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The Factorization of Braided Hopf Algebras

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Abstract We obtain the double factorization of braided bialgebras or braided Hopf algebras and give the relations among integrals and semisimplicity of braided Hopf algebra and its factors.

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1. Introduction

It is well-known that the factorization of domain plays an important role in ring theory. Majid in [1, Theorem 7.2.3] studied the factorization of Hopf algebra and showed that $H \cong A \bowtie B$ for two sub-bialgebras A and B when multiplication m_H is bijective. Zhang, Yang and Ren[2] generalized these results to braided cases.

Braided tensor categories become more and more important. They have been applied to conformal field, vertex operator algebras and isotopy invariants of links[3–8].

In this paper, we obtain the double factorization of braided bialgebras or braided Hopf algebras, i.e., we give the conditions to factorize a braided Hopf algebra into the double cross products of sub-bialgebras or sub-Hopf algebras. We give the relations among integrals and semisimplicity of braided Hopf algebra and its factors.

Throughout this paper, we work in braided tensor category $(\mathcal{C}, C)$, where $\mathcal{C}$ is a concrete category and underlying set of every object in $\mathcal{C}$ is a vector space over a field k. For example, Yetter-Drinfeld category over Hopf algebras with invertible antipode and some important categories in [5] are such categories.

2. Preliminaries

We assume that H and A are two braided bialgebras with morphisms:

$$\alpha : H \otimes A \rightarrow \quad A \quad , \qquad \beta : H \otimes A \rightarrow H,$$
$$\phi : A \rightarrow H \otimes \quad A \quad , \qquad \psi : H \rightarrow H \otimes A$$

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such that (A,α) is a left H-module coalgebra, (H,β) is a right A-module coalgebra, (A,ϕ) is a left H-comodule algebra, and (H,ψ) is a right A-comodule algebra.

We define the multiplication m_D, unit η_D, comultiplication Δ_D and counit ϵ_D in $A\otimes H$ as follows:

$$\Delta_D=(\mathrm{id}_A\otimes m_H\otimes m_A\otimes \mathrm{id_H})(\mathrm{id_A}\otimes \mathrm{id_H}\otimes \mathrm{C_{A,H}}\otimes \mathrm{id_A}\otimes \mathrm{id_H})(\mathrm{id_A}\otimes \phi\otimes\psi\otimes \mathrm{id_H})(\Delta_\mathrm{A}\otimes\Delta_\mathrm{H}),$$

$$m_D=(m_A\otimes m_H)(\mathrm{id_A}\otimes\alpha\otimes\beta\otimes \mathrm{id_H})(\mathrm{id_A}\otimes \mathrm{id_H}\otimes \mathrm{C_{H,A}}\otimes \mathrm{id_A}\otimes \mathrm{id_H})(\mathrm{id_A}\otimes\Delta_\mathrm{H}\otimes\Delta_\mathrm{A}\otimes \mathrm{id_H}),$$

and $\epsilon_D=\epsilon_A\otimes\epsilon_H$, $\eta_D=\eta_A\otimes\eta_H$. We denote $(A\otimes H, m_D,\eta_D,\Delta_D,\epsilon_D)$ by $A^\phi_\alpha\bowtie^\psi_\beta H$, which is called the double bicrossproduct of A and H and denoted by $A\overset{b}{\bowtie}H$ or $A\bowtie H$ for short. When ϕ and ψ are trivial, we denote $A^\phi_\alpha\bowtie^\psi_\beta H$ by $A_\alpha\bowtie_\beta H$ or $A\bowtie H$, called a double cross product[9−11].

$\mathcal{V}ect(k)$ denotes the braided tensor category of all vector spaces over field k, equipped with ordinary tensor and unit $I=k$, as well as ordinary twist map as braiding. ${}^B_B\mathcal{YD}$ denotes Yetter-Drinfeld category[11,Preliminaries].

3. The factorization of braided bialgebras or braided Hopf algebras

In this section, we obtain the factorization of braided bialgebras or braided Hopf algebras.

The associative law does not hold for double cross products in general, i.e., the equation

$$((A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2)\ {}_{\alpha_2}\bowtie_{\beta_2} A_3)=(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} (A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3))$$

does not hold in general. Therefore, we denote a method of adding brackets for n factors by σ. For example, when $n=3$,

$$\sigma_1(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3)=((A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2)\ {}_{\alpha_2}\bowtie_{\beta_2} A_3)$$

and

$$\sigma_2(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3)=(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} (A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3))$$

Definition 3.1 *Let $A_1,A_2,\dots,A_n$ and H be braided bialgebras or braided Hopf algebras, σ be a method of adding brackets for n factors. If $j_{A_1}(A_1)j_{A_2}(A_2)\cdots j_{A_n}(A_n)=H$, where $j_{A_i}:A_i\to H$ is a braided bialgebra or Hopf algebra morphism for $i=1,2,\dots,n$, and for every pair of brackets of σ:*

$$(A_l\otimes A_{l+1}\otimes\cdots\otimes A_{l+t}),$$

$j_{A_l}(A_l)j_{A_{l+1}}(A_{l+1})\cdots j_{A_{l+t}}(A_{l+t})$ is a sub-bialgebra or sub-Hopf algebra of H and $m^t_H(j_{A_l}\otimes j_{A_{l+1}}\otimes\cdots\otimes j_{A_{l+t}})$ is a bijective map from $A_l\otimes A_{l+1}\otimes A_{l+2}\otimes\cdots\otimes A_{l+t}$ onto $j_{A_l}(A_l)j_{A_{l+1}}(A_{l+1})\cdots j_{A_{l+t}}(A_{l+t})$, then $\{j_{A_1},j_{A_2},\dots,j_{A_n}\}$ is called a double factorization of H with respect to σ. If for every method σ of adding brackets, $\{j_{A_1},j_{A_2},\dots,j_{A_n}\}$ is a double factorization of H with respect to σ, then $\{j_{A_1},j_{A_2},\dots,j_{A_n}\}$ is called a double factorization of H. If A_t is a sub-bialgebra or sub-Hopf algebra of H and j_{A_t} is an inclusion map from A_t to H by sending a to a for any $a\in A_t$, $t=1,2,\dots,n$, then $H=A_1A_2\cdots A_n$ is called an inner double factorization of H.

Theorem 3.2 *Let $A_1,A_2,\dots,A_n$ and H be braided bialgebras or braided Hopf algebras, and let*

σ be a method of adding brackets for n factors. Assume that j_{A_i} is a bialgebra or Hopf algebra morphism from A_i to H for $i = 1, 2, \dots, n$. If $\{j_{A_1}, j_{A_2}, \dots, j_{A_n}\}$ is a double factorization of H with respect to σ, then in braided tensor category $(\mathcal{C}, C)$, there exists a set $\{\alpha_i, \beta_i \mid i = 1, 2, \dots, n\}$ of morphisms such that

$$\sigma(A_1 \ {}_{\alpha_1}\bowtie_{\beta_1} A_2 \ {}_{\alpha_2}\bowtie_{\beta_2} A_3 \ {}_{\alpha_3}\bowtie_{\beta_3} \cdots {}_{\alpha_{n-1}}\bowtie_{\beta_{n-1}} A_n) \cong H \quad \textit{(as bialgebras or Hopf algebras)}$$

and the isomorphism is $m_H^{n-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_n})$.

Proof We use induction for n. When $n = 2$, we can obtain the proof by [2, Theorem 2.1] (i.e. the factorization theorem). For $n > 2$, we can assume that

$$\sigma(A_1 \otimes A_2 \otimes A_3 \cdots \otimes A_n) = \sigma_1(A_1 \otimes A_2 \otimes \cdots \otimes A_t) \otimes \sigma_2(A_{t+1} \otimes \cdots \otimes A_n).$$

Next we consider t in following three cases.

(i) If $1 < t < n-1$, let $B_1 = j_{A_1}(A_1) j_{A_2}(A_2) \cdots j_{A_t}(A_t)$ and $B_2 = j_{A_t+1}(A_{t+1}) j_{A_{t+2}}(A_{t+2}) \cdots j_{A_n}(A_n)$. It follows from the inductive assumption that $\{j_{A_1}, j_{A_2}, \dots, j_{A_t}\}$ is a double factorization of B_1 with respect to σ_1 and $\{j_{A_{t+1}}, j_{A_{t+2}}, \dots, j_{A_n}\}$ is a double factorization of B_2 with respect to σ_2. Thus, there exists a set $\{\alpha_i, \beta_i \mid i = 1, 2, \dots, n, i \neq t\}$ of morphisms such that

$$\sigma_1(A_1 \ {}_{\alpha_1}\bowtie_{\beta_1} A_2 \ {}_{\alpha_2}\bowtie_{\beta_2} \cdots \alpha_{t-1} \bowtie_{\beta_{t-1}} A_t) \cong B_1 (\text{as bialgebras or Hopf algebras}) \tag{3.1}$$

and

$$\sigma_2(A_{t+1} \ {}_{\alpha_{t+1}}\bowtie_{\beta_{t+1}} \cdots {}_{\alpha_n}\bowtie_{\beta_n} A_n) \cong B_2 \text{ (as bialgebras or Hopf algebras) .} \tag{3.2}$$

The isomorphisms of (3.1) and (3.2) are $m_H^{t-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_t})$ and $m_H^{n-t}(j_{A_{t+1}} \otimes j_{A_{t+2}} \otimes \cdots \otimes j_{A_n})$, respectively. Let j_{B_1} and j_{B_2} denote the contain-map of B_1 and B_2 in H, respectively. We can get that $m_H(j_{B_1} \otimes j_{B_2})$ is a bijective since $m_H^{n-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_n})$ is a bijective. In fact, $m_H(j_{B_1} \otimes j_{B_2}) = m_H^{n-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_n})((m_H^{t-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_t}))^{-1} \otimes (m_H^{n-t-1}(j_{A_{t+1}} \otimes j_{A_{t+2}} \otimes \cdots \otimes j_{A_n}))^{-1})$.

(ii) If $t = 1$, let $B_1 = A_1$, $B_2 = j_{A_2}(A_2) j_{A_3}(A_3) \cdots j_{A_n}(A_n)$, $j_{B_1} = j_{A_1}$ and j_{B_2} be a contain-map from B_2 to H. We can get that $m_H(j_{B_1} \otimes j_{B_2})$ is a bijective by the way similar to (i).

(iii) If $t = n-1$, let $B_1 = j_{A_1}(A_1) j_{A_2}(A_2) \cdots j_{A_{n-1}}(A_{n-1})$ and $B_2 = A_n$. We can also get that $m_H(j_{B_1} \otimes j_{B_2})$ is a bijective.

Consequently, by [2, Factorization Theorem], there exist α_t and β_t such that $B_{1\alpha_t} \bowtie_{\beta_t} B_2 \cong j_{B_1}(B_1) j_{B_2}(B_2) = H$. Considering relations (3.1) and (3.2), we have that

$$\sigma(A_1 \ {}_{\sigma_1}\bowtie_{\beta_1} A_2 \ {}_{\alpha_2}\bowtie_{\beta_2} A_3 \ {}_{\alpha_3}\bowtie_{\beta_3} \cdots {}_{\alpha_{n-1}}\bowtie_{\beta_{n-1}} A_n) \cong H \quad \text{(as bialgebras or Hopf algebras)}$$

and the isomorphism is $m_H^{n-1}(j_{A_1} \otimes j_{A_2} \otimes \cdots \otimes j_{A_n})$. The proof is completed. □

Corollary 3.3 *Let X, A and H be braided bialgebras or braided Hopf algebras. Assume j_A and j_H are bialgebra or Hopf algebra morphisms from A to X and H to X, respectively. Then $\{j_A, j_H\}$ is a double factorization of X iff in braided tensor category $(\mathcal{C}, C)$, there exist morphisms α and β such that*

$$A \ {}_{\alpha}\bowtie_{\beta} H \cong X \quad \textit{(as bialgebras or Hopf algebras)}$$

and the isomorphism is $m_X(j_A \otimes j_H)$.

Corollary 3.4 *Let* $A_1, A_2, \ldots, A_n$ *be braided sub-Hopf algebras of a finite-dimensional Hopf algebra* H, *and* $H = A_1A_2\cdots A_n$. *Assume that* σ *is a method of adding brackets for* n *factors. Then the following statements are equivalent.*

(i) $H = A_1A_2\cdots A_n$ *is an inner double factorization of* H *with respect to* σ.

(ii) $\dim H = \dim(A_1)\dim(A_2)\cdots\dim(A_n)$, *and for every pair of brackets in* σ: $(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l})$, $(A_tA_{t+1}\cdots A_{t+l})$ *is a sub-Hopf algebra of* H.

(iii) For every pair of bracket in σ : $(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l})$, $(A_tA_{t+1}\cdots A_{t+l})$ *is a sub-Hopf algebra of* H, *and there exists a set* $\{\alpha_i, \beta_i \mid i = 1, 2, \ldots, n\}$ *of morphisms such that*

$$\sigma(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3\ {}_{\alpha_3}\bowtie_{\beta_3} \cdots {}_{\alpha_{n-1}}\bowtie_{\beta_{n-1}} A_n) \cong H \quad \textit{(as bialgebras or Hopf algebras)}$$

and the isomorphism is m_H^{n-1}.

Proof Obviously, (iii) implies (ii). By Theorem 3.2, (i) implies (iii). It suffices to show that (ii) implies (i). Assume that $(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l})$ is a pair of brackets in σ. We only need to show that m_H^l is a bijective map from $(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l})$ onto $A_tA_{t+1}\cdots A_{t+l}$. Since $\dim(A_1A_2\cdots A_n) = \dim(A_1)\dim(A_2)\cdots\dim(A_n)$, we have that $\dim(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l}) = \dim(A_tA_{t+1}\cdots A_{t+l})$, which implies that m_H^l is a bijective map from $(A_t \otimes A_{t+1} \otimes \cdots \otimes A_{t+l})$ onto $A_tA_{t+1}\cdots A_{t+l}$. The proof is completed. □

Lemma 3.5 *Let* A *and* B *be braided sub-Hopf algebras of braided Hopf algebra* H.

(i) If $AB = BA$, *then* AB *is sub-Hopf algebra of* H.

(ii) If the antipode of H *is invertible, and* AB *or* BA *is a braided sub-Hopf algebra of* H, *then* $AB = BA$.

Proof (i) It is clear.

(ii) We can assume that AB is a braided sub-Hopf algebra of H without loss of generality. For any $a \in A$ and $b \in B$, we see that $S(S^{-1}(a)S^{-1}(b)) = ba$. Thus $BA \subseteq AB$ since AB is a braided sub-Hopf algebra. For any $x \in AB$, there exist $a_i \in A$ and $b_i \in B$ such that $S(x) = \sum a_ib_i$. We see that $x = S^{-1}S(x) = S^{-1}(\sum a_ib_i) = \sum S^{-1}(b_i)S^{-1}(a_i) \in BA$. Thus $AB \subseteq BA$. Consequently, $AB = BA$. □

Corollary 3.6 *Let* $A_1, A_2, \ldots, A_n$ *be braided sub-Hopf algebras of a finite-dimensional braided Hopf algebra* H, *and* $H = A_1A_2\cdots A_n$. *Then the following statements are equivalent.*

(i) $H = A_1A_2\cdots A_n$ *is an inner double factorization of* H.

(ii) $\dim H = \dim(\mathrm{A}_1)\dim(\mathrm{A}_2)\cdots\dim(\mathrm{A}_n)$ *and* $A_uA_v = A_vA_u$ *for* $1 \leq u < v \leq n$.

(iii) For $1 \leq u < v \leq n$, $A_uA_v = A_vA_u$, *and for any method* σ *of adding brackets, there exists a set* $\{\alpha_i, \beta_i \mid i = 1, 2, \ldots, n\}$ *of morphisms such that*

$$\sigma(A_1\ {}_{\alpha_1}\bowtie_{\beta_1} A_2\ {}_{\alpha_2}\bowtie_{\beta_2} A_3\ {}_{\alpha_3}\bowtie_{\beta_3} \cdots {}_{\alpha_{n-1}}\bowtie_{\beta_{n-1}} A_n) \cong H \quad \textit{(as Hopf algebras)}$$

and the isomorphism is m_H^{n-1}.

(iv) $H = A_{i_1}A_{i_2}\cdots A_{i_n}$ *is an inner double factorization of* H*, where* $\{i_1, i_2, \ldots, i_n\} = \{1, 2, 3, \ldots, n\}$.

Proof (i)$\Rightarrow$ (ii). For $1 \le u < v \le n$, there exists a method of adding brackets σ such that $(A_u \otimes A_v)$ is a pair of brackets in σ. Thus A_uA_v is a sub-Hopf algebra of H. By Lemma 3.5, $A_uA_v = A_vA_u$. It follows from Corollary 3.4 that $\mathrm{dimH} = \mathrm{dim(A_1)dim(A_2)\cdots dim(A_n)}$.

(ii)$\Rightarrow$ (i) follows from Corollary 3.4.

Similarly, (ii) and (iv) are equivalent.

By Corollary 3.4, we also have that (ii) and (iii) are equivalent. □

If H is an almost commutative braided Hopf algebra, in particular, H is a coquasitriangular braided Hopf algebra, then $AB = BA$ for any braided sub-Hopf algebras A and B of H. Note that every quantum commutative braided Hopf algebra H is a coquasitriangular braided Hopf algebra with coquasitriangular structure $r = \epsilon_H \otimes \epsilon_H$.

Example[12,Lemma 3.4] Assume that Γ is a commutative group and $\hat{\Gamma}$ is the character group of Γ with $g_i \in \Gamma$, $\chi_i \in \hat{\Gamma}$, $\chi_i(g_j)\chi_j(g_i) = 1$, $1 < N_i$, where N_i denotes the order of $\chi_i(g_i)$, $1 \le i < j \le \theta$. Let H denote the algebra generated by set $\{x_i \mid 1 \le i \le \theta\}$ with relation:

$$x_l^{N_l} = 0,\ x_ix_j = \chi_j(g_i)x_jx_i \ \text{ for } 1 \le i, j, l \le \theta \text{ with } i \ne j. \tag{3.3}$$

Define coalgebra operations and kG-(co-)module operations in H as follows:

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon(x_i) = 0,$$

$$\delta^-(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)x_i.$$

Then H is called a quantum linear space in ${}^{k\Gamma}_{k\Gamma}\mathcal{YD}$. By [12, Lemma 3.4], H is a braided Hopf algebra with $\mathrm{dimH} = \mathrm{N_1N_2\cdots N_\theta}$. Let H_i be the sub-algebra generated by x_i in H. It is easy to check that H_i is a braided sub-Hopf algebra of H with $\mathrm{dimH_i} = \mathrm{N_i}$ and $H = H_1H_2\cdots H_\theta$ is an inner double factorization of H by Corollary 3.6 (ii). Furthermore, when $\theta = 1$ and $N_1 = p$ is prime, then H is not commutative with $\mathrm{dimH = p}$.

By the way, it is well-known that the eighth Kaplansky's conjecture is that if the dimension of Hopf algebra H is prime, then H is commutative and cocommutative. Zhu[13] gave the positive answer. Now the above example shows that braided version of the 8th Kaplansky's conjecture does not hold, i.e., there exists a noncommutative braided Hopf algebra H with prime dimension.

If there are two non-trivial sub-bialgebras or sub-Hopf algebras A and B of H such that $H = AB$ is an inner double factorization of H, then H is called a double factorizable bialgebra or Hopf algebra. Otherwise, H is called a double infactorisable bialgebra or Hopf algebra.

A bialgebra (Hopf algebra) H is said to satisfy the a.c.c. on sub-bialgebras (sub-Hopf algebras) if for every chain $A_1 \subseteq A_2 \subseteq \cdots$ of sub-bialgebras (sub-Hopf algebras) of H there is an integer n such that $A_i = A_n$, for all $i > n$. Similarly, we can define d.c.c..

If H satisfies the d.c.c. or a.c.c. on sub-bialgebras (sub-Hopf algebras), then H can be factorized into a product of finite double infactorisable sub-bialgebras (sub-Hopf algebras).

By Corollary 3.3, we can easily know that Sweedler's 4-dimensional Hopf algebra H_4 over field k is double infactorisable in category $\mathcal{V}ect(k)$. In fact, if H_4 is double factorisable, then there exist two non-trivial sub-Hopf algebras A and B such that $H = AB$ and $H \cong A \bowtie B$. It is clear that A and B are 2-dimensional. Thus they are commutative, which implies that H_4 is commutative, leading to a contradiction. Thus H_4 is double infactorisable. Similarly, if p and q are two prime numbers and H is a non-commutative Hopf algebra with $\dim H = pq$, then H is double infactorisable. Consequently, every Taft algebra H with dimH = p^2 is double infactorisable in $\mathcal{V}ect(k)$.

4. The factorization of braided bialgebras or braided Hopf algebras in Yetter-Drinfeld categories

Throughout this section, we work in a Yetter-Drinfeld category ${}^B_B\mathcal{YD}$. In this section, we give the relation among integrals and semisimplicity of braided Hopf algebra and its factors.

If H is a finite-dimensional braided Hopf algebra in ${}^B_B\mathcal{YD}$, then $\int_H^l$ and $\int_H^r$ are one-dimensional by [14] or [11, Theorem 2.2.1]. So there exist a non-zero left integral Λ_H^l and a non-zero right integral Λ_H^r.

Proposition 4.1 *If A, H and $D = A \overset{b}{\bowtie} H$ are finite dimensional braided Hopf algebras, then*

(i) There are $u \in H$ and $v \in A$ such that

$$\Lambda_D{}^l = \Lambda_A^l \otimes u, \quad \Lambda_D^r = v \otimes \Lambda_H^r;$$

(ii) If $D = A \overset{b}{\bowtie} H$ is semisimple, then A and H are semisimple;

(iii) If $D = A \overset{b}{\bowtie} H$ is unimodular, i.e., $\Lambda_D^l = \Lambda_D^r$, then there exists a non-zero $x \in k$ such that $\Lambda_D = x\Lambda_A^l \otimes \Lambda_H^r$, and $A \overset{b}{\bowtie} H$ is semisimple iff A and H are semisimple.

Proof (i) Let $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ and $h^{(1)}, h^{(2)}, \ldots, h^{(m)}$ be the bases of A and H, respectively. Assuming

$$\Lambda_D^l = \sum k_{ij}(a^{(i)} \otimes h^{(j)}),$$

where $k_{ij} \in k$, we have that

$$a\Lambda_D^l = \epsilon(a)\Lambda_D^l$$

and

$$\sum \epsilon(a)k_{ij}(a^{(i)} \otimes h^{(j)}) = \sum k_{ij}(aa^{(i)} \otimes h^{(j)}),$$

for any $a \in A$. Let $x_j = \sum_i k_{ij}a^{(i)}$. Considering that $\{h^{(j)}\}$ is a base of H, we get x_j is a left integral of A and there exists $k_j \in k$ such that $x_j = k_j\Lambda_A^l$ for $j = 1, 2, \ldots, m$. Thus

$$\Lambda_D^l = \sum_j k_j(\Lambda_A^l \otimes h^{(j)}) = \Lambda_A^l \otimes u,$$

where $u = \sum_j k_j h^{(j)}$.

Similarly, we have that $\Lambda_D^r = v \otimes \Lambda_H^r$.

(ii) and (iii) follow from (i). □

Remark 4.2 For braided version of biproduct $A \bowtie\!\!\!\!\triangleleft H$ (see [10, Corollary 2.17] or [11, Chapter 4]), Proposition 4.1 also holds. For example, in Example 3.7, $\int_H^l = \int_H^r = kx_1^{N_1-1}x_2^{N_2-1}\cdots x_\theta^{N_\theta-1}$ of quantum linear space H. By Proposition 4.1, the integral of the biproduct $D = H \bowtie\!\!\!\!\triangleleft k\Gamma$ is $\int_D = kx_1^{N_1-1}x_2^{N_2-1}\cdots x_\theta^{N_\theta-1} \otimes (\sum_{g\in\Gamma} g)$.

5. The factorization of ordinary Hopf algebras

Throughout this section, we work in the category $\mathcal{V}ect(k)$. In this section, using the results in preceding section, we give the relation among semisimplicity of Hopf algebra and its factors.

Lemma 5.1 *Assume that H is a finite-dimensional Hopf algebra. Then*

(i) H is cosemisimple iff H^* semisimple.

(ii) H is semisimple iff H^* cosemisimple.

Proof (i) If H is cosemisimple, then there exists $T \in \int_{H^*}^l$ such that $\epsilon_{H^*}(T) \neq 0$ by [15, Theorem 2.4.6, i.e., dual Maschke theorem]. Therefore H^* is semisimple. Conversely, if H^* is semisimple, then there exists $T \in \int_{H^*}^l$ such that $\epsilon_{H^*}(T) \neq 0$. Using again [15, dual Maschke theorem], we have that H is cosemisimple.

Similarly, we get (ii). □

Lemma 5.2[8],[15] *If H is a finite dimensional Hopf algebra with* char$k = 0$, *then the following conditions are equivalent.*

(i) H is semisimple;

(ii) H is cosemisimple;

(iii) H is semisimple and cosemisimple;

(iv) $S_H^2 = \mathrm{id}_H$;

(v) $\mathrm{tr}(S_H^2) \neq 0$.

Proof (i)$\Rightarrow$ (ii). If H is semisimple, then H^* is cosemisimple by Lemma 5.1. So H^* is semisimple by [15, Theorem 2.5.2]. Therefore H is cosemisimple. It follows from [15, Theorem 2.5.2] that (ii) implies (i). Consequently, (i), (ii) and (iii) are equivalent. Using formula $\mathrm{tr}(S_H^2) = \epsilon_H(\Lambda_H^l)\Lambda_{H^*}^r(1_H)$ in [8, Proposition 2 (c)], we have that (iii) and (v) are equivalent. (iii) implies (iv) by [15, Theorem 2.5.3]. Obviously, (iv) implies (v). □

Lemma 5.3[8],[15] *If H is a finite dimensional Hopf algebra with* char$k > (\mathrm{dimH})^2$, *then the following conditions are equivalent.*

(i) H is semisimple and cosemisimple;

(ii) $S_H^2 = \mathrm{id}_H$;

(iii) $\mathrm{tr}(S_H^2) \neq 0$.

Proof Using formula $\mathrm{tr}(S_H^2) = \epsilon_H(\Lambda_H^l)\Lambda_{H^*}^r(1_H)$ in [8, Proposition 2 (c)], we have that (i) and (iii) are equivalent. (i) implies (ii) by [15, Theorem 2.5.3]. Obviously, (ii) implies (iii). □

Proposition 5.4 *Assume that $D = A \overset{b}{\bowtie} H$ are finite dimensional Hopf algebras. If $A \overset{b}{\bowtie} H$ is (co)semisimple, then A and H are (co)semisimple.*

Proof If D is semisimple, then A and H are semisimple by Proposition 4.1. If D is cosemisimple, by Lemma 5.1, D^* is semisimple. Considering $(A \overset{b}{\bowtie} H)^* \cong A^* \overset{b}{\bowtie} H^*$ (see [11, Proposition 3.1.2], note that the evaluations in [11, Proposition 3.1.12] or [9, Proposition 1.11] are not the same in this paper), we have that A^* and H^* are semisimple by Proposition 4.1. Consequently, A and H are cosemisimple. □

Theorem 5.5 *Assume that $\{j_{A_1}, j_{A_2}, \ldots, j_{A_n}\}$ is a double factorization of finite-dimensional Hopf algebra H with respect to some method σ of adding brackets. Then*

(I) H is semisimple and cosemisimple iff A_i is semisimple and cosemisimple for $i = 1, 2, \ldots, n$, iff $\mathrm{tr}(S^2_{A_i}) \neq 0$ for $i = 1, 2, \ldots, n$, iff $\mathrm{tr}(S^2_H) \neq 0$.

(II) If H is (co)semisimple, then A_i is (co)semisimple for $i = 1, 2, \ldots, n$.

(III) If A_i is involutory for $i = 1, 2, \ldots, n$, then H is involutory.

(IV) H admits a coquasitriangular structure iff A_i admits a coquasitriangular structure for $i = 1, 2, \ldots, n$.

(V) If $\mathrm{char}k = 0$, then the following are equivalent.

(i) H is semisimple and cosemisimple; (i)′ H is semisimple; (i)″ H is cosemisimple.

(ii) A_i is semisimple and cosemisimple for $i = 1, 2, \ldots, n$; (ii)′ A_i is semisimple for $i = 1, 2, \ldots, n$; (ii)″ A_i is cosemisimple for $i = 1, 2, \ldots, n$.

(iii) A_i is involutory for $i = 1, 2, \ldots, n$.

(iv) H is involutory.

(v) $\mathrm{tr}(S^2_H) \neq 0$.

(vi) $\mathrm{tr}(S^2_{A_i}) \neq 0$ for $i = 1, 2, \ldots, n$.

(VI) If $\mathrm{char}k > (\mathrm{dimH})^2$, or $\mathrm{char}k > (\mathrm{dimA_i})^2$ for $i = 1, 2, \ldots, n$, then the following are equivalent.

(i) H is semisimple and cosemisimple.

(ii) A_i is semisimple and cosemisimple for $i = 1, 2, \ldots, n$.

(iii) A_i is involutory for $i = 1, 2, \ldots, n$.

(iv) H is involutory.

(v) $\mathrm{tr}(S^2_H) \neq 0$.

(vi) $\mathrm{tr}(S^2_{A_i}) \neq 0$ for $i = 1, 2, \ldots, n$.

Proof By Theorem 3.2, there exists $\{\alpha_i, \beta_i \mid i = 1, 2, \ldots, n\}$ such that $D =: \sigma(A_1 {}_{\alpha_1}\bowtie_{\beta_1} A_2 {}_{\alpha_2}\bowtie_{\beta_2} A_3 {}_{\alpha_3}\bowtie_{\beta_3} \cdots {}_{\alpha_{n-1}}\bowtie_{\beta_{n-1}} A_n) \cong H$ (as Hopf algebras) and the isomorphism is m^{n-1}. It is clear that

$$S^2_D = S^2_{A_1} \otimes S^2_{A_2} \otimes \cdots \otimes S^2_{A_n} \tag{5.4}$$

(see [9, Proposition 1.6]) and

$$\mathrm{tr}(S^2_D) = \mathrm{tr}(S^2_{A_1})\mathrm{tr}(S^2_{A_2}) \cdots \mathrm{tr}(S^2_{A_n}) \tag{5.5}$$

(see [16, Theorem XIV.4.2].

(I) If H is semisimple and cosemisimple, then $\mathrm{tr}(S_H^2) \neq 0$. So $\mathrm{tr}(S_{A_i}^2) \neq 0$ by formula (5.5) for $i = 1, 2, \ldots, n$. It follows from [8, Proposition 2 (c)] that A_i is semisimple and cosemisimple for $i = 1, 2, \ldots, n$. Similarly, we can show the others.

(II) It follows from Proposition 5.4.

(III) It follows from formula (5.4).

(IV) It follows from the dual result of [17, Corollary 2.3] or [11, Corollary 7.4.7°].

(V) By Lemma 5.2, (i), (i)$'$ and (i)$''$ are equivalent; (ii), (ii)$'$ and (ii)$''$ are equivalent; (ii), (iii) and (vi) are equivalent; (i), (iv) and (v) are equivalent. By (I), (i), (ii), (v) and (vi) are equivalent.

(VI) By Lemma 5.2, (ii), (iii) and (vi) are equivalent. By (I), (i), (ii), (v) and (vi) are equivalent. By (III), (iii) implies (iv).

Now we show that (iv) implies (v). Let $\mathrm{char}k = p$. Since p is prime and $\mathrm{dim}H = (\mathrm{dim}A_1)(\mathrm{dim}A_2) \cdots (\mathrm{dim}A_n)$ with $p > (\mathrm{dim}A_i)^2$, we have that p does not divide $\mathrm{dim}H$. Therefore, $\mathrm{tr}(S_H^2) = \mathrm{tr}(\mathrm{id}_H) \neq 0$. □

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