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Three Nonnegative Solutions of Three-Point Boundary Value Problem for Second-Order Impulsive Differential Equations

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Abstract The paper studies the existence of three nonnegative solutions to a type of three-point boundary value problem for second-order impulsive differential equations, and obtains the sufficient conditions for existence of three nonnegative solutions by means of the Leggett-Williams's fixed point theorem.

Keywords impulsive; three-point boundary value problem; Leggett-Williams's fixed point theorem; nonnegative solutions.

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1. Introduction

In this paper, we study the existence of three nonnegative solutions to a type of three-point boundary value problem for the second-order impulsive differential equation

$$\begin{cases} -x''(t) = f(t, x(t)) \quad \text{for} \quad t \in (0,1)\backslash\{t_1, \ldots, t_m\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \Delta x'|_{t=t_k} = 0, \quad k = 1, 2, \ldots, m, \\ x(0) = \beta x(\xi), \quad x'(1) = 0, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times R_+, R_+)$, $R_+ = [0, +\infty)$, $0 < t_1 < t_2 < \cdots < t_m < 1$, β, $\xi \in (0,1)$, $I_k \in C(R_+, R_+)$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $x(t_k^+)$ and $x(t_k^-)$ denote the right limit and left limit of $x(t)$ at $t = t_k$, $k = 1, 2, \ldots, m$, respectively. Also $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$.

By means of the Leggett-Williams's fixed point theorem, we obtain the sufficient conditions for existence of three nonnegative solutions in which there is a positive solution at least.

In many problems of science and technology, the impulsive phenomenon exists widely, especially in engineering, physics, communication, science of life and economic field. Impulsive differential equations describe processes which experience a sudden change of their state at certain moments. They, under some circumstances, could express the certain regulation of the matters

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more accurately than the classical differential equations[1]. Therefore, it is very important to study impulsive differential equations.

Guo[2] used fixed point index theory for cone mappings to investigate the existence of multiple positive solutions of a boundary value problem

$$\begin{cases} -x''(t) = f(t, x(t)), & t \neq t_k, k = 1, 2, \dots, m, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, m, \\ ax(0) - bx'(0) = \theta, & cx(1) + dx'(1) = \theta. \end{cases}$$

In [3], by using the Leggett Williams fixed point theorem, some results were obtained which guarantee the existence of three nonnegative solutions to the second order impulsive differential equations

$$\begin{cases} y''(t) + \phi(t)f(y(t)) = 0 \quad \text{for} \quad t \in (0,1)\backslash\{t_1, \dots, t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \\ \Delta y'|_{t=t_k} = J_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \\ y(0) = y(1) = 0. \end{cases}$$

He[4] used the method of upper and lower solutions and monotone iterative to investigate the existence of maximal and minimal solutions of the periodic boundary value problem for first order impulsive functional differential equations. Zhang[5] obtained results of existence for first order non-homogeneous boundary value problem of impulsive differential equations by means of the method of upper and lower solutions coupled with the monotone iterative technique. In [6] was investigated the existence of solutions to the second order impulsive integro-differential equations by using Leray-Schauder continuous theorem of the condensing mapping,

2. Preliminaries

Denote $J = [0, 1]$ and $PC[J, R] = \{x | x : J \to R,\ x$ is continuous for $t \neq t_k$, right limit $x(t_k^+)$ and left limit $x(t_k^-)$ exist, and $x(t_k) = x(t_k^-)$ at $t = t_k$, for $k = 1, 2, \dots, m\}$. Obviously, $PC[J, R]$ is a Banach space with norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$. Denote $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \dots, J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 1]$, and $J' = J\backslash\{t_1, t_2, \dots, t_m\}$.

Definition 2.1 *x is said to be a solution of boundary value problem (1.1), if $x \in PC[J, R] \bigcap C^2[J', R]$ and satisfies (1.1).*

Lemma 2.1[7] *$H \subset PC[J, R]$ is a relatively compact set if and only if H is uniform bounded in J and equicontinuous in all J_k, $k = 1, 2, \dots, m$.*

Let

$$G(t, s) = \frac{1}{1-\beta} \begin{cases} s, & s < \xi, s < t; \\ \beta s + (1-\beta)t, & t \leq s \leq \xi; \\ \beta\xi + s(1-\beta), & \xi \leq s \leq t; \\ \beta\xi + t(1-\beta), & t < s, \xi < s. \end{cases}$$

Lemma 2.2 *$x \in PC[J, R] \bigcap C^2[J', R]$ is a solution of the boundary value problem (1.1) if and* *only if $x \in PC[J,R]$ is a solution of the integral equation*

$$x(t) = \int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k)). \tag{2.1}$$

Proof Suppose $x \in PC[J,R]\bigcap C^2[J',R]$ is a solution of the boundary value problem (1.1), and let $t_0 = 0$. By virtue of mean value theorem of differentials, we obtain

$$x(t_k) - x(t_k - h) = x'(\xi_k)h \quad \text{for} \quad 0 < h < t_k - t_{k-1},$$

where $\xi_k \in (t_k - h, t_k)$. By (1.1), we have $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-) = 0$. Then the left derivative $x'_-(t_k)$ exists, and

$$x'_-(t_k) = \lim_{h\to 0^+} \frac{x(t_k) - x(t_k - h)}{h} = \lim_{\xi_k \to t_k^-} x'(\xi_k) = x'(t_k^-).$$

Set $x'(t_k) = x'(t_k^-)$, $k = 1, 2, \ldots, m$. Then

$$x'(t) = x'(0) - \int_0^t f(s,x(s))\mathrm{d}s \quad \text{for} \quad 0 \le t \le t_1$$

$$x'(t_1) = x'(0) - \int_0^{t_1} f(s,x(s))\mathrm{d}s. \tag{2.2}$$

It follows from (1.1) and (2.2) that

$$x'(t) = x'(t_1^+) - \int_{t_1}^t f(s,x(s))\mathrm{d}s = x'(0) - \int_0^t f(s,x(s))\mathrm{d}s + x'(t_1^+) - x'(t_1)$$

for $t_1 < t \le t_2$. Similarly, we have

$$x'(t) = x'(0) - \int_0^t f(s,x(s))\mathrm{d}s + \sum_{0<t_k<t}[x'(t_k^+) - x'(t_k)] \quad \text{for all} \quad t \in J.$$

Therefore

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)f(s,x(s))\mathrm{d}s \quad \text{for} \quad 0 \le t \le t_1,$$

$$x(t_1) = x(0) + x'(0)t_1 - \int_0^{t_1} (t_1 - s)f(s,x(s))\mathrm{d}s$$

and for $t_1 < t \le t_2$

$$\begin{aligned} x(t) =& x(t_1^+) + x'(0)(t-t_1) - \int_0^{t_1}(t-t_1)f(s,x(s))\mathrm{d}s - \int_{t_1}^t (t-s)f(s,x(s))\mathrm{d}s + (x'(t_1^+) - x'(t_1))(t-t_1) \\ =& x(0) + x'(0)t - \int_0^t (t-s)f(s,x(s))\mathrm{d}s + (x'(t_1^+) - x'(t_1))(t-t_1) + (x(t_1^+) - x(t_1)). \end{aligned}$$

In the same way, we can show

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)f(s,x(s))\mathrm{d}s + \sum_{0<t_k<t}[x'(t_k^+) - x'(t_k)](t-t_k) + \sum_{0<t_k<t}[x(t_k^+) - x(t_k^-)]$$

for all $t \in J$.

Since $\Delta x|_{t=t_k} = I_k(x(t_k))$, $\Delta x'|_{t=t_k} = 0, k = 1, 2, \dots, m$, we have

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)f(s, x(s))\mathrm{d}s + \sum_{0<t_k<t} I_k(x(t_k)) \quad \text{for all} \quad t \in J. \tag{2.3}$$

Combining the boundary conditions of the problem (1.1), we obtain

$$x'(1) = x'(0) - \int_0^1 f(s, x(s))\mathrm{d}s, \quad x'(0) = \int_0^1 f(s, x(s))\mathrm{d}s.$$

By (2.3), we have $x(\xi) = x(0) + x'(0)\xi - \int_0^\xi (\xi - s)f(s, x(s))\mathrm{d}s + \sum_{0<t_k<\xi} I_k(x(t_k))$. But $x(0) = \beta x(\xi)$, we get

$$x(0) = \frac{\beta}{1-\beta}\Big[\xi \int_0^1 f(s, x(s))\mathrm{d}s - \int_0^\xi (\xi - s)f(s, x(s))\mathrm{d}s + \sum_{0<t_k<\xi} I_k(x(t_k))\Big].$$

Hence

$$\begin{aligned} x(t) =& \frac{\beta\xi}{1-\beta}\int_0^1 f(s, x(s))\mathrm{d}s - \frac{\beta}{1-\beta}\int_0^\xi (\xi - s)f(s, x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \\ & t\int_0^1 f(s, x(s)\mathrm{d}s - \int_0^t (t-s)f(s, x(s)\mathrm{d}s + \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned}$$

1^0 For all $t \le \xi$

$$\begin{aligned} x(t) =& \int_0^t \frac{s}{1-\beta} f(s, x(s))\mathrm{d}s + \int_t^\xi \frac{\beta s + t(1-\beta)}{1-\beta} f(s, x(s))\mathrm{d}s + \int_\xi^1 \frac{\beta\xi + t(1-\beta)}{1-\beta} f(s, x(s))\mathrm{d}s + \\ & \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned}$$

2^0 For all $t \ge \xi$

$$\begin{aligned} x(t) =& \int_0^\xi \frac{s}{1-\beta} f(s, x(s))\mathrm{d}s + \int_\xi^t \frac{\beta\xi + s(1-\beta)}{1-\beta} f(s, x(s))\mathrm{d}s + \int_t^1 \frac{\beta\xi + t(1-\beta)}{1-\beta} f(s, x(s))\mathrm{d}s + \\ & \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned}$$

Then for all $t \in J$

$$x(t) = \int_0^1 G(t, s)f(s, x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k)).$$

On the other hand, if $x \in PC[J, R]$ is a solution of the integral equation (2.1), it is easy to obtain $x \in PC[J, R] \bigcap C^2[J', R]$ is a solution of the boundary value problem (1.1) by (2.1). □

The operator $A : PC[J, R] \to PC[J, R]$ is defined by

$$(Ax)(t) = \int_0^1 G(t, s)f(s, x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k)) \quad \text{for} \quad t \in J. \tag{2.4}$$

Obviously, we have the following

Lemma 2.3 *$x \in PC[J, R] \bigcap C^2[J', R]$ is a solution of the boundary value problem (1.1) if and* *only if $x \in PC[J, R]$ is a fixed point of A.*

Lemma 2.4 *The function G satisfies*

(1) $G(t,s) \le \frac{s}{1-\beta}$ for all $t, s \in J$, and $G(t,s) \ge \frac{\beta}{1-\beta}\min\{s,\xi\}$ for all $t \in J$;

(2) $\int_0^1 G(t,s)\mathrm{d}s = \frac{\beta\xi(2-\xi)}{2(1-\beta)} + t - \frac{1}{2}t^2$, and $\min_{t\in[\delta,1]}\int_0^1 G(t,s)\mathrm{d}s = \frac{\beta\xi(2-\xi)}{2(1-\beta)} + \frac{\delta(2-\delta)}{2}$ for $\max\{t_m, \xi\} < \delta < 1$.

Lemma 2.5 *$A : PC[J, R] \to PC[J, R]$ is a completely continuous operator.*

Proof By (2.4), it is easy to see that A is a continuous operator. Since

$$(Ax)'(t) = \int_0^1 \frac{\partial G}{\partial t} f(s, x(s))\mathrm{d}s, \quad t \in J, \quad t \ne t_k, \quad k = 1, 2, \ldots, m$$

and $|\frac{\partial G}{\partial t}| \le 1$, $A(S)$ is uniform bounded in J and equicontinuous in all J_k for any bounded set $S \in PC[J, R]$, $k = 1, 2, \ldots, m$.

Therefore, it follows from Lemma 2.1 that A is a completely continuous operator. □

Let $E = (E, \|\cdot\|)$ be a Banach space and $P \subset E$ be a cone on E. A continuous mapping $\omega : P \longrightarrow [0, +\infty)$ is said to be a concave nonnegative continuous functional on P, if ω satisfies $\omega\big(\lambda x + (1-\lambda)y\big) \ge \lambda\omega(x) + (1-\lambda)\omega(y)$ for all $x, y \in P$ and $\lambda \in [0, 1]$.

Let $a, b, d > 0$ be constants. Define $P_d = \{x \in P : \|x\| < d\}$, $\overline{P}_d = \{x \in P : \|x\| \le d\}$ and $P(\omega, a, b) = \{x \in P : \omega(x) \ge a, \|x\| \le b\}$. In order to prove our main results, we need the following Legget-Williams fixed point theorem[7,8].

Lemma 2.6 *Let $(E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E and $c > 0$ be a constant. Suppose there exists a concave nonnegative continuous functional ω on P with $\omega(x) \le \|x\|$ for all $x \in \overline{P}_c$. Let $A : \overline{P}_c \longrightarrow \overline{P}_c$ be a completely continuous operator. Assume there are numbers a, b and d with $0 < d < a < b \le c$ such that*

(H_1) $\{x \in P(\omega, a, b) : \omega(x) > a\} \ne \varnothing$ and $\omega(Ax) > a$ for all $x \in P(\omega, a, b)$;

(H_2) $\|Ax\| < d$ for all $x \in \overline{P}_d$;

(H_3) $\omega(Ax) > a$ for all $x \in P(\omega, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore,

$$x_1 \in P_a;\ x_2 \in \{x \in P(\omega, b, c) : \omega(x) > b\};\ x_3 \in \overline{P}_c \setminus (P(\omega, b, c) \cup \overline{P}_a).$$

3. Main results

Let $E = PC[J, R]$ and $P = PC[J, R_+] = \{x | x : J \to R_+,\ x$ is continuous for $t \ne t_k$, $x(t_k^+)$ and $x(t_k^-)$ exist, $x(t_k) = x(t_k^-)$, $k = 1, 2, \ldots, m\}$. Then P is a cone of $E = PC[J, R]$.

Take δ satisfying $\max\{t_m, \xi\} < \delta < 1$. Define $\omega : P \to R_+$ with $\omega(x) = \min_{t\in[\delta,1]} x(t)$. Then ω is a concave nonnegative continuous functional on P, and satisfies $\omega(x) \le \| x \|$ for all $x \in P$.

Denote $\sigma = \max_{(t,s)\in J\times J} G(t,s)$, $i_k = \inf_{x\in[0,+\infty)} I_k(x)$, $k = 1, 2, \ldots, m$ and $a = 1 + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} i_k + \sum_{k=1}^m i_k$.

(A$_0$) There exist constants $0 < \gamma$, $\gamma_k < +\infty$ such that $\overline{\lim}_{x\to+\infty}\frac{f(t,x)}{x} < \gamma$ holds uniformly for t, and $\overline{\lim}_{x\to+\infty}\frac{I_k(x)}{x} < \gamma_k$, for $k = 1, 2, \dots, m$.

Theorem 3.1 *Suppose A_0 holds, and $\sigma\gamma + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi}\gamma_k + \sum_{k=1}^m \gamma_k < 1$. There exist constants b and d with $0 < d < a < \frac{a}{\beta\xi} < b$, such that the following conditions hold*

(A$_1$) $f(t,x) < (1-\beta)d$ for all $(t,x) \in J\times[0,d]$;

(A$_2$) $I_k(x) \le \min\{\frac{d}{4m}, \frac{(1-\beta)d}{4m\beta}\}$, $k = 1, 2, \dots, m$ for all $x \in [0,d]$;

(A$_3$) $f(t,x) > \frac{2(1-\beta)}{\beta\xi(2-\xi)+\delta(2-\delta)(1-\beta)}$ for all $(t,x) \in [\delta,1]\times[a,b]$.

Then the boundary value problem (1.1) has at least three nonnegative solutions x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore the nonnegative solutions $x_1 \in P_a$ and $x_2 \in \{x \in P(\omega,b,c) : \omega(x) > b\}$, the positive solution $x_3 \in \overline{P}_c \setminus (P(\omega,b,c)\cup\overline{P}_a)$.

Proof It follows from (A$_0$) that there exists $\tau > 0$ such that $0 \le f(t,x) \le \gamma x$, $I_k(x) \le \gamma_k x$ for all $t \in J$ and $x > \tau$.

Let $M = \max_{(t,x)\in J\times[0,\tau]} f(t,x)$, $M_k = \max_{x\in[0,\tau]} I_k(x)$. Then

$$0 \le f(t,x) \le \gamma x + M \quad \text{for} \quad t \in J \quad \text{and} \quad x \ge 0$$

$$I_k(x) \le \gamma_k x + M_k \quad \text{for} \quad x \ge 0, \quad k = 1, 2, \dots, m.$$

Take $c > \frac{\sigma M + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} M_k + \sum_{k=1}^m M_k}{1-(\sigma\gamma + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi}\gamma_k + \sum_{k=1}^m \gamma_k)}$. Then as $\|x\| \le c$, for all $t \in J$, we have

$$\begin{aligned}
0 \le |(Ax)(t)| &\le \sigma(\gamma\|x\| + M) + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi}(\gamma_k\|x\| + M_k) + \sum_{k=1}^m(\gamma_k\|x\| + M_k) \\
&\le (\sigma\gamma + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi}\gamma_k + \sum_{k=1}^m \gamma_k)c + (\sigma M + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} M_k + \sum_{k=1}^m M_k) \\
&< c.
\end{aligned}$$

Therefore, $\|Ax\| < c$.

By Lemma 2.5, we obtain that $A : \overline{P}_c \to \overline{P}_c$ is a completely continuous operator.

Take $u_0 = \frac{a+b\beta\xi}{2\beta\xi}$. Then $\omega(u_0) = \frac{a+b\beta\xi}{2\beta\xi} > a$ and $\|u_0\| = \frac{a+b\beta\xi}{2\beta\xi} < b$. Thus $u_0 \in \{x \in P(\omega,a,b) : \omega(x) > a\} \ne \varnothing$. For $x \in P(\omega,a,b)$, by (A$_3$), we have

$$\begin{aligned}
\omega(Ax) &= \min_{t\in[\delta,1]}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big) \\
&> \min_{t\in[\delta,1]}\Big(\frac{2(1-\beta)}{\beta\xi(2-\xi)+\delta(2-\delta)(1-\beta)}\int_0^1 G(t,s)\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} i_k + \sum_{k=1}^m i_k\Big) \\
&= 1 + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} i_k + \sum_{k=1}^m i_k = a.
\end{aligned}$$

So the condition (H$_1$) of Lemma 2.6 holds.

It follows from Lemma 2.4, (A$_1$) and (A$_2$) that for $x \in \overline{P}_d = \{x \in P : \|x\| \le d\}$

$$\|Ax\| = \sup_{t\in J}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big)$$

$$\leq \frac{1}{1-\beta}\int_0^1 sf(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{k=1}^{m} I_k(x(t_k))$$
$$< d\int_0^1 s\mathrm{d}s + \frac{\beta}{1-\beta}\frac{md(1-\beta)}{4\beta m} + \frac{dm}{4m} = d.$$

Hence, the condition (H_2) of Lemma 2.6 holds.

Since $\|x\| \leq c$, $\omega(x) \geq a$ and $\|Ax\| > b$ as $x \in P(\omega, a, c)$ and $\|Ax\| > b$, by Lemma 2.4 and (A_3), we can show that

$$\omega(Ax) = \min_{t\in[\delta,1]}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big)$$
$$\geq \min_{t\in[\delta,1]}\frac{1}{1-\beta}\Big(\int_0^\xi \beta sf(s,x(s))\mathrm{d}s + \int_\xi^1 \beta\xi f(s,x(s))\mathrm{d}s\Big) + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{k=1}^{m} I_k(x(t_k))$$
$$> \frac{\beta\xi}{1-\beta}\Big(\int_0^\xi sf(s,x(s))\mathrm{d}s + \int_\xi^1 sf(s,x(s))\mathrm{d}s\Big) + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{k=1}^{m} I_k(x(t_k))$$
$$\geq \beta\xi\sup_{t\in J}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big)$$
$$= \beta\xi\|Ax\| > a.$$

Therefore the condition (H_3) of Lemma 2.6 holds.

Then the boundary value problem (1.1) has at least three nonnegative solutions x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore the nonnegative solutions $x_1 \in P_a$ and $x_2 \in \{x \in P(\omega, b, c) : \omega(x) > b\}$, and the positive solution $x_3 \in \overline{P}_c \setminus (P(\omega, b, c) \cup \overline{P}_a)$.

Therefore the theorem is proved. □

When the condition (A_0) holds, it is easy to see that for all $b_0 > b$ and $x \in [b, b_0]$, there is no restriction on the growth of f and I_k, $k = 1, 2, \ldots, m$. If the restriction of the condition (A_0), for f and $I_k, k = 1, 2, \ldots, m$ as $x \to +\infty$, is removed, we can get

Theorem 3.2 *Suppose that there exist constants b, c, d with $0 < d < a < \frac{a}{\beta\xi} < b < c$ such that (A_1) and (A_3) hold, and*

(A_4) $f(t,x) < (1-\beta)c$ for all $(t,x) \in J \times [0,c]$;

(A_5) $I_k(x) \leq \frac{d}{4m}\min\{1, \frac{(1-\beta)}{\beta}\}$ for all $x \in [0,c]$, $k = 1, 2, \ldots, m$.

Then the boundary value problem (1.1) has at least three nonnegative solutions x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore the nonnegative solutions $x_1 \in P_a$ and $x_2 \in \{x \in P(\omega, b, c) : \omega(x) > b\}$, and the positive solution $x_3 \in \overline{P}_c \setminus (P(\omega, b, c) \cup \overline{P}_a)$.

Proof Since $\frac{2(1-\beta)}{\beta\xi(2-\xi)+\delta(2-\delta)(1-\beta)} < \frac{(1-\beta)}{\beta\xi} < \frac{(1-\beta)}{\beta\xi} < (1-\beta)c$, conditions ($A_3$) and ($A_4$) are reasonable. The condition (A_4) implies (A_2).

As $\|x\| \leq c$, we can show that

$$\|Ax\| = \sup_{t\in J}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big)$$

$$\leq \frac{1}{1-\beta}\int_0^1 sf(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{k=1}^{m} I_k(x(t_k))$$

$$< c\int_0^1 s\mathrm{d}s + \frac{\beta}{1-\beta}\frac{mc(1-\beta)}{4\beta m} + \frac{cm}{4m} < c.$$

Therefore, $\|Ax\| < c$.

The following proof is similar to that in Theorem 3.1, and the theorem is proved. □

Theorem 3.3 *Suppose that there exists a constant b with $0 < a < \frac{a}{\beta\xi} < b$ such that (A_3) holds. Either (A_0) holds, or both (A_4) and (A_5) hold. And*

(A_6) $f(t,0) = 0$ for $t \in J$, $\overline{\lim}_{x\to 0^+}\frac{f(t,x)}{x} < 1-\beta$ holds uniformly on t. $I_k(0) = 0$, $\overline{\lim}_{x\to 0^+}\frac{I_k(x)}{x} < \frac{1}{4m}\min\{1\ \ \frac{(1-\beta)}{\beta}\}$, $k = 1, 2, \ldots, m$;

Then the boundary value problem (1.1) has at least three nonnegative solutions x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore the $x_1 \in P_a$ and $x_2 \in \{x \in P(\omega, b, c) : \omega(x) > b\}$, and the positive solution $x_3 \in \overline{P}_c \setminus (P(\omega, b, c) \cup \overline{P}_a)$.

Proof By (A_6), there exists $0 < d \leq 1$, such that $0 \leq f(t,x) \leq (1-\beta)x$ for all $t \in J$ and $0 \leq x \leq d$. $I_k(x) \leq \frac{1}{4m}\min\{1\ \ \frac{(1-\beta)}{\beta}\}x$ for $0 \leq x \leq d$, $k = 1, 2, \ldots, m$.

As $\|x\| \leq d$, it is easy to see

$$\|Ax\| = \sup_{t\in J}\Big(\int_0^1 G(t,s)f(s,x(s))\mathrm{d}s + \frac{\beta}{1-\beta}\sum_{0<t_k<\xi} I_k(x(t_k)) + \sum_{0<t_k<t} I_k(x(t_k))\Big)$$

$$< \Big(\int_0^1 s\mathrm{d}s + \frac{\beta}{1-\beta}\frac{m(1-\beta)}{4\beta m} + \frac{m}{4m}\Big)\|x\| = \|x\|.$$

We get $\|Ax\| < d$.

The following proof is similar to that in Theorem 3.1, and the theorem is proved. □

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