$

Also $\langle \bigcup_{i=1}^{n-1} T_i \rangle$ is isomorphic to $K_{5,n-1}$. By Lemma 1 and Equations (1) and (3), we get

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) =& \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) \\
=& \mathrm{cr}_\phi(T_n \cup E_0, \bigcup_{i=1}^{n-1} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
=& \mathrm{cr}_\phi(T_n \cup E_0, \bigcup_{t_i \in A_1} T_i) + \mathrm{cr}_\phi(T_n \cup E_0, \bigcup_{t_i \in A_2} T_i) + \\
& \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
=& \sum_{t_i \in A_1} \mathrm{cr}_\phi(T_n \cup E_0, T_i) + \sum_{t_i \in A_2} \mathrm{cr}_\phi(T_n \cup E_0, T_i) + \\
& \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
\geq& 4|A_1| + 5|A_2| + 5 + Z(5,n-1) \\
=& Z(5,n-1) + 4|A_1| + 5(n-1-|A_1|) + 5 \\
=& Z(5,n-1) + 5(n-1) - |A_1| + 5 \\
>& Z(5,n-1) + 5n - \frac{n+1}{3} \\
\geq& Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\end{aligned}$$

Subcase 2.2 For any t_i $(1 \leq i \leq n)$, $\mathrm{cr}_\phi(T_i, E_0) \geq 1$, and there exists a vertex t_i $(1 \leq i \leq n)$ such that $\mathrm{cr}_\phi(T_i, E_0) = 1$.

Without loss of generality, suppose $\mathrm{cr}_\phi(T_n, E_0) = 1$. Let ϕ_2 be the subdrawing of $\langle T_n \cup E_0 \rangle$ induced by ϕ. Now we will explain there is only one good drawing ϕ_2 of $\langle T_n \cup E_0 \rangle$: First, suppose the edge $t_n v_1$ of T_n and the edge $v_3 v_4$ of K_5 cross each other. We can suppose the vertices t_n, v_1, v_3, v_4 are located on the plane R^2 as in Figure 5(a), and the other three vertices of K_5 can be located arbitrarily around the vertices t_n, v_1, v_3, v_4. But as there is no crossing on the other 4 edges which are incident with vertex t_n, and the edges incident with v_1 of K_5 cannot cross the edges of T^n, these edges can be drawn as shown in Figure 5(a) and the edge $v_2 v_3$ cannot be drawn as the dotted line as shown in the Figure 5(a) or $\mathrm{cr}_\phi(T_n, E_0) \geq 2$ does not hold. As for the other edges we can draw them analogously, and the only difference is that they are connected by straight lines or by arcs, but they are isomorphic to each other. So the subdrawing ϕ_2 of

$\langle T_n \cup E_0 \rangle$ must be isomorphic to Figure 5(b).

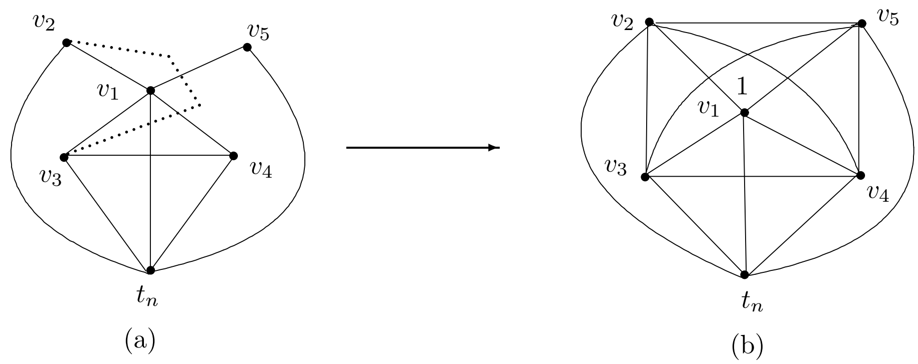

Figure 5 A good drawing ϕ_2 of $\langle T_n \cup E_0 \rangle$ when $\mathrm{cr}_\phi(T_n, E_0) = 1$

In Figure 5(b), if t_i $(1 \le i \le n-1)$ is located in the region marked with 1, then $\mathrm{cr}_\phi(T_i, E_0) \ge 4$, and with hypothesis $\mathrm{cr}_\phi(T_i, T_j) \ge 1$ $(1 \le i < j \le n)$, $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$; if t_i $(1 \le i \le n-1)$ is located in the other region, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$. So no matter which region the vertex t_i $(1 \le i \le n-1)$ is located in, we always have $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$. Also, $\mathrm{cr}_\phi(E_0 \cup T_n) = 4$ and $\bigcup_{i=1}^{n-1} T_i$ is isomorphic to $K_{5,n-1}$. By Lemma 1, we get

$$
\begin{aligned}
\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) \\
&= \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{i=1}^{n-1} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
&= \sum_{i=1}^{n-1} \mathrm{cr}_\phi(E_0 \cup T_n, T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
&\ge 5(n-1) + 4 + Z(5, n-1) \\
&\ge Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\end{aligned}
$$

Subcase 2.3 For any t_i $(1 \le i \le n)$, $\mathrm{cr}_\phi(T_i, E_0) \ge 2$, and there exists a vertex t_i $(1 \le i \le n)$ such that $\mathrm{cr}_\phi(T_i, E_0) = 2$.

Without loss of generality, suppose $\mathrm{cr}_\phi(T_n, E_0) = 2$. Let ϕ_3 of $\langle T_n \cup E_0 \rangle$ be induced by ϕ. By using the same method as in Subcase 2.2, we divide the subdrawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ into 3 different cases:

1) Two edges of T_n cross with one edge of E_0. In this case, there is only one drawing of ϕ_3, see Figure 6;

2) One edge of T_n crosses with two edges of E_0. In this case, there are three different drawings of ϕ_3, see Figure 7, Figure 8 and Figure 9; Figure 7 presents the case that one edge of T_n crosses with two adjacent edges of E_0; Figure 8 presents the case that one edge of T_n crosses with two unadjacent edges e_1 and e_2 of E_0, where e_1 and e_2 do not cross each other; Figure 9

presents the case that one edge of T_n crosses with two unadjacent edges e_1 and e_2 of E_0, where e_1 and e_2 cross each other.

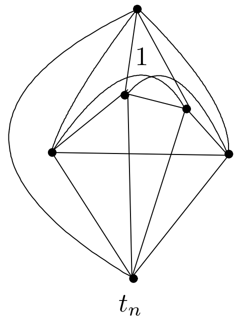

Figure 6

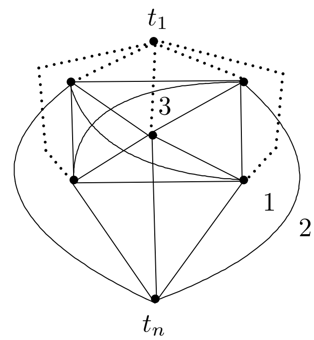

Figure 7

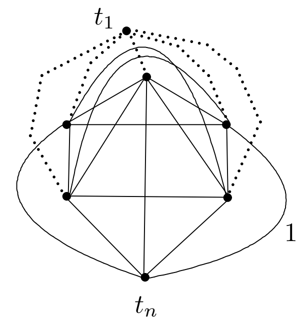

Figure 8

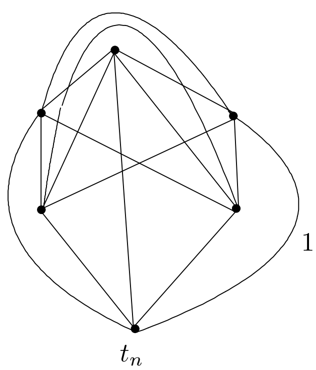

Figure 9

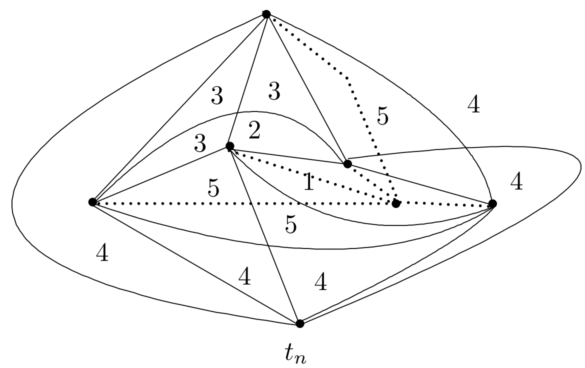

Figure 10

Five cases of good drawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ when $cr_\phi(T_n, E_0) = 2$

3) Two edges of T_n cross with two edges of E_0. In this case, there is only one drawing of ϕ_3, see Figure 10.

In the following we will discuss the different drawings of ϕ_3.

1) The subdrawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ is isomorphic to Figure 6. By Figure 6, we know, if t_i $(1 \le i \le n-1)$ is located in the region marked with 1, then $\mathrm{cr}_\phi(T_i, E_0) \ge 4$, and by hypothesis that $\mathrm{cr}_\phi(T_i, T_j) \ge 1$ $(1 \le i < j \le n)$, $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$; Also if t_i $(1 \le i \le n-1)$ is located in the other regions, it is easy to obtain that $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$. So no matter which region of Figure 6 the vertex t_i $(1 \le i \le n-1)$ is located in, there always holds $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \ge 5$. As $\mathrm{cr}_\phi(E_0 \cup T_n) = 5$ and $\bigcup_{i=1}^{n-1} T_i$ is isomorphic to complete bipartite graph $K_{5,n-1}$, we have by Lemma 1

$$\mathrm{cr}_\phi(H_n) = \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)$$

$$\begin{aligned}
&= \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{i=1}^{n-1} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
&= \sum_{i=1}^{n-1} \mathrm{cr}_\phi(E_0 \cup T_n, T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
&\geq 5(n-1) + 5 + Z(5, n-1) \\
&\geq Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\end{aligned}$$

2) The subdrawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ is isomorphic to Figure 7. First assume that $\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$. By Figure 7, if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 1, then we can also prove $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5$ with the hypothesis that $\mathrm{cr}_\phi(T_i, E_0) \geq 2$; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 2, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 4$, and only if $\mathrm{cr}_\phi(T_i, E_0) = 2$ and $\mathrm{cr}_\phi(T_i, T_n) = 2$, there holds $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4$, see t_1 in Figure 7; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 3, then $\mathrm{cr}_\phi(T_i, E_0) \geq 4$. By the hypothesis that $\mathrm{cr}_\phi(T_i, T_j) \geq 1$ $(1 \leq i, j \leq n; i \neq j)$, we have $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5$; if t_i $(1 \leq i \leq n-1)$ is located in the other regions, it is easy to know that $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5$.

Let $B_1 := \{t_i | \mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4, 1 \leq i \leq n-1\}$ and $B_2 := \{t_i | \mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5, 1 \leq i \leq n-1\}$. From the definition above, we know that if $t_i \in B_1$, then $\mathrm{cr}_\phi(T_i, E_0) = 2$, $\mathrm{cr}_\phi(T_i, T_n) = 2$ and $B_1 \cup B_2 = \{t_i, 1 \leq i \leq n-1\}$. Also $\mathrm{cr}_\phi(E_0 \cup T_n) = 5$, and $(\bigcup_{i=1}^{n-1} T_i)$ is isomorphic to complete bipartite graph $K_{5,n-1}$. So by Lemma 1, we have

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) =& \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) \\
=& \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in B_1} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in B_2} T_i) + \\
& \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
\geq& 4|B_1| + 5|B_2| + 5 + Z(5, n-1) \\
=& 5(n-1) - |B_1| + 5 + Z(5, n-1).
\end{aligned}$$

By the assumption at the beginning $\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$ and by a simple calculation, we have

$$|B_1| > Z(5, n-1) + 5(n-1) + 5 - (Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1) \geq \lfloor \frac{n}{2} \rfloor + 2. \tag{4}$$

So $|B_1| \neq \emptyset$. Let $t_1 \in B_1$. By the analogous discussion, for any t_i $(2 \leq i \leq n)$, $\mathrm{cr}_\phi(T_i, E_0 \cup T_1) \geq 4$. If $t_i \in B_1$ $(2 \leq i \leq n-1)$, then we have by observation, $\mathrm{cr}_\phi(T_i, E_0 \cup T_1) \geq 6$.

Let $B_3 := \{t_i | \mathrm{cr}_\phi(T_i, E_0 \cup T_1) = 4, 2 \leq i \leq n\}$.

Let $B_4 := \{t_i | \mathrm{cr}_\phi(T_i, E_0 \cup T_1) \geq 5, 2 \leq i \leq n, \}$.

Obviously, $B_1 \cap B_3 = \emptyset$, $B_3 \cup B_4 = \{t_i, 2 \leq i \leq n\}$ and $(B_1 \cup B_3) \subseteq \{t_i, 1 \leq i \leq n\}$, so

$$|B_1 \cup B_3| \leq n. \tag{5}$$

By discussing analogously to Equation (4), we also have

$$|B_3| \geq \lfloor \frac{n}{2} \rfloor + 2. \tag{6}$$

Combining (4) and (6), we get

$$|B_1 \cup B_3| = |B_1| + |B_3| \geq n + 3. \tag{7}$$

This contradicts with Equation (5). So $\mathrm{cr}_\phi(H_n) \geq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$.

3) The subdrawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ is isomorphic to Figure 8. By Figure 8, if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 1, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 4$, and only if $\mathrm{cr}_\phi(T_i, E_0) = 2$ and $\mathrm{cr}_\phi(T_i, T_n) = 2$, the equation $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4$ holds, just like the t_1 in Figure 8; if t_i $(1 \leq i \leq n-1)$ is located in the other regions, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5$. So by using an analogous method as in Figure 7, one can also obtain that $\mathrm{cr}_\phi(H_n) \geq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$. As for Figure 9, the discussion is the same as Figure 8.

4) The subdrawing ϕ_3 of $\langle T_n \cup E_0 \rangle$ is isomorphic to Figure 10. First assume that

$$\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1. \tag{8}$$

In Figure 10, if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 5, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 5$; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 4, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 6$; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 3, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 4$ and $\mathrm{cr}_\phi(T_i, E_0) \geq 3$; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 2, then $\mathrm{cr}_\phi(T_i, E_0) \geq 3$, and with the hypothesis $\mathrm{cr}_\phi(T_i, T_n) \geq 1$, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 4$; if t_i $(1 \leq i \leq n-1)$ is located in the region marked with 1, then $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) \geq 4$, and if $\mathrm{cr}_\phi(T_i, E_0) = 2$, we must have $\mathrm{cr}_\phi(T_i, T_n) = 2$ and $\mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4$.

Let $\Omega := \{t_i, 1 \leq i \leq n-1\}$.

Let C_1 and C_2 denote the sets of the vertices $t_i, 1 \leq i \leq n-1$, located in region 4 and in region 5, respectively.

Let $D = \Omega \backslash (C_1 \cup C_2)$.

Let $C_3 := \{t_i | \mathrm{cr}_\phi(T_i, E_0 \cup T_n) = 4, \mathrm{cr}_\phi(T_i, T_n) = 2, \mathrm{cr}_\phi(T_i, E_0) = 2, 1 \leq i \leq n-1\}$. Obviously, C_3 denotes the set of the vertices located in region 1 and has exact two crossings with edge set E_0 and T_n respectively, and the lines connecting the vertices of C_3 and the 5 vertices of K_5 are all like the dotted line as shown in Figure 10. But the vertex which is located in region 1 and does not belong to C_3 has at least 3 crossings with the edges set E_0.

By the observation, if $t_i \in C_1, t_j \in C_2, t_k, t_l \in C_3$, then

$$\mathrm{cr}_\phi(T_i, E_0 \cup T_k) \geq 4, \tag{9}$$

$$\mathrm{cr}_\phi(T_j, E_0 \cup T_k) \geq 5, \tag{10}$$

$$\mathrm{cr}_\phi(T_l, E_0 \cup T_k) \geq 6. \tag{11}$$

First we assume that $C_3 = \varnothing$. As $C_1 \cap C_2 = \varnothing$, $(\bigcup_{i=1}^{n-1} T_i)$ is isomorphic to complete bipartite graph $K_{5,n-1}$ and $\mathrm{cr}_\phi(E_0 \cup T_n) = 3$. So by Equation (1) and Lemma 1, we obtain

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) =& \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) \\
=& \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in (C_1 \cup C_2)} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in D} T_i) + \\
& \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
=& \sum_{t_i \in (C_1 \cup C_2)} \mathrm{cr}_\phi(E_0 \cup T_n, T_i) + \sum_{t_i \in D} \mathrm{cr}_\phi(E_0 \cup T_n, T_i) + \\
& \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
\geq& 5|C_1 \cup C_2| + 4|D| + 3 + Z(5, n-1) \\
=& 5(|C_1| + |C_2|) + 4(n - 1 - |C_1| - |C_2|) + 3 + Z(5, n-1) \\
=& 4(n-1) + (|C_1| + |C_2|) + 3 + Z(5, n-1).
\end{aligned}$$

Also with the hypothesis that $\mathrm{cr}_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$, we have

$$|C_1| + |C_2| \leq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) - 1 \leq \lfloor \frac{n}{2} \rfloor - 1. \tag{12}$$

Under the assumption $C_3 = \emptyset$, if $t_i \in D$, then $\mathrm{cr}_\phi(T_i, E_0) \geq 3$. Combining (1), (12) and Lemma 1, we get

$$\begin{aligned}
\mathrm{cr}_\phi(H_n) =& \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n} T_i) \\
=& \mathrm{cr}_\phi(E_0, \bigcup_{t_i \in (C_1 \cup C_2)} T_i) + \mathrm{cr}_\phi(E_0, \bigcup_{t_i \in D} T_i) + \\
& \mathrm{cr}_\phi(E_0, T_n) + \mathrm{cr}_\phi(E_0) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n} T_i) \\
\geq& 2(|C_1| + |C_2|) + 3(n - 1 - |C_1| - |C_2|) + 2 + 1 + Z(5,n) \\
=& 3(n-1) - (|C_1| + |C_2|) + 3 + Z(5,n) \\
\geq& 3(n-1) - (\lfloor \frac{n}{2} \rfloor - 1) + 3 + Z(5,n) \\
=& Z(5,n) + 2n + \lceil \frac{n}{2} \rceil + 1 \\
\geq& Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\end{aligned}$$

It is a contradiction with Equation (8). Then $C_3 \neq \emptyset$.

Let $t_1 \in C_3$, $C_3' := C_3 \backslash \{t_1\}$, $\Omega' := \{t_i, 2 \leq i \leq n\}$ and $D' = \Omega' \backslash (C_3' \cup C_2)$. By the definition of sets C_1, C_2, D, we have the following inequality:

$$\mathrm{cr}_\phi(H_n) = \mathrm{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)$$

$$
\begin{aligned}
&= \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in C_1} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in C_2} T^i) + \\
&\quad \mathrm{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in D} T_i) + \mathrm{cr}_\phi(E_0 \cup T_n) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \\
&\geq 6|C_1| + 5|C_2| + 4|D| + 3 + Z(5, n-1) \\
&= 6|C_1| + 5|C_2| + 4(n-1-|C_1|-|C_2|) + 3 + Z(5, n-1) \\
&= 4(n-1) + 2|C_1| + |C_2| + 3 + Z(5, n-1).
\end{aligned}
$$

By the assumption (8), we get:

When n is odd

$$2|C_1| + |C_2| < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) = \lfloor \frac{n}{2} \rfloor. \tag{13}$$

When n is even

$$2|C_1| + |C_2| < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) = \lfloor \frac{n}{2} \rfloor - 2. \tag{14}$$

Furthermore, by the definition of set C_2, C_3', D' and Equations (9), (10) and (11), we also have

$$
\begin{aligned}
\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(E_0 \cup T_1 \cup \bigcup_{i=2}^{n} T_i) \\
&= \mathrm{cr}_\phi(E_0 \cup T_1, \bigcup_{t_i \in C_3'} T_i) + \mathrm{cr}_\phi(E_0 \cup T_1, \bigcup_{t_i \in C_2} T_i) + \\
&\quad \mathrm{cr}_\phi(E_0 \cup T_1, \bigcup_{t_i \in D'} T_i) + \mathrm{cr}_\phi(E_0 \cup T_1) + \mathrm{cr}_\phi(\bigcup_{i=2}^{n} T_i) \\
&\geq 6|C_3'| + 5|C_2| + 4|D'| + 3 + Z(5, n-1) \\
&= 6|C_3'| + 5|C_2| + 4(n-1-|C_3'|-|C_2|) + 3 + Z(5, n-1) \\
&= 4(n-1) + 2|C_3'| + |C_2| + 3 + Z(5, n-1).
\end{aligned}
$$

By the assumption (8), there also holds the following:

When n is odd

$$2|C_3'| + |C_2| < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) = \lfloor \frac{n}{2} \rfloor. \tag{15}$$

When n is even

$$2|C_3'| + |C_2| < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) = \lfloor \frac{n}{2} \rfloor - 2. \tag{16}$$

So when n is odd, by adding (13) to (15), and then dividing 2, we get

$$|C_1| + |C_2| + |C_3'| \leq \lfloor \frac{n}{2} \rfloor - 1 = \lceil \frac{n}{2} \rceil - 2. \tag{17}$$

When n is even, by adding (14) to (16) and then dividing 2, we get

$$|C_1| + |C_2| + |C_3'| \leq \lfloor \frac{n}{2} \rfloor - 3 = \lceil \frac{n}{2} \rceil - 3. \tag{18}$$

Let $\Lambda = (C_1 \cup C_2 \cup C_3' \cup \{t_1, t_n\})$ and $\Gamma = \Omega \backslash (C_1 \cup C_2 \cup C_3\})$. By the definition of the sets above, it is easy to know: if $t_i \in \Lambda$, then $\mathrm{cr}_\phi(E_0, t_i) \geq 2$; if $t_i \in \Gamma$, then $\mathrm{cr}_\phi(E_0, t_i) \geq 3$. So by Lemma

1, we get

$$\begin{aligned}\mathrm{cr}_\phi(H_n) =&\mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n} T_i)\\ =&\mathrm{cr}_\phi(E_0, \bigcup_{t_i\in\Lambda} T_i) + \mathrm{cr}_\phi(E_0, \bigcup_{t_i\in\Gamma} T_i) + \mathrm{cr}_\phi(E_0) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n} T_i)\\ \geq&2(|C_1| + |C_2| + |C_3'| + 2) + 3(n-2-|C_1|-|C_2|-|C_3'|) + 1 + Z(5,n)\\ =&3n - (|C_1| + |C_2| + |C_3'|) + Z(5,n) - 1.\end{aligned}$$

Combining the assumption (8), we also have

$$|C_1| + |C_2| + |C_3'| > Z(5,n) + 3n - 1 - (Z(5,n) + 2n + \lfloor\frac{n}{2}\rfloor + 1) = \lceil\frac{n}{2}\rceil - 2. \tag{19}$$

But they are contradicting with Equations (17) and (18). So we have $\mathrm{cr}_\phi(H_n) \geq Z(5,n) + 2n + \lfloor\frac{n}{2}\rfloor + 1$.

Subcase 2.4 For any t_i $(1 \leq i \leq n)$, $\mathrm{cr}_\phi(T_i, E_0) \geq 3$.

By Lemma 1 and Equation (1), it is easy to get

$$\begin{aligned}\mathrm{cr}_\phi(H_n) = \mathrm{cr}_\phi(E_0 \cup \bigcup_{i=1}^{n} T_i) &= \mathrm{cr}_\phi(E_0, \bigcup_{i=1}^{n} T_i) + \mathrm{cr}_\phi(E_0) + \mathrm{cr}_\phi(\bigcup_{i=1}^{n} T_i)\\ &\geq 3n + 1 + Z(5,n).\end{aligned}$$

Now the proof is completed. □

Let H be a graph isomorphic to K_5. Consider a graph G_H obtained by joining all vertices of H to five vertices of a 3-edge connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained by contracting all the edges of H to a vertex h.

Lemma 5 $\mathrm{cr}(G_H^*) \leq \mathrm{cr}(G_H) - 3$.

Proof Let ψ be the optimal drawing of G_H. By joining all the 5 vertices of H to a vertex z, we will obtain a graph isomorphic to K_6. As $\mathrm{cr}(K_6) \geq 3$, in ψ there are at least 3 crossings on the edges of H. Let edge set $E_2 := \{v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5\}$, see Figure 11 and Figure 12. Then we divide into 2 cases by number of crossings on the edge set E_2:

Case 1 On ψ, there are at least 3 crossings on E_2. As H includes a subgraph S_4 with the vertex v_1 as a 4-vertex, $G_H \backslash E_2$ is isomorphic to G_H^*, and $\mathrm{cr}(G_H^*) = \mathrm{cr}(G_H\backslash E_2) \leq \mathrm{cr}_\psi(G_H\backslash E_2) \leq \mathrm{cr}_\psi(G_H) - 3 = \mathrm{cr}(G_H) - 3$, see Figure 11.

Case 2 On ψ, there are at most 2 crossings on E_2. Then by the parity of crossing numbers[14], $\mathrm{cr}_\psi(H) = 1$, and there is only one subdrawing of graph H under ψ (see Figure 12). So there are at most one crossing on the edge set $\{v_2v_3, v_3v_4, v_4v_5, v_5v_2\}$ (see Figure 12), say, on the edge v_4v_5. On the edges incident with vertex v_1, contracting along the edge, which has the minimum crossings, to the vertex h as shown in Figure 12 will also decrease at least 3 crossings. In Figure

12, we assume the edge v_1v_3 has the minimum crossing. □

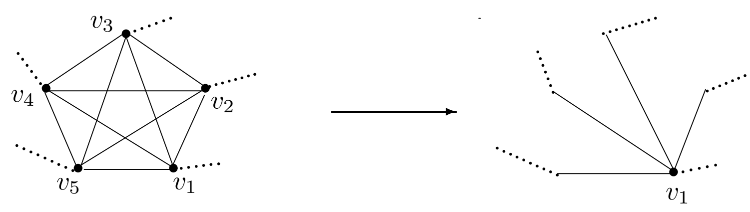

Figure 11 $G_H \backslash E_2$ isomorphic to G_H^* when there are at least 3 crossings on E_2

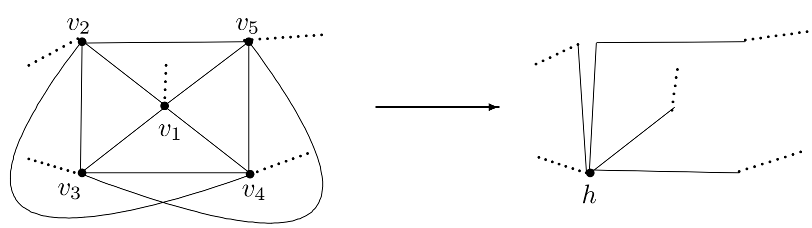

Figure 12 The contraction when here are at most 2 crossings on E_2

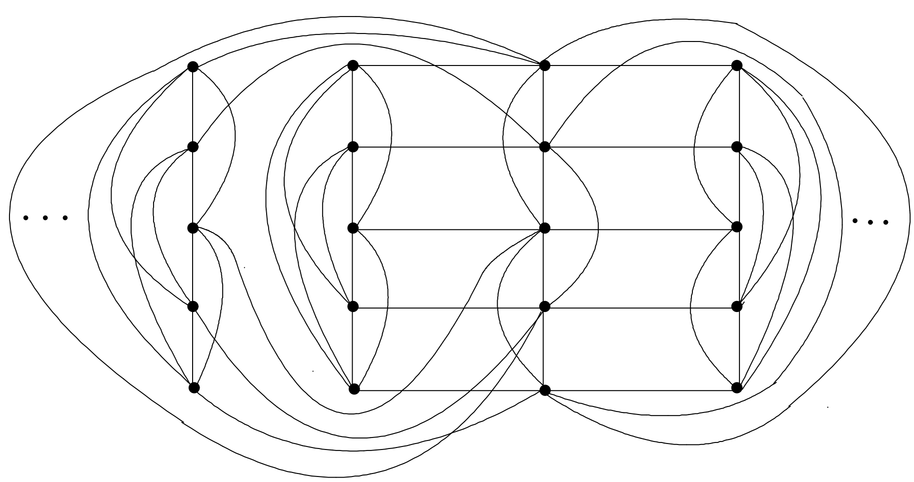

Figure 13 An optimal drawing of $K_5 \times S_n$

Theorem 2 $\mathrm{cr}(K_5 \times S_n) = Z(5,n) + 5n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1.$

Proof In Figure 13, by contracting each copy K_5^i $(i = 1, 2 \ldots, n)$ to a vertex t_i, we will obtain a graph which is isomorphic to H_n, and the drawing is just an optimal drawing of H_n. Through the contracting, each copy K_5^i $(i = 1, 2, \ldots, n)$ decreases exactly 3 crossings, so $\mathrm{cr}(K_5 \times S_n) \leq Z(5,n) + 5n + \lfloor \frac{n}{2} \rfloor + 1$. In the following, we will prove the opposite inequality holds. Let φ be the optimal drawing of $K_5 \times S_n$. On φ, contracting each K_5^i to a vertex t_i $(i = 1, 2 \ldots, n)$ yields a graph isomorphic to H_n. According to Theorem 1 and by using Lemma 5 repeatedly, we have

$$\begin{aligned}\mathrm{cr}(K_5 \times S_n) &= \mathrm{cr}_\varphi(K_5 \times S_n) \geq \mathrm{cr}(H_n) + 3n \\ &= Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 + 3n = Z(5, n) + 5n + \lfloor \frac{n}{2} \rfloor + 1.\end{aligned}$$

The proof is completed. □

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