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Nonabsolute Fuzzy Integrals, Absolute Integrability and Its Absolute Value Inequality

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Abstract Absolute integrability and its absolute value inequality for fuzzy-number-valued functions are worth to be considered. In this paper, absolute integrability and its absolute value inequality for fuzzy-number-valued functions are discussed by means of the characteristic theorems of nonabsolute fuzzy integrals and the embedding theorem, i.e., the fuzzy number space can be embedded into a concrete Banach space. Several necessary and sufficient conditions and examples are given.

Keywords fuzzy numbers; fuzzy-number-valued functions; absolute integrability.

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Since the concept of fuzzy set was originally introduced by Zadeh in 1965, the fuzzy mathematics has developed rapidly as a new mathematic branch. For fuzzy analysis, in order to calculate the expectation of fuzzy random variables and to meet the need of solving fuzzy differential equations, the continuity, differentiability, integrability of fuzzy-number-valued functions and the relations between them have been investigated. In 1986, in order to study the expectation of fuzzy random variables, Puri and Ralescu[1] proposed the integral of fuzzy-number-valued functions as an extension of Aumann integral. In 1987, Kaleva[2] used this integral to discuss Cauchy problem of fuzzy differential equations and pointed out that a continuous fuzzy-number-valued function is integrable and its integral primitive is differentiable everywhere. Furthermore the derivative of the primitives is equal to its integrand function. In 1992, Wu and Ma[3] characterized differentiability and integrability of fuzzy-number-valued functions by using both embedding theorem from fuzzy-number space to concrete Banach space, and Bochner and Pettis integrals of Banach-valued functions. Subsequently, Wu and Gong[4−8] discussed the nonabsolute integrals of fuzzy-number-valued functions and found several results different from the traditional real analysis. For instance, differentiability, absolute integrability, and so on.

For absolute-valued inequality, there are two different forms based on different problems. One is in the sense of fuzzy absolute-value and another is in the sense of the metric (or the norm) as

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follows.

$$|\int_a^b \tilde{f}| \le \int_a^b |\tilde{f}|,$$

$$\|\int_a^b \tilde{f}\| \le \int_a^b \|\tilde{f}\|.$$

To make our analysis possible, first we will recall some basic results of fuzzy numbers and fuzzy calculus. Section 2 is devoted to discussing the absolute integrability and its absolute value inequality of fuzzy Aumann-Henstock integral. In Sections 3 and 4 we shall focus our attentions on the absolute integrability and its absolute-value inequality of the fuzzy Henstock integral and strongly fuzzy Henstock integral, respectively.

1. Preliminaries

Let $F(R)$ be a fuzzy set on R. For $\tilde{A} \in F(R)$, if $\tilde{A}$ is normal, convex, upper semi-continuous and the support $[A]_0 = \overline{\{x \in R : A(x) > 0\}}$ is compact, then $\tilde{A}$ is called a fuzzy number. Denote E^1 as fuzzy number space[3−5].

For $\tilde{A}, \tilde{B} \in E^1$, $k \in R$, $\tilde{A} + \tilde{B} = \tilde{C}$ is defined by $A_\lambda + B_\lambda = C_\lambda, \lambda \in [0,1]$, i.e., for any $\lambda \in [0,1]$, $A_\lambda^+ + B_\lambda^+ = C_\lambda^+, A_\lambda^- + B_\lambda^- = C_\lambda^-$. $[kA]_\lambda = kA_\lambda, \lambda \in [0,1]$, here $A_\lambda = \{x | A(x) \ge \lambda\}$. A_λ is a closed interval[3,4,5], and denoted as $[A_\lambda^-, A_\lambda^+]$.

Define $D(\tilde{A}, \tilde{B}) = \sup_{\lambda \in [0,1]} \max(|A_\lambda^- - B_\lambda^-|, |A_\lambda^+ - B_\lambda^+|)$ as the distance[3−5] between fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Based on the background of various problems, two definitions of integration for the fuzzy-number-valued functions are proposed as follows: one is an extension of Aumann integral of the set-valued functions, such as Kaleva integral[2] which was defined firstly by Puri and Ralescu[1] in 1986 to meet the need of calculating the expectations of fuzzy random variables, and successfully applied to discuss fuzzy differential equations, and fuzzy Henstock integral which as a nonabsolute integral is an extension of Kaleva integral defined by Wu and Gong[4]. Another integral is Riemann-type by first taking the sum and then the limit which is known as a constructive definition, e.g., Riemann type integral[10] which was defined by Goetschel and Voxman in 1986, and the integral defined by Nanada[11] by means of obtaining its upper and lower sums. However, these integrals mentioned above are all the special cases of the fuzzy Henstock integral defined by Wu and Gong[5] in 2001. Therefore, we first introduce these two fuzzy Henstock integrals.

Let $\delta : [a,b] \to R^+$. A division $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine division (δ-fine (M) division), if the following conditions are satisfied:

(1) $a = x_0 < x_1 < \cdots < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ $([x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))), i = 1, 2, \ldots, n$.

Definition 1.1[5] *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-valued function. $\tilde{f}$ is said to be fuzzy Henstock integrable on $[a,b]$ and integral value is $\tilde{A} \in E^1$ if for every $\varepsilon > 0$, there is a function $\delta(x) > 0$*

such that for any δ-fine division $P = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$D(\sum_i \tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{A}) < \varepsilon.$$

As usual, we write $(FH)\int_a^b \tilde{f} = \tilde{A}$ and $\tilde{f} \in FH[a, b]$.

Remark 1.1 If the fuzzy-number-valued function $\tilde{f}$ and the fuzzy number $\tilde{A}$ in Definition 1.1 are replaced by a real-valued function f and a real number A, respectively, then the real-valued function f is said to be Henstock integral on $[a, b]$, and we write $f \in H[a, b]$.

Henstock integral, as a Riemann-type integral, is a nonabsolute integral which equals to the Perron and the Denjoy integrals, and includes Riemann, improper Riemann, Lebesgue and Newton integrals[12].

Remark 1.2 A fuzzy-number-valued function $\tilde{f} \in FH[a, b]$ if and only if f_λ^-, f_λ^+ are Henstock integrable uniformly for $\lambda \in [0, 1]$, i.e., δ is independent of $\lambda \in [0, 1]$[5].

Remark 1.3 If the fuzzy-number-valued function $\tilde{f}$, the fuzzy number $\tilde{A}$ and the $\delta-$fine division in Definition 1.1 are replaced by a real-valued function f, a real number A and a δ-fine (M) division respectively, then the real-valued function f is said to be McShane integral on $[a, b]$, and we write $f \in M[a, b]$. McShane integral is equivalent to Lebesgue integral, but its definition is Riemann-type[12].

Remark 1.4 If the δ-fine division in Definition 1.1 is replaced by a δ-fine (M) division, then the fuzzy-number-valued function f is said to be fuzzy McShane integral on $[a, b]$, and we write $f \in FM[a, b]$. From [13], $\tilde{f} \in FM[a, b]$ if and only if f_λ^-, f_λ^+ are McShane integrable uniformly for $\lambda \in [0, 1]$.

Definition 1.2[4] *A fuzzy-number-valued function $\tilde{f} : [a, b] \to E^1$ is said to be I-bounded if there exist two integrable functions $h, g \in H[a, b]$, such that for any $s(x) \in [\tilde{f}(x)]_0$ we have $g(x) \le s(x) \le h(x)$.*

An I-bounded function $\tilde{f} : [a, b] \to E^1$ is said to be Aumann-Henstock integrable over $[a, b]$ if

$$\{(H)\int_a^b s, \quad s(x) \text{ is a Henstock integrable selection for } [\tilde{f}(x)]_\lambda \ \}(\lambda \in [0, 1])$$

determines a unique fuzzy number A, denoted as $(FAH)\int_a^b \tilde{f} = \tilde{A}$ or $\tilde{f} \in FAH[a, b]$.

Remark 1.5[4] A measurable fuzzy-number-valued function $\tilde{f} \in FAH[a, b]$ if and only if $f_\lambda^-, f_\lambda^+ \in H[a, b]$ for any $\lambda \in [0, 1]$.

Remark 1.6 If the inequality $g(x) \le s(x) \le h(x)$ and Henstock integrable selection in Definition 1.2 are replaced by $s(x) \le |h(x)|$ and Lebesgue integrable selection respectively, then fuzzy-number-valued function $\tilde{f}$ is said to be Kaleva integrable on $[a, b]$. We write $\tilde{f} \in K[a, b]$. And $\tilde{f} \in K[a, b]$ if and only if $f_\lambda^-, f_\lambda^+ \in L[a, b]$ for any $\lambda \in [0, 1]$, where $L[a, b]$ denotes the Lebesgue integrable functions space on $[a, b]$. From [4], we can also find that the fuzzy Aumann-Henstock integral is an extension of the Kaleva integral.

2. The absolute integrability of fuzzy Aumann-Henstock integral

Lemma 2.1[7] *If $\tilde{A} \in E^1$, then the class of closed intervals $\{\{|r| : r \in A_\lambda\} : \lambda \in [0,1]\}$ determines a unique fuzzy number, and this fuzzy number is called the absolute value of $\tilde{A}$, denoted as $|\tilde{A}|$. Furthermore, $|\tilde{A}|_\lambda = [|\tilde{A}|^-_\lambda, |\tilde{A}|^+_\lambda]$, where*

$$|\tilde{A}|^-_\lambda = \frac{1}{2}\max\{|A^-_\lambda| + A^-_\lambda, |A^+_\lambda| - A^+_\lambda\},$$

$$|\tilde{A}|^+_\lambda = \max\{|A^-_\lambda|, |A^+_\lambda|\}.$$

Lemma 2.2[7] *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function. Then $\tilde{f} \in K[a,b]$ if and only if $|\tilde{f}| \in FAH[a,b]$, and*

$$|(FAH)\int_a^b \tilde{f}| \leq (K)\int_a^b |\tilde{f}|.$$

Theorem 2.1 *Let $\tilde{f} \in FAH[a,b]$. Then $|\tilde{f}| \in FAH[a,b]$ if and only if*

$$\|\tilde{f}\|_{E^1} = D(\tilde{f}, \tilde{0}) = \sup_{\lambda\in[0,1]} \max\{|f^-_\lambda|, |f^+_\lambda|\}$$

is Lebesgue integrable on $[a,b]$ (here $\|\cdot\|_{E^1} = D(\cdot, \tilde{0})$ does not stand for the norm of E^1. For brevity, we denote it as $\|\cdot\|$) and

$$\|(FAH)\int_a^b \tilde{f}\| \leq (L)\int_a^b \|\tilde{f}\|.$$

Proof Since $|\tilde{f}| \in FAH[a,b]$, by Lemma 2.2, $\tilde{f} \in K[a,b]$, i.e., $f^-_\lambda, f^+_\lambda \in L[a,b]$ for any $\lambda \in [0,1]$. Notice that

$$\begin{aligned}\|\tilde{f}\|_{E^1} &= \sup_{\lambda\in[0,1]} \max\{|f^-_\lambda|, |f^+_\lambda|\} = \sup_{\lambda_n\in[0,1]} \max\{|f^-_{\lambda_n}|, |f^+_{\lambda_n}|\}\\ &\leq h(x),\end{aligned}$$

where $\{\lambda_n\}$ is the set of all rational numbers on $[0,1]$. From the measurability of $\|\tilde{f}\|_{E^1}$, we have $\|\tilde{f}\|_{E^1} \in L[a,b]$.

Conversely, if $\|\tilde{f}\|_{E^1} \in L[a,b]$, then $|f^-_\lambda|, |f^+_\lambda| \in L[a,b]$ for any $\lambda \in [0,1]$. Thus $|\tilde{f}|^-_\lambda, |\tilde{f}|^+_\lambda \in L[a,b]$, i.e., $|\tilde{f}| \in K[a,b] \subset FAH[a,b]$, and

$$\begin{aligned}\|(FAH)\int_a^b \tilde{f}\| &= D((FAH)\int_a^b \tilde{f}, \tilde{0})\\ &= \sup_{\lambda\in[0,1]} \max\{|(H)\int_a^b f^-_\lambda|, |(H)\int_a^b f^+_\lambda|\}\\ &\leq \sup_{\lambda\in[0,1]} \max\{(L)\int_a^b |f^-_\lambda|, (L)\int_a^b |f^+_\lambda|\}\\ &\leq (L)\int_a^b D(\tilde{f}, \tilde{0}) = (L)\int_a^b \|\tilde{f}\|.\end{aligned}$$

We can find that Theorem 2.1 is easier to apply compared with Lemma 2.2.

3. Absolute integrability of the fuzzy Henstock integral

Let $f:[a,b]\to X$ be a Banach-valued function, X a Banach space, and X^* the conjugate space of X. Then f is called Henstock integrable on $[a,b]$[14] (McShane integrable[15]) and its integral value is $A\in X$ if for every $\varepsilon>0$, there exists a function $\delta(x)>0$ such that for any δ-fine division (δ-fine (M)division) $P=\{[x_{i-1},x_i];\xi_i\}$, we have

$$\|\sum_i f(\xi_i)(x_i-x_{i-1})-A\|<\varepsilon.$$

We write $(VH)\int_a^b f=A((VM)\int_a^b f=A)$ or $f\in VH[a,b](f\in VM[a,b])$.

f is said to be Pettis integrable on $[a,b]$ and integral value is A if x^*f is Lebesgue integrable for any $x^*\in X^*$ and there is an $A\in X$ such that $(L)\int_E x^*f=x^*A$ for any measurable set $E\in[a,b]$. We write $(P)\int_a^b f=A$.

Lamma 3.1[16] *Let $f:[a,b]\to X$ be a Banach-valued function. Then f is McShane integrable on $[a,b]$ if and only if f is Henstock and Pettis integrable on $[a,b]$.*

Lemma 3.2[3] *Let $\tilde{A}\in E^1$ and define $j(\tilde{A})=(A^-,A^+)$. Then $j(E^1)$ is a closed convex cone with vertex θ in $\overline{C}[0,1]\times\overline{C}[0,1]$ (Here $\overline{C}[0,1]$ stands for the class of all real-valued bounded functions f on $[0,1]$ such that f is left continuous for any $x\in(0,1]$ and f has a right limit for any $x\in[0,1)$, especially f is right continuous at $x=0$, and with the norm $\|f\|=\sup_{x\in[a,b]}|f(x)|$, $\overline{C}[0,1]$ is a Banach space. $\overline{C}[0,1]\times\overline{C}[0,1]$ with the norm $\|(\cdot,\cdot)\|=\max(\|\cdot\|,\|\cdot\|$ is a Banach space), and $j(E^1)\to\overline{C}[0,1]\times\overline{C}[0,1]$ satisfies*

(1) $j(s\tilde{A}+t\tilde{B})=sj(\tilde{A})+tj(\tilde{B})$ for all $\tilde{A},\tilde{B}\in E^1, s\geq 0, t\geq 0$.

(2) $D(\tilde{A},\tilde{B})=\|j(\tilde{A})-j(\tilde{B})\|$ for all $\tilde{A},\tilde{B}\in E^1$,

i.e., j embeds E^1 into $\overline{C}[0,1]\times\overline{C}[0,1]$ isometrically and isomorphically.

Lemma 3.3[5] *Let $\tilde{f}$ be fuzzy-number-valued function. Then $\tilde{f}\in FH[a,b]$ ($\tilde{f}\in FM[a,b]$) if and only if $j(\tilde{f})=(f^-,f^+)\in VH[a,b]$ ($j(\tilde{f})=(f^-,f^+)\in VM[a,b]$), and*

$$j((FH)\int_a^b \tilde{f})=(VH)\int_a^b j(\tilde{f}),$$

$$(j((FM)\int_a^b \tilde{f})=(VM)\int_a^b j(\tilde{f}).$$

Theorem 3.1 *Let $\tilde{f}:[a,b]\to E^1$ be a fuzzy-number-valued function, and $\tilde{f}\in FH[a,b]$. Then $\tilde{f}\in FM[a,b]$ if and only if $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable, and*

$$\|(FH)\int_a^b \tilde{f}\|\leq (L)\int_a^b \|\tilde{f}\|.$$

Proof Only if: For $\tilde{f}\in FM[a,b]$, by Remark 1.4, we infer that f_λ^- and f_λ^+ are McShane integrable uniformly for $\lambda\in[0,1]$. On the other hand, since

$$f_0^-(x)\leq f_\lambda^-(x)\leq f_\lambda^+(x)\leq f_0^+(x),$$

we have

$$\|\tilde{f}\|_{E^1}=\sup_{\lambda\in[0,1]}\max\{|f_\lambda^-|,|f_\lambda^+|\}=\sup_{\lambda_n\in[0,1]}\max\{|f_{\lambda_n}^-|,|f_{\lambda_n}^+|\}$$

$$\leq \max\{|f_0^-(x)|, |f_0^+(x)|\},$$

where $\{\lambda_n\}$ is the set of all rational numbers on $[0,1]$. By the measurability of $\|\tilde{f}\|_{E^1}$, we can obtain the Lebesgue integrability of $|f_0^-|, |f_0^+|$. So $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable on $[a,b]$.

If: Since $\tilde{f} \in FH[a,b]$, we denote $j(\tilde{f}) = (f^-, f^+)$, by Lemma 3.3, we can infer that $j(\tilde{f}) \in VH[a,b]$. Here we only discuss the case of f^-. For any $x^* \in \mathcal{B}(\mathcal{X}^*)$, real-valued function $x^* f^-$ is measurable. Since $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable, and note that $x^* \in \mathcal{B}(\mathcal{X}^*)$, we have

$$\int_E |x^* f^-| \leq \int_E \|f^-\| \leq \int_E \|\tilde{f}\|_{E^1} < \infty,$$

for every measurable set E. Thus, f^- is Dunford integrable.

Let $v(E)$ be Dunford integral value. For any subinterval $[c,d] \subset [a,b]$, by the Henstock integrability of f^-, we have $v([c,d]) \in X$. Besides, we can prove that $v(E) \in X$ for any measurable set E. In fact, given $\varepsilon > 0$, by the Lebesgue integrability of $\|f^-\|$ and $\|\tilde{f}\|_{E^1}$, there exists a $\eta > 0$, such that $\int_E \|f^-\| \leq \int_E \|\tilde{f}\|_{E^1} < \varepsilon$ when $\mu(E) < \eta$. Hence,

$$\|v(E)\| = \sup_{x^* \in \mathcal{B}(X^*)} |(L) \int_E x^* f^-| \leq \sup_{x^* \in \mathcal{B}(X^*)} (L) \int_E |x^* f^-| \leq (L) \int_E \|f^-\| \leq (L) \int_E \|\tilde{f}\|_{E^1} < \varepsilon,$$

when $\mu(E) < \eta$. Furthermore, by the Proposition $2B$ in [17], we have $v(E) \in X$. Therefore, f^- is Pettis integrable. Finally, by Lemma 3.1, we prove that f^- is McShane integrable on $[a,b]$.

Similarly, f^+ is McShane integrable on $[a,b]$. By Lemma 3.3, $\tilde{f} \in FM[a,b]$. Similar to the proof of Theorem 2.1, we have

$$\|(FH) \int_a^b \tilde{f}\| \leq (L) \int_a^b \|\tilde{f}\|.$$

Theorem 3.2 *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function, and $\tilde{f} \in FH[a,b]$. Then $\tilde{f}$ is absolute Henstock integrable (i.e., $|\tilde{f}| \in FH[a,b]$) if and only if $\tilde{f} \in FM[a,b]$, and*

$$|(FH) \int_a^b \tilde{f}| \leq (M) \int_a^b |\tilde{f}|.$$

Proof Sufficiency. Since $\tilde{f} \in FM[a,b]$, $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable. For any $x^* \in \mathcal{B}(X^*)$, real-valued function $x^* j(|\tilde{f}|)$ is measurable. Define

$$j(|\tilde{f}|) = (|\tilde{f}|^-, |\tilde{f}|^+) = (\frac{1}{2} \max\{|f^-| + f^-, |f^+| - f^+\}, \max\{|f_\lambda^-|, |f_\lambda^+|\}).$$

Since $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable, and note that for any $x^* \in \mathcal{B}(X^*)$, we have

$$\int_E |x^* |\tilde{f}|^+| \leq \int_E \||\tilde{f}|^+\| \leq \int_E \|\tilde{f}\|_{E^1} < \infty,$$

$$\int_E |x^* |\tilde{f}|^-| \leq \int_E \||\tilde{f}|^-\| \leq \int_E \|\tilde{f}\|_{E^1} < \infty,$$

for any measurable set E.

Similar to the sufficiency proof of Theorem 3.1, we can obtain that $|\tilde{f}|^-$ and $|\tilde{f}|^+$ are McShane integrable on $[a,b]$. Hence, $|\tilde{f}|^-$ and $|\tilde{f}|^+$ are Henstock integrable on $[a,b]$. Then, by Lemma 3.3, we prove that $\tilde{f}$ is absolute Henstock integrable.

Necessity. Since $|\tilde{f}| \in FH[a,b]$, $|\tilde{f}|^+_\lambda = \max\{|f^-_\lambda|, |f^+_\lambda|\}$ is Henstock integrable on $[a,b]$ uniformly for $\lambda \in [0,1]$. In addition,

$$\|\tilde{f}\|_{E^1} = \sup_{\lambda\in[0,1]} \max\{|f^-_\lambda|, |f^+_\lambda|\} = \sup_{\lambda_n\in[0,1]} \max\{|f^-_{\lambda_n}|, |f^+_{\lambda_n}|\} \le \max\{|f^-_0|, |f^+_0|\},$$

where $\{\lambda_n\}$ is the set of all rational numbers on $[0,1]$. By the Henstock integrability of $\max\{|f^-_0|, |f^+_0|\}$ (it is equivalent to Lebesgue integrability when it is nonnegative) and the measurability of $\|\tilde{f}\|_{E^1}$, we prove that $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable. By Theorem 3.1, $\tilde{f} \in FM[a,b]$.

By Theorem 2.1, absolute value inequality holds.

By the necessity of the proof of Theorem 2.1, we can obtain the following corollary.

Corollary 3.1 *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy -number-valued function. Then $|\tilde{f}| \in FM[a,b]$ if and only if $\tilde{f} \in FM[a,b]$.*

4. Absolute integrability of the fuzzy strong Henstock integral

The fuzzy Henstock integral is introduced based on solving the non-continuous fuzzy system and completing the theory of fuzzy calculus. For fuzzy Henstock integral, however the integral primitive is not differentiable almost everywhere[8] no matter whether the definition is given by means of Aumann's method or Riemann-type definition by first taking the sum and then the limit, and even for Kaleva integral. Therefore, in order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, we have defined strong fuzzy Henstock integral[8]. For a real function, strong Henstock integral and Henstock integral are equivalent in real analysis, but as far as fuzzy-number-valued function is concerned, they are not equivalent[8]. In this section, we shall discuss the absolute integrability of the fuzzy strong Henstock integral.

Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function. Then $\tilde{f}$ is said to be strongly Henstock integrable on $[a,b]$ if there exists interval additive fuzzy-number-valued function and for every $\varepsilon > 0$, there is a $\delta(x) > 0$ such that for any $\delta-$fine division $P = \{[x_{i-1}, x_i]; \xi_i\}$, we have

$$\sum_i D(\tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{F}([x_{i-1}, x_i])) < \varepsilon,$$

where $\tilde{F}([s,t]) = \tilde{F}(t) - \tilde{F}(s)$ is H-difference. The integral is written as $(SFH)\int_a^b \tilde{f} = \tilde{F}(b) - \tilde{F}(a)$ or $\tilde{f} \in SFH[a,b]$.[8]

Remark 4.1 If the δ-fine division in above definition is replaced by δ-fine (M) division, then $\tilde{f}$ is called fuzzy strongly McShane integrable on $[a,b]$, and we write $\tilde{f} \in SFM[a,b]$.

Remark 4.2[15] If in Remark 4.1, the fuzzy-number-valued function $\tilde{f}$ and $\tilde{F}$ are replaced by Banach-valued function f and F respectively, and the distance D is replaced by the norm of difference, then function f is said to be strongly McShane integrable on $[a,b]$.

Lemma 4.1[15] *Let $f : [a,b] \to X$ be a Banach-valued function. Then f is strongly McShane integrable on $[a,b]$ if and only if f is Bochner integrable on $[a,b]$, i.e., f is strongly measurable* *on $[a,b]$ and*

$$\int_a^b \|f\| < \infty.$$

Lemma 4.2 *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function. Then $\tilde{f}$ is strongly fuzzy McShane integrable on $[a,b]$ if and only if $j(\tilde{f})$ is strongly McShane integrable.*

By Lemma 3.2, the proof is the same as [5].

Theorem 4.1 *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-numbed-valued function, and $\tilde{f} \in SFH[a,b]$. Then $\tilde{f} \in SFM[a,b]$ if and only if $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable, and*

$$\|(SFH)\int_a^b \tilde{f}\| \leq (L)\int_a^b \|\tilde{f}\|.$$

Proof Since $\tilde{f} \in SFM[a,b] \subset FM[a,b]$, by Theorem 3.1, $\|\tilde{f}\|_{E^1}$ is Lebesgue integrable.

Conversely, notice that

$$\int_a^b \|j(\tilde{f})\| = \int_a^b \|\tilde{f}\|_{E^1} < \infty,$$

by the strong measurability of $j(\tilde{f})$, we have $j(\tilde{f})$ is Bochner integrable. In addition, by Lemma 4.1, $j(\tilde{f})$ is strongly McShane integrable.
Hence, $\tilde{f}$ is fuzzy strongly McShane integrable, i.e., $\tilde{f} \in SFM[a,b]$.

Corollary 4.1 *Let $\tilde{f} : [a,b] \to E^1$ be a fuzzy-number-valued function. Then $\tilde{f} \in SFM[a,b]$ if and only if $\tilde{f} \in FM[a,b]$.*

As is well known, for a real-valued function, the Henstock integral is equivalent to the strong Henstock integral and the McShane integral is equivalent to the strong McShane integral. However, they are not equivalent for Banach-valued functions. For the fuzzy-number-valued function, by the example in [4], we have found that the fuzzy Henstock integral is unequal to the fuzzy strong Henstock integral. However, by Corollary 4.1, we can find that the fuzzy McShane integral is equal to the fuzzy strong McShane integral. Furthermore, the following example shows that the fuzzy strong Henstock integral is unequal to the fuzzy strong McShane integral.

Example 4.1 Let $\{\tilde{A}_n\}, n = 1, 2, 3, \ldots$, be a series of fuzzy numbers, $\|\sum_{n=1}^{\infty} \tilde{A}_n\|$ be convergent, and $\sum_{n=1}^{\infty} \|\tilde{A}_n\|$ be divergent. Define $\tilde{f} : [0,1] \to E^1$ as follows.

$$\tilde{f}(x) = \begin{cases} 2^n \tilde{A}_n, & x \in (2^{-n}, 2^{-n+1}), n = 1, 2, 3, \ldots, \\ \tilde{0}, & x \in [0,1] \backslash (2^{-n}, 2^{-n+1}). \end{cases}$$

Obviously, $\int_0^1 \|\tilde{f}\|_{E^1} = \sum_{n=1}^{\infty} \|\tilde{A}_n\|_{E^1} = \infty$. By Theorem 4.1, we infer that $\tilde{f}$ is not fuzzy strongly McShane integrable, but we can prove that $\tilde{f}(x)$ is fuzzy strongly Henstock integrable.

In fact, define $\tilde{F} : [0,1] \to E^1$ as follows.

$$\tilde{F}(x) = \begin{cases} 2^n(x - 2^{-n})\tilde{A}_n + \sum\limits_{k=n+1}^{\infty} A_k, & x \in (2^{-n}, 2^{-n+1}], n = 1, 2, 3, \ldots, \\ \tilde{0}, & x = 0. \end{cases}$$

Given $0 < \varepsilon < 1$ and N, such that when $n > N$, $\| \sum_{k=n}^{\infty} \tilde{A}_n \| < \frac{\varepsilon}{5}$, $\|\tilde{A}_n\| < \frac{\varepsilon}{5}$, and there exists $M > 1$, such that $\|\tilde{A}_n\| < \frac{\varepsilon}{5}$, for any n.

Define $\delta : [0,1] \to R^+$ as follows

$$\delta(x) = \begin{cases} \min(x - 2^{-n}, x - 2^{-n+1}), & x \in (2^{-n}, 2^{-n+1}), n = 1, 2, 3, \ldots, \\ \dfrac{\varepsilon}{5M4^n}, & x = 2^{-n+1}, \\ \dfrac{1}{2^N}, & x = 0. \end{cases}$$

For any $\delta-$fine division $P = \{[x_{i-1}, x_i]; \xi_i\}$, we suppose that $\xi_0 = 0$. Then there is a $\beta > 0$, such that $([0,\beta]; 0) \in P$.

(1) When $\xi_i \in (2^{-n}, 2^{-n+1})$, since $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset (2^{-n}, 2^{-n+1})$, we have

$$D(\tilde{f}(\xi_i)(x_i - x_{i-1}), F([x_{i-1}, x_i])) = D(2^n \tilde{A}_n (x_i - x_{i-1}), 2^n \tilde{A}_n (x_i - x_{i-1})) = 0.$$

(2) When $\xi_i = 2^{-n+1}$,

$$\begin{aligned} &D(\tilde{f}(\xi_i)(x_i - x_{i-1}), F([x_{i-1}, x_i])) \\ &= D(2^n (x_i - \frac{1}{2^n})\tilde{A}_n + \sum_{k=n+1}^{\infty} \tilde{A}_k, 2^{n+1}(x_{i-1} - \frac{1}{2^{n+1}})\tilde{A}_{n+1} + \sum_{k=n+2}^{\infty} \tilde{A}_k) \\ &= D(2^n (x_i - \frac{1}{2^n})\tilde{A}_n, 2^{n+1}(x_{i-1} - \frac{1}{2^n})\tilde{A}_{n+1}) \\ &\le D(2^n (x_i - \frac{1}{2^n})\tilde{A}_n, \tilde{0}) + D(2^{n+1}(x_{i-1} - \frac{1}{2^n})\tilde{A}_{n+1}, \tilde{0}) \\ &\le 2^n \|\tilde{A}_n\| \frac{\varepsilon}{5M4^n} + 2^{n+1} \|\tilde{A}_{n+1}\| \frac{\varepsilon}{5M4^n} \\ &\le \frac{\varepsilon}{5 \cdot 2^n} + \frac{\varepsilon}{5 \cdot 2^{n-1}} = \frac{3\varepsilon}{5 \cdot 2^n}. \end{aligned}$$

(3) When $\xi_i = 0$, let $m > N$, such that $\beta \in (2^{-m}, 2^{-m+1}]$. Then

$$\begin{aligned} D(\tilde{f}(0)\beta, F([0,\beta])) &= D(\tilde{0}, 2^m (\beta - \frac{1}{2^m})\tilde{A}_m + \sum_{k=m+1}^{\infty} \tilde{A}_k) \\ &\le 2^m \|\tilde{A}_m\| (\beta - \frac{1}{2^m} + \| \sum_{k=m+1}^{\infty} \tilde{A}_k \| \le \|\tilde{A}_m\| + \frac{\varepsilon}{5} = \frac{2\varepsilon}{5}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\sum_{i=1}^{p} D(\tilde{f}(\xi_i)(x_i - x_{i-1}), F([x_{i-1}, x_i])) \\ &= D(\tilde{f}(0)\beta, F([0,\beta])) + \sum_{i=2}^{p} D(\tilde{f}(\xi_i)(x_i - x_{i-1}), F([x_{i-1}, x_i])) \\ &= \frac{2\varepsilon}{5} + \sum_{n=1}^{\infty} \frac{3\varepsilon}{5 \cdot 2^n} = \varepsilon. \end{aligned}$$

Hence $\tilde{f}$ is fuzzy strongly Henstock integrable on $[0,1]$.

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