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Singular Nonlinear Boundary Value Problems Arising in the Boundary Layer Behind Expansion Wave

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Abstract A class of singular nonlinear boundary value problems arising in the boundary layer behind expansion wave are studied. Sufficient conditions for the existence and uniqueness of positive solutions to the problems are established by utilizing the monotonic approaching technique. And a theoretical estimate formula for skin friction coefficient is presented. The numerical solution is presented by using the shoot method. The reliability and efficiency of the theoretical prediction are verified by numerical results.

Keywords expansion wave; boundary layer; singular boundary value problems; existence and uniqueness; positive solution; skin friction; estimation.

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1. Introduction

Fluid dynamicists have long known that the appearance of boundary layers was not restricted to the canonical problem of the motion of a body through a viscous fluid. A technologically important source of boundary layer phenomenon is the flow behind expansion wave traveling over smooth surfaces. When a plane expansion wave advances into a stationary fluid, with a plane wall perpendicular to the wave front, a boundary layer is established along the wall behind the wave. This boundary layer is often important in the studies of phenomena involving non-stationary waves. Most outstanding and representative contribution on the problem had been made by Mires[1], For a list of the key references of a vast literature concerning this subject we refer to the references[2–5]. In this field, Most of work tends to similarity solutions of boundary layer equations, and many people had paid attention to the numerical solutions of the equations of similarity solutions. The qualitative properties of the solutions and heat transfer are studied in this paper, and a theoretical estimate formula for skin friction coefficient denoted by the velocity ratio parameter is presented.

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2. Laminar boundary layer equations

Consider a plane laminar flow with spatial coordinates (x, y) which is established along the wall behind expansion wave, corresponding velocity components (u, v) and $\mathrm{d}p/\mathrm{d}x = 0$. For steady flow, the boundary layer equations for $x > 0$ on the mass conservation, momentum conservation and energy conservation can be written as[1−2].

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}), \tag{2}$$

$$\rho C_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = \frac{\partial}{\partial y}(\kappa\frac{\partial T}{\partial y}) + \mu(\frac{\partial u}{\partial y})^2, \tag{3}$$

$$P = \rho RT. \tag{4}$$

The boundary conditions are

$$u(x,0) = u_w, u(x,\infty) = u_e, \tag{5}$$

$$v(x,0) = 0, \tag{6}$$

$$T(x,0) = T_w, T(x,\infty) = T_e. \tag{7}$$

Where μ is coefficient of viscosity, κ is thermal conductivity and C_p is specific heat at constant pressure.

3. Nonlinear boundary value problems

3.1 Stream function and similarity variables

Let us introduce a stream function $\Psi(x, y)$ and a similarity variable η by the expressions

$$\Psi = \sqrt{2u_e x\nu_w}f(\eta),\ \eta = \sqrt{\frac{u_e}{2x\nu_w}}\int_0^Y \frac{T_w}{T(x,y)}\mathrm{d}y. \tag{8}$$

Substituting (8) into (1)–(7), and furthermore assuming dimensionless temperature is only a function of η, we can obtain similarity equations as follows:

Momentum equations:

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \quad 0 < \eta < +\infty, \tag{9}$$

$$f(0) = 0,\ f'(0) = \xi,\ f'(+\infty) = 1. \tag{10}$$

Energy equations:

$$\overline{T}''(\eta) + Pr\cdot f(\eta)\overline{T}'(\eta) = -\frac{Pr\cdot u_e^2}{C_{p,\,w}T_e}(f''(\eta))^2, \quad 0 < \eta < +\infty, \tag{11}$$

$$\overline{T}(0) = \lambda,\ \overline{T}(+\infty) = 1. \tag{12}$$

Here $\xi = f'(0) = \dfrac{u_w}{u_e}$ is the velocity ratio parameter, $\lambda = T_w/T_e$ is the temperature ration parameter, $Pr = \mu C_p/\kappa$ is the Prandtl number, and $0 \le \xi < 1$ for a expansion wave[1,2,4].

3.2 Crocco variables transformation

Introducing a transformation as[7–8]

$$g(t) = f''(\eta) \ \ \text{(dimensionless shear stress)}, \tag{13}$$

$$t = f'(\eta) \ \ \text{(dimensionless tangential velocity)}, \tag{14}$$

$$w(t) = \overline{T}(\eta) \ \ \text{(dimensionless temperature)}, \tag{15}$$

and substituting (13)–(15) into (9)–(12), in terms of $f''(\eta) > 0$, $0 < \eta < +\infty$, $f''(+\infty) = 0$, we arrive at the following singular nonlinear two-point boundary value problems:

Momentum equations:

$$g(t)g''(t) + t = 0, \quad 0 \le \xi < t < 1, \tag{16}$$

$$g(1) = 0, \quad g'(\xi) = 0. \tag{17}$$

Energy equations:

$$w''(t) + (1 - Pr)w'(t)g'(t)/g = -\frac{Pr \cdot u_e^2}{C_{p,\,w}T_e}, \quad 0 \le \xi < t < 1, \tag{18}$$

$$w(1) = 1, \ w(\xi) = \lambda. \tag{19}$$

Clearly, the boundary value problems (16)–(17) are de-coupled and may be considered firstly, the solutions then may be used to solve the boundary value problems (18)–(19). It may be seen from the derivation process that only the positive solutions of the boundary value problems (16)–(17) are physically significant.

3.3 The solutions of the boundary value problems (16)–(17)

Since the boundary value problems (16)–(17) is singular at $t = 1$, we consider firstly the boundary problems as follows

$$\begin{cases} g''(t) = -\dfrac{t}{g(t)}, \ 0 \le \xi < t < 1, \\ g'(\xi) = 0, \ g(1) = h > 0. \end{cases} \tag{20}$$

Denote the solution of the boundary value problem (20) by $g_h(t)$, we can obtain the following lemmas.

Lemma 1 *If $h_1 > h_2 > 0$, then $g_{h_1}(t) \ge g_{h_2}(t)$.*

Proof If the inequality is not true, then there exists a point $t_0 \in [\xi, 1)$ such that $g_{h_1}(t_0) < g_{h_2}(t_0)$. We consider only two cases.

(i) $g_{h_1}(\xi) < g_{h_2}(\xi)$.

Choose $t_0 = \xi$, since $g_{h_1}(1) > g_{h_2}(1) > 0$, there exists a maximal interval $[\xi, k]$ $(k < 1)$ such that $g_{h_1}(t) < g_{h_2}(t)$ for $t \in [\xi, k)$ and $g_{h_1}(k) = g_{h_2}(k) = m > 0$. $g_{h_1}(t)$ and $g_{h_2}(t)$ are both the positive solutions of the integral equation

$$g(t) = m + \int_{\xi}^{k} G_1(t, s)\frac{s}{g(s)}\mathrm{d}s, \tag{21}$$

where

$$G_1(t,s)=\begin{cases} k-t, & 0\le \xi\le s\le t\le k<1,\\ k-s, & 0\le \xi\le t\le s\le k<1.\end{cases}$$

equation (21) implies

$$0<g_{h_2}(t)-g_{h_1}(t)=\int_\xi^k G_1(t,s)\Big[\frac{s}{g_{h_2}(s)}-\frac{s}{g_{h_1}(s)}\Big]\mathrm{d}s<0,$$

which is a contradiction.

(ii) $g_{h_1}(\xi)\ge g_{h_2}(\xi)$.

Since $g_{h_1}(1)>g_{h_2}(1)>0$, there exists a maximal interval $[a,b]\,(\xi\le a<b<1)$, which contains the point t_0 such that $g_{h_1}(a)=g_{h_2}(a)$ and $g_{h_1}(b)=g_{h_2}(b)$, and $g_{h_1}(t)<g_{h_2}(t)$ for $t\in(a,b)$. Let $g_{h_1}(a)=g_{h_2}(a)=\alpha$ and $g_{h_1}(b)=g_{h_2}(b)=\beta$. Then for $t\in[a,b]$, $g_{h_1}(t)$ and $g_{h_2}(t)$ are both the positive solutions of the integral equation

$$g(t)=\frac{b\alpha-a\beta}{b-a}+\frac{\beta-\alpha}{b-a}t+\int_a^b G_2(t,s)\frac{s}{g(s)}\mathrm{d}s, \tag{22}$$

where

$$G_2(t,s)=\begin{cases} \dfrac{(b-t)(s-a)}{b-a}, & \xi\le a\le s\le t\le b<1,\\ \dfrac{(b-s)(t-a)}{b-a}, & \xi\le a\le t\le s\le b<1.\end{cases}$$

equation (22) implies

$$0<g_{h_2}(t)-g_{h_1}(t)=\int_a^b G_2(t,s)\Big[\frac{s}{g_{h_2}(s)}-\frac{s}{g_{h_1}(s)}\Big]\mathrm{d}s<0,$$

which is also a contradiction.

Lemma 2 *For any fixed $h>0$, the boundary value problem (20) has at most one positive solution.*

Proof Suppose the boundary value problem (20) has two different positive solutions $g_1(t)$ and $g_2(t)$ for each fixed $h>0$. Then, without loss of generality, we may assume that there exists a point $t_0\in[\xi,\,1]$ such that $g_1(t_0)>g_2(t_0)$. Since $g_1(1)=g_2(1)=h$, there exists a maximal close interval $[a_1,b_1]\subseteq[\xi,\,1]$ such that $g_1(t)>g_2(t)$ for $t\in[a_1,b_1]$.

(i) If $a_1=\xi$, then $g_1(t)\ge g_2(t)$ for $t\in[\xi,b_1]\subseteq[\xi,\,1]$ and $g_1(b_1)=g_2(b_1)$.

(ii) If $a_1\ne\xi$, then $g_1(a_1)=g_2(a_1)$ and $g_1(b_1)=g_2(b_1)$ for $t\in[a_1,b_1]\subset[\xi,\,1]$, and $g_1(t)>g_2(t)$ for $t\in(a_1,b_1)$.

Along the same lines as in the cases (i) and (ii) in Lemma 1, we may show that this is impossible.

Lemma 3 *For any fixed $h>0$, the boundary value problem (20) has one positive solution.*

Proof For any fixed $h>0$, if $g(t)$ is the positive solution of the boundary value problem (20), then $g(t)$ is convex on $[\xi, 1]$ and must be a positive solution of the following integral equation

$$g(t) = h + \int_{\xi}^{1} G_3(t,s)\frac{s}{g(s)}\mathrm{d}s, \tag{23}$$

where

$$G_3(t,s) = \begin{cases} 1-t, & \xi \le s \le t \le 1, \\ 1-s, & \xi \le t \le s \le 1. \end{cases}$$

We define a Mapping $\phi : \Omega \to \Omega$ by

$$(\phi g)(t) = h + \int_{\xi}^{1} G_3(t,s)\frac{s}{g(s)}\mathrm{d}s,$$

where $\Omega = \{g(t) \in C[\xi, 1] : h \le g(t) \le (\phi g)(t)\}$, and $C[\xi, 1]$ is the set of all real-valued continuous functions defined on $[\xi, 1]$. Then ϕ is a compactly continuous mapping from Ω to Ω. The Schauder Fixed Point Theorem asserts that the mapping ϕ has at least one fixed point $g_h(t)$ in Ω, which implies that $g_h(t)$ is a positive solution of the boundary value problem (20).

Denote $g(\xi) = \sigma$ and consider the initial value problem

$$\begin{cases} g''(t) = \dfrac{-t}{g(t)}, \ 0 \le \xi < t < 1, \\ g(\xi) = \sigma > 0, \ g'(\xi) = 0. \end{cases} \tag{24}$$

Let $g(t)$ be the positive solution of the boundary value problem (16)–(17) and $[\xi, t_\sigma^*)$ be the maximal interval of existence of solutions of the initial value problem (24). Then we may establish the following lemmas.

Lemma 4 *(i) Let g_1 and g_2 be the solutions of the initial value problem* (24) *for $\sigma = \sigma_1$ and $\sigma = \sigma_2$, respectively. If $\sigma_1 < \sigma_2$, then $t_{\sigma_1}^* < t_{\sigma_2}^*$.*

(ii) t_σ^* is a continuous function of σ and $t_\sigma^* \to +\infty$ as $\sigma \to +\infty$.

The proof of this lemma is similar to that of [9, Lemmas 1, 2 and 3], so we omit it here.

Lemma 5 *For any fixed $h > 0$, the positive solution $g_h(t)$ of the boundary value problem (20) satisfies*

$$g_h(\xi) > \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}, \quad 0 \le \xi < 1.$$

Proof In terms of the initial value problem (24), for $t \in (\xi, 1)$

$$g(t) < \sigma - \frac{1}{6\sigma}(t^3 - 3\xi^2 t + 2\xi^3) < \sigma - \frac{1}{6\sigma}(t^3 - 3\xi^2 + 2\xi^3).$$

Let $f(t) = \sigma - \frac{1}{6\sigma}(t^3 - 3\xi^2 + 2\xi^3)$. Then the positive solution of the initial value (24) satisfies $g(t) < f(t)$ for $t \in (\xi, 1)$. In terms of Lemma 4, assume $f(t)$ intersects the t -axis at the point t_0^*. Then $t_0^* = \sqrt[3]{6\sigma^2 + 3\xi^2 - 2\xi^3}$. Especially for $t_0^* = 1$ this yields

$$\sigma = \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}, \quad 0 \le \xi < 1.$$

Similar to Lemma 1, we may show the positive solution of the initial value problem (24) is increasing with σ, so the positive solutions $g(t, \sigma)$ of the initial value problem (24) cannot intersect the point 1 for $\sigma \leq \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}$. This implies that for $\sigma \leq \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}$, the positive solution of the initial value problem (24) can not satisfy $g(1) \geq 0$. This shows that for any fixed $h > 0$, the positive solution $g_h(t)$ of the boundary value problem (20) satisfies

$$g_h(\xi) > \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}, \quad 0 \leq \xi < 1.$$

Theorem *The boundary value problems (16)–(17) has a unique positive solution, for any* $\xi \in [0, 1)$*, and satisfies the following estimate formula*

$$\sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]} < g(\xi) < \frac{3(1-\xi)(1-\xi^2)}{\sqrt{6[1-\xi^2(3-2\xi)]}}.$$

Proof Lemmas 2 and 3 show that for any $h > 0$, the boundary value problem (20) has a unique positive solution. Then for any $h_2 > h_1 > 0$, in terms of equation (23) and Lemma 1,

$$0 < g_{h_2}(t) - g_{h_1}(t) = h_2 - h_1 + \int_\xi^1 G_3(t,s)\Big[\frac{s}{g_{h_2}(s)} - \frac{s}{g_{h_1}(s)}\Big]\mathrm{d}s \leq h_2 - h_1.$$

This indicates the series of positive solutions $\{g_h(t)\}$ converges uniformly to a limit with h on $[\xi, 1]$, denoted by $g_0(t)$. Then

$$\lim_{h\to 0} g_h(t) = g_0(t),\ t \in [\xi,\ 1].$$

Lemma 5 implies $g_0(\xi) > \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}$, $0 \leq \xi < 1$. For any $h \geq 0$, by the convexity of $g_h(t)$, this yields

$$\begin{aligned} g_h(t) &\geq h + \frac{g_h(\xi)-h}{\xi-1}\ (t-1) = h + \frac{g_h(\xi)}{\xi-1}\ (t-1) - \frac{h}{\xi-1}\ (t-1) \\ &\geq \frac{\sqrt{6[1-\xi^2(3-2\xi)]}}{6(\xi-1)}\ (t-1). \end{aligned} \tag{25}$$

It follows from inequality (25) for $h \geq 0$ and the right integral function of equation (23) that

$$G_3(t,s)\frac{s}{g_h(s)} \leq G_3(t,s)\frac{6(\xi-1)s}{\sqrt{6[1-\xi^2(3-2\xi)]}(s-1)}. \tag{26}$$

We get

$$g_h(t) = h + \int_\xi^1 G_3(t,s)\frac{s}{g_h(s)}\mathrm{d}s.$$

Let $h \to 0^+$ and use the Monotone Convergence Theorem[10], we obtain:

$$g_0(t) = \lim_{h\to 0}\int_\xi^1 G_3(t,s)\frac{s}{g_h(s)}\mathrm{d}s = \int_\xi^1 \lim_{h\to 0} G_3(t,s)\frac{s}{g_h(s)}\mathrm{d}s,$$

i.e.,

$$g_0(t) = \int_\xi^1 G_3(t,s)\frac{s}{g_0(s)}\mathrm{d}s. \tag{27}$$

The above arguments indicate that the boundary value problems (16)–(17) have a unique positive solution $g_0(t)$. Furthermore, we can obtain the following formula by (26) and (27)

$$\sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]} < g(\xi) < \frac{3(1-\xi)(1-\xi^2)}{\sqrt{6[1-\xi^2(3-2\xi)]}}, \quad 0 \leq \xi < 1. \qquad \square$$

This proves that the boundary value problems (16)–(17) have a unique positive solution $g(t)$, satisfying

$$\sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]} < g(\xi) < \frac{3(1-\xi)(1-\xi^2)}{\sqrt{6[1-\xi^2(3-2\xi)]}}, \quad 0 \le \xi < 1. \tag{28}$$

In order to illustrate the reliability and efficiency of the proposed theoretical results. we compare the results of the estimate formula (28) with numerical results by solving the boundary value problems (16)–(17) for different velocity ratio parameter ξ. For numerical results we can refer to Table 1 and Figure 1.

For the sake of simplicity of comparison, denote the skin friction $g(\xi)$ obtained by numerical calculation by $\sigma_{com} = g(\xi)$, and the estimated results are obtained by estimate formula (28) by $\sigma_{\text{lower-bound}} = \sqrt{\frac{1}{6}[1-\xi^2(3-2\xi)]}$ and $\sigma_{\text{upper-bound}} = \frac{3(1-\xi)(1-\xi^2)}{\sqrt{6[1-\xi^2(3-2\xi)]}}$, respectively. The skin friction coefficients obtained by numerical calculation for different velocity ratio parameter ξ and by formula (28) are presented in Table 1. The reliability and efficiency of the theoretical estimate formula (28) are verified by numerical results.

ξ	$\sigma_{\text{lower-bound}}$	$\sigma_{com} = g(\xi)$	$\sigma_{\text{upper-bound}}$
ξ=0.0	0.4082	0.4683	1.2247
ξ=0.2	0.3684	0.4421	0.9937
ξ=0.4	0.3286	0.3743	0.7668
ξ=0.6	0.2442	0.2747	0.5285
ξ=0.8	0.1317	0.1489	0.2734
ξ=0.9	0.0683	0.0772	0.1391

Table 1 The The skin friction coefficient obtained numerically and by formula (28)

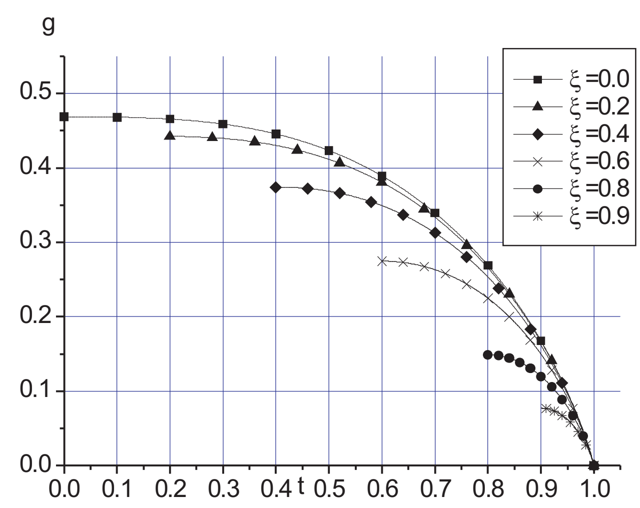

Figure 1 Dimensionless shear stress profiles for $\xi = 0.0$ to 0.9

Table 1 shows that all numerical results lie in the range that are estimated by formula (28).

When velocity ratio parameter ξ is smaller, the error that is estimated by lower-bound of formula (28) is bigger. But, with the increasing of velocity ratio parameter ξ, the formula (28) goes more and more reliable and efficient. Especially, for appropriately big velocity ratio parameter ξ, we can consider the results that are estimated by lower-bound of formula (28) as approximate value of skin fiction coefficient.

Fig.1 shows the characteristics of dimensionless shear stress $g(t)$ for $\xi = 0.0$ to 0.9 by solving the boundary value problems (16)–(17). The results indicate that the skin friction σ decreases with increasing of the velocity ratio parameter ξ, i.e., the skin friction is a decrease function of velocity ratio parameter. For each fixed value of ξ, dimensionless shear stress is a decrease function of dimensionless tangential velocity in $[\xi, 1]$, and $\xi = 0$ for classical Balasis solution. Clearly, all results are completely consistent with the results obtained by theoretical analysis.

3.4 The solutions of the boundary value problems (18)–(19)

Utilizing the unique analytical solution of the boundary value problems (16)–(17), the solution of the boundary value problems (18)–(19) is established and represented as follows:

$$\begin{aligned} w(t) = &-\frac{Pr\cdot u_e^2}{C_{p,\,w}T_e}\int_\xi^t (g(s))^{1-Pr}ds\int_\xi^t (g(s))^{Pr-1}\mathrm{d}s+ \\ &\frac{1-\lambda-\dfrac{Pr\cdot u_e^2}{C_{p,\,w}T_e}\displaystyle\int_\xi^1 (g(s))^{1-Pr}\Big(\int_\xi^s (g(x))^{Pr-1}\mathrm{d}x\Big)\mathrm{d}s}{\int_\xi^1 g^{Pr-1}(s)\mathrm{d}s}\int_\xi^t (g(s))^{Pr-1}\mathrm{d}s+ \\ &\frac{Pr\cdot u_e^2}{C_{p,\,w}T_e}\int_\xi^1 (g(s))^{1-Pr}\mathrm{d}s\int_\xi^1 (g(s))^{Pr-1}\mathrm{d}s+ \\ &\frac{Pr\cdot u_e^2}{C_{p,\,w}T_e}\int_\xi^t (g(s))^{1-Pr}\Big(\int_\xi^s (g(x))^{Pr-1}\mathrm{d}x\Big)\mathrm{d}s+\lambda. \end{aligned}$$

For $pr = 1$, we can obtain

$$w(t) = -\frac{u_e^2}{2C_{p,\,w}T_e}(t-\xi)^2 + \frac{1-\lambda+\dfrac{u_e^2}{2C_{p,\,w}T_e}(1-\xi)^2}{1-\xi}(t-\xi)+\lambda.$$

This shows that the temperature distribution $w(t)$ has a parabolic distribution with tangential velocity t.

4. Conclusions

A class of singular nonlinear boundary value problems arising in the boundary layer behind expansion wave are studied. Sufficient conditions for the existence and uniqueness of positive solutions to the problems are established by utilizing the monotonic approaching technique. And a theoretical estimate formula for skin friction coefficient is presented. The numerical solution is presented by using the shoot method. The reliability and efficiency of the theoretical prediction are verified by numerical results.

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