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Asymptotically Isometric Copies of l^∞ in Some Banach Lattices

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Abstract In this paper, we show that any σ-complete Banach lattice, with a σ-order semi-continuous but not σ-order continuous norm, contains an asymptotically isometric copy of l^∞. We also get that the Fenchel-Orlicz space with the Orlicz norm may not contain an asymptotically isometric copy of l^∞.

Keywords asymptotically isometric copy; Banach lattice; Fenchel-Orlicz space.

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1. Introduction

Banach spaces X and Y are said to be isomorphic if there is a one-to-one bounded linear operator T from X onto Y. Two Banach spaces X and Y are said to be $(1+\varepsilon)$-isometric provided there exists an isomorphism $T : X \to Y$ such that $\|x\|_X \le \|Tx\|_Y \le (1+\varepsilon)\|x\|_X$ for all $x \in X$. A Banach space X is said to contain an almost isometric copy of Y if for any $\varepsilon > 0$ there is a subspace Z in X such that Z and Y are $(1+\varepsilon)$-isometric. A Banach space is said to contain an asymptotically isometric copy of l^∞ if there is a null sequence $\{\varepsilon_n\}$ in $(0,1)$ and a bounded linear operator $T : l^\infty \to X$ so that

$$\sup(1-\varepsilon_n)|\xi_n| \le \|T((\xi_n)_n)\| \le \sup|\xi_n|$$

for all $(\xi_n)_n$ in l^∞. Among the above definitions, the latter can imply the former, but the converse is not true.

For two Banach lattices X and Y, we shall say that they are (lattice) isomorphic if there is a mapping $T : X \to Y$ which is an isomorphism between Banach spaces and is order preserving, that is, $Tx \ge 0$ if and only if $x \ge 0$. Furthermore, we can give the corresponding definitions of the above notions for two Banach lattices, that is, the mappings in the above definitions are all order preserving.

Let us recall the relative results. Krivine[1] proved that if a Banach space X contains l_n^p's $(1+\varepsilon)$-isometric uniformly for some $\varepsilon > 0$ and some $1 \le p \le \infty$, then it also contains them

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almost isometrically. James[2] showed that a Banach space X contains an almost isometric copy of l^1 or c_0, whenever it contains an isomorphic copy of l^1 or c_0. For l^p $(1 < p < \infty)$, Odell and Schlumprecht[3] proved that there is a Banach space X isomorphic to l^p $(1 < p < \infty)$, but it does not contain any almost isometric copy of l^p. Partinton[4] showed that a Banach space X contains an almost isometric copy of l^∞, whenever it contains an isomorphic copy of l^∞. Hudzik and Mastylo[5] showed that any σ-complete Banach lattice, with a order semi-continuous norm containing a isomorphic copy of l^∞, contains an almost isometric copy of l^∞.

Let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+_e$ stand for the reals, nonnegative reals and extended (by $+\infty$) nonnegative reals. (Ω, Σ, μ) denotes a σ-finite measure space. A continuous, convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an Orlicz function if

$$\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$$

and

$$\lim_{t\to \infty} \frac{\varphi(t)}{t} = \infty.$$

Let X be a Banach space. Let $L^0(\Omega, X)$ be the set of all X valued (equivalence classes of) strongly Σ-measurable functions defined on Ω. The Orlicz function φ generates a functional $I_\varphi : L^0(\Omega, X) \to \mathbb{R}^+_e$ by

$$I_\varphi(f) = \int_\Omega \varphi(\|f(t)\|)\mathrm{d}\mu.$$

The set

$$L_\varphi(\Omega, X) = \{f \in L^0(\Omega, X); I_\varphi(\lambda f) < +\infty \quad \text{for some } \lambda > 0\}$$

is called an Orlicz space. The Orlicz function φ generates the Luxemburg norm defined for $f \in L_\varphi(\Omega, X)$ by

$$\|f\|_\varphi := \inf\{\lambda > 0; I_\varphi(\frac{f}{\lambda}) \le 1\},$$

as well as the Orlicz norm

$$\|f\|^0_\varphi := \inf\{\lambda^{-1}(1 + I_\varphi(\lambda f)); \lambda > 0\}.$$

Note that $(L_\varphi(\Omega, X), \|\cdot\|_\varphi)$ is a Banach space. The Luxemburg norm and the Orlicz norm generated by the same Orlicz function φ are equivalent, satisfying

$$\|f\|_\varphi \le \|f\|^0_\varphi \le 2\|f\|_\varphi \quad \text{for any } f \in L_\varphi(\Omega, X).$$

Let ψ be the complementary function to φ in the sense of Young, that is,

$$\psi(\|x^*\|) = \sup\{\langle x, x^*\rangle - \varphi(\|x\|); x \in X\} \quad \text{for any } x^* \in X^*.$$

It is well known that $(L_\varphi(\Omega), \|\cdot\|_\varphi)^* = (L_\psi(\Omega), \|\cdot\|^0_\psi)$.

2. Main results

Since we use one of the results proved by Hudzik and Mastylo in [5], for convenience, we state it as a lemma here.

Lemma 2.1 *A σ-complete Banach lattice X, which is not σ-order continuous, contains a subspace isomorphic to l^∞.*

Considering the asymptotic isometric operator, we add the condition that X is σ-order semi-continuous and get Theorem 2.2. A σ-complete Banach lattice X is said to be σ-order semi-continuous whenever for every upward non-negative sequence $\{x_n\}_n$ in X with $\bigvee x_n = x$ and $x \in X$, we have $\lim_{n\to\infty} \|x_n\| = \|x\|$.

Theorem 2.2 *A σ-complete Banach lattice X, which is σ-order semi-continuous but not σ-order continuous, contains a sublattice asymptotically isometric to l^∞.*

Proof From Lemma 2.1, X contains a subspace isomorphic to l^∞. It follows that there exists a bounded sequence $\{u_k\}$ of mutually disjoint elements of X, so that $1 = \inf \|u_k\| > 0$ and $0 \le u_k \le x$ for all $k \in \mathbb{N}$. Put

$$K_n = \sup\{\|\sum_{i=n}^{m} u_i\|; m \in \mathbb{N}, m \ge n\} \quad \text{for all } n \in \mathbb{N}.$$

Since $\{u_k\}$ are disjoint positive elements of X, $\{K_n\}$ is a nonincreasing sequence satisfying $1 \le K_n \le \|x\|$ for all $n \in \mathbb{N}$. Then we have

$$1 \le K = \lim_n K_n \le \|x\|.$$

Let $\{\varepsilon_n\}$ be a null sequence in $(0,1)$. Take integers θ_n and η $(n = 1, 2, \ldots)$, with $0 < \theta_n < 1 < \eta$ and $\theta_n \eta^{-1} \ge (1 - \varepsilon_n)$. Put $k_1 \in \mathbb{N}$ such that $K_{k_1} < \eta K$. From the definition of K_n, there is $k_2 > k_1$ such that

$$\|\sum_{i=k_1}^{k_2-1} u_i\| > \theta_1 K_{k_1} > \theta_1 K.$$

By induction, we get an increasing sequence $\{k_n\}$ of integers, such that

$$\|\sum_{i=k_n}^{k_{n+1}-1} u_i\| > \theta_n K,$$

and

$$K_{k_n} < \eta K.$$

Let

$$x_n = \sum_{i=k_n}^{k_{n+1}-1} (\eta K)^{-1} u_i.$$

Then $x_n \wedge x_m = 0$ $(n \ne m)$. $x_n < (\eta K)^{-1} x$ for all $n \in \mathbb{N}$.

For each $(\xi_n) \in (l^\infty)^+$, we define

$$T((\xi_n)) = \bigvee_{i=1}^{\infty} \xi_i x_i.$$

The supremum exists because X is σ-complete and $\xi_i x_i \le \|(\xi_n)\|_\infty (\eta K)^{-1} x$. It is easy to check that T is an isomorphism from the positive cone of $(l^\infty)^+$ into that of X. So it can extend uniquely to a lattice isomorphism $\overline{T}$ from l^∞ into X.

For any $(\xi_n) \in (l^\infty)$, since X is σ-order semi-continuous, we have

$$\begin{aligned}\|\overline{T}(\xi_n)\| &= \|\overline{T}|(\xi_n)|\| = \|T|(\xi_n)|\| \\ &= \lim_{n\to\infty} \|\sum_{i=1}^{n} |\xi_i| x_i\| \leq \|(\xi_i)\|_\infty \sup_n \|\sum_{i=1}^{n} x_i\| \\ &\leq (\eta K)^{-1} K_{k_1} \|(\xi_i)\|_\infty \leq \|(\xi_i)\|_\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\overline{T}(\xi_n)\| &= \||\overline{T}(\xi_n)|\| = \|T((\xi_n)^+) + T((\xi_n)^-)\| \\ &\geq |\xi_n| \|x_n\| \geq \theta\eta^{-1}|\xi_n| \geq (1-\varepsilon_n)|\xi_n|, \quad n \in \mathbb{N}.\end{aligned}$$

Then

$$\|\overline{T}(\xi_n)\| \geq \sup_n (1-\varepsilon_n)|\xi_n|.$$

Hence, X contains an asymptotically isometric copy of l^∞ as a sublattice.

The following result gives an example of a space containing an almost isometric copy of l^∞ but not containing asymptotically isometric copies of l^∞.

Theorem 2.3 *Let φ be an Orlicz function defined on $\mathbb{R}$ and ψ be the complementary function to φ in the sense of Young. If $(L_\psi(\Omega), \|\cdot\|_\psi)^* = (L_\varphi(\Omega), \|\cdot\|_\varphi^0)$. Then $(L_\varphi(\Omega), \|\cdot\|_\varphi^0)$ need not contain an asymptotically isometric copy of l^∞.*

Proof Suppose $(L_\varphi(\Omega), \|\cdot\|_\varphi^0)$ must contain an asymptotically isometric copy of l^∞. Since $(L_\psi(\Omega), \|\cdot\|_\psi)^* = (L_\varphi(\Omega), \|\cdot\|_\varphi^0)$, by [6,Theorem 1], it must contain an isometric copy of l^∞. However, this contradicts the fact that the Orlicz space $(L_\varphi(\Omega), \|\cdot\|_\varphi^0)$ need not contain an isometric copy of l^∞[7]. □

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