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All Prime Cubes Are Anti-Sociable Numbers

LI Wei Xun
(Department of Mathematics, Maoming University, Guangdong 525000, China)
(E-mail: liweixun@163.com)

Abstract In this paper, by using the basic properties of arithmetic function $\sigma(n)$, the existence of amicable pairs is discussed. We prove that all prime cubes are anti-sociable numbers.

Keywords prime; power; amicable pair; anti-sociable number.

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1. Introduction

For any positive integer n, let $\sigma(n)$ be the sum of the divisors of n. A pair of positive integers (a, b) is called an amicable pair number if

$$\sigma(a) = \sigma(b) = a + b. \tag{1}$$

Conversely, for certain a, if there does not exist positive integer b satisfying (1), then a is called an anti-sociable number. When an amicable pair (a, b) satisfies $a = b$, a is a famous perfect number. Therefore, the amicable pair number and the anti-soisble number are always a conspicuous topic in Number Theory[1,2]. In 2000, Luck[3] proved that all Fermat numbers are anti-sociable numbers. In 2007, Li[4] proved that all mersenne numbers M_p are anti-sociable numbers. In this paper, we shall discuss the anti-sociable number on the power of an odd prime.

Let positive integer n be a prime power, that is,

$$n = p^r, \tag{2}$$

where p is a prime and r is a positive integer. In [4, Lemma 6], when $r = 1$, n is an anti-sociable number. In [5] and [6], n is an anti-sociable number, where $p = 2$ or p is an odd prime number and $r = 2$.

In this paper, we will prove the more universal result about anti-sociable numbers.

Therorem *If $p \geq 2r^2$, n is an anti-sociable number.*

On the basis of the above Theorem, we have the following deduction.

Deduction *All prime cubes are anti-sociable numbers.*

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Also, we get a guess of the anti-sociable numbers from the known results.

Guess *All prime powers are anti-sociable numbers.*

2. Preliminaries

Lemma 1 *Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the standard facorization of the positive integer a,*

$$\sigma(a) = a \sum_{i=1}^{k} (1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{r_i}}).$$

Proof This follows immediately from [7, Therorem 1.9.1].

Lemma 2 *Let $a > 2$. Then $\sigma(a) < 2a \log a$.*

Proof This follows immediately from [8, Lemma 3].

Lemma 3 *Let r be positive integers with $r > 2$. If $x \geq 2r^2$, then we have*

$$x > 2 \log x^r. \tag{3}$$

Proof Let $f(x) = x - 2\log x^r$. Then $f'(x) = 1 - \frac{2r}{x}$. When $x > 2r$, $f(x)$ is an increasing function. Let $r \geq 3$. We get $2r^2 > 2r$, moreover

$$f(2r^2) = 2r(r - \log 2 - 2\log r) > 0.$$

If $x \geq 2r^2$, we get $f(x) > 0$. Therefore, inequality (3) satisfies $x > 2r^2$. This completes the proof. □

3. Proof of the Theorem

Proof Let $a = p^r$, where p is a prime and r is a positive integer. According to the result of the references [4] and [6], we know that the theorem satisfies $r = 1$ or $r = 2$. So we only need to discuss the condition for $r \geq 3$.

If a is not an anti-sociable number, we can find proper positive integer b satisfying (1). By Lemma 1, we have

$$\sigma(a) = \sigma(p^r) = 1 + p + \cdots + p^{r-1} + p^r. \tag{4}$$

So, from (1) and (4), we have

$$b = 1 + p + \cdots + p^{r-1}, \tag{5}$$

and

$$\sigma(b) = \sigma(a) = pb + 1. \tag{6}$$

From (5), we get $b = \frac{p^r - 1}{p-1} < \frac{p^r}{p-1} \leq p^r$. By Lemma 2, we get

$$\sigma(b) < 2b \log b < 2b \log p^r. \tag{7}$$

Combining with (6) and (7), we have

$$p < 2 \log p^r. \tag{8}$$

However, by Lemma 3, if $p \geq 2r^2$, inequality (8) is impossible. We can draw the conclusion that: If $p \geq 2r^2$, $a = p^r$ must be an anti-sociable number. This completes the proof. $\square$

4. Proof of the Deduction

Proof From the Theorem in this paper, we only need to prove p^3 is an anti-sociable number when $p < 18$.

When $p < 18$, only $p = 2, 3, 5, 7, 11, 13$ and 17.

If $a = p^3$ is not an anti-sociable number, we have $b \in \{7, 13, 31, 57, 133, 183, 307\}$ from (1) which satisfies

$$\sigma(b) = \sigma(a). \tag{9}$$

From the simple calculation, we know (9) is impossible. So, p^3 is an anti-sociable number. This completes the proof. $\square$

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