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A Note on Zeros of Characters of Finite Groups

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Abstract The aim of this note is to classify the finite meta-abelian groups in which every irreducible character vanishes on at most three conjugacy classes.

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1. Introduction

Let G be a finite group. $\mathrm{Irr}_1(G)$ denotes the set of non-linear irreducible complex characters of G, and p always denotes a prime. For $\chi \in \mathrm{Irr}(G)$, set $\upsilon(\chi) := \{g \in G \mid \chi(g) = 0\}$. Clearly, $\upsilon(\chi)$ is a union of some conjugacy classes of G. An old theorem of Burnside asserts that $\upsilon(\chi)$ is not empty for any $\chi \in \mathrm{Irr}_1(G)$. In this paper, we consider the following problem: given the number of zeros in character table of a finite group G, what can be said about the structure of G? Our aim is to classify the finite meta-abelian group G satisfying the following hypothesis:

(HY) Each $\chi \in \mathrm{Irr}_1(G)$ vanishes on at most three conjugacy classes.

The main result of this paper is as follows.

Theorem *A finite meta-abelian group G satisfies (HY) if and only if G is one of the following groups:*

(1) G is a Frobenius group with abelian kernel G' and a complement of order 2 or 3.

(2) $G \cong D_8$ or Q_8.

(3) $G = G'P$, where G' is a normal and abelian 2-complement of G, $P \in Syl_2(G)$, $|P| = 4$, $|Z(G)| = 2$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and a complement $P/Z(G)$ of order 2.

(4) G is a Frobenius group with kernel G' and a cyclic complement of order 4.

(5) $G = (G'\langle t\rangle) \times \langle u\rangle$, where $\langle u\rangle$ is a cyclic group of order 3, t is an involution and $G'\langle t\rangle$ is a Frobenius group with kernel G' and a complement of order 2.

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We will call a conjugacy class of G as a G-class. The rest of our notation is standard and taken from [1].

2. Proof of Theorem

First, we give some lemmas for proving the theorem.

Lemma 1 *Let G be a meta-abelian group. If $[G:G']=p$, then G is a Frobenius group with kernel G' and a complement of order p.*

Proof As G' is abelian, the Hall p'-subgroup K of G' is clearly a normal p-complement of G. So $G=KP$ with $P\in \mathrm{Syl}_p(G)$. Next $[P:P']=[G/K:(G/K)']=[G/K:G'/K]=p$, forcing $P'=1$ and so $K=G'$. Note that $C_{G'}(P)\subseteq Z(G)\cap G'=1$ (See [1, Theorem 5.6]). This implies that P acts as fixed point freely on G', and we are done.

For a finite group G, if $G'<G$ and $|C_G(g)|=|C_{G/G'}(G'g)|$ holds for any $g\in G-G'$, then (G,G') is called a Camina-pair.

Lemma 2[2, Theorem 2.1] *Let (G,G') be a Camina-pair. Suppose that G is not a p-group. Then either G is a Frobenius group with kernel G' or G/G' is a p-group for some prime p. In this case, G has a normal p-complement $M, M<G'$ and $C_G(m)\subseteq G'$ for all $m\in M-\{1\}$.*

Lemma 3[3, Lemma 19.1] *Let P be a p-group of class ≤ 2 and suppose that P acts non-trivially on some p'-group Q such that $C_P(x)\subseteq P'$ for all $x\in Q-\{1\}$. Then the action is Frobenius and P is either cyclic or isomorphic to Q_8.*

Proof of Theorem The sufficiency is obvious. We need only to prove the necessity. Take $\varphi\in \mathrm{Irr}_1(G)$. Since G' is abelian by the hypothesis, $\varphi_{G'}$ is not irreducible. It follows by [1, Theorem 6.22] that there exists a linear character λ of a subgroup H with $G'\leq H<G$ such that $\varphi=\lambda^G$. Then $G-H\subseteq v(\varphi)$.

Assume that $[H:G']=m$ and $[G:H]=r$. Then we have

$$G=H+Hx_1+\cdots+Hx_{r-1},\ x_i\notin H,$$

and

$$H=G'+G'y_1+\cdots+G'y_{m-1},\ y_j\notin G'.$$

It follows that

$$G-H=\sum_{i=1}^{r-1}\sum_{j=1}^{m-1}G'y_jx_i+\sum_{i=1}^{r-1}G'x_i. \tag{*}$$

For $x\notin G'$, $G'x$ is a G-class or a union of some G-classes, and so we conclude by the above equality $(*)$ that $G-H$ consists of at least $m(r-1)$ G-classes. Bearing in mind that $G-H\subseteq v(\varphi)$, then $G-H$ consists of at most 3 G-classes (since $v(\varphi)$ consists of at most 3 G-classes by the hypothesis), and thus we obtain that $m(r-1)\leq 3$, that is, $[H:G']([G:H]-1)\leq 3$.

Since $[H:G']([G:H]-1)\leq 3$, one of the following three cases occurs: (i) $[G:G']=2$ or $[G:G']=3$; (ii) $[G:G']=4$; (iii) $[H:G']=3$, $[G:H]=2$.

(I) Suppose that $[G:G']=2$ or $[G:G']=3$.

In this case, by Lemma 1, we can easily conclude that G satisfies (1) of the theorem.

(II) Suppose that $[G:G']=4$.

In this case, we have $G-G'=G'x\cup G'y\cup G'z$, where $x,y,z\in G-G'$. Note that G' is abelian, we obtain that $G=KP$, where K is an abelian normal 2-complement of G and $P\in \mathrm{Syl}_2(G)$. Clearly, $K\le G'$. Then $|P|\ge 4$ and $G/K\cong P$. In particular, every element of $\mathrm{Irr}(P)$ vanishes on at most 3 P-classes.

Suppose first that $|P|\ge 8$. Then P is of maximal class (see [4, P.375]). Assume that $|P|\ge 16$. As P is of maximal class, one of the upper central series member must have index 16. Now every group of order 16 has a non-linear irreducible character which vanishes on at least 4 classes (see [5, P.300]). Thus P is either a group of order 4 or a non-abelian group of order 8.

Now suppose that $G/K\cong P$ is a non-abelian group of order 8. Let χ be the unique element of $\mathrm{Irr}_1(G/K)$. We can easily conclude that $G-G'\subseteq \upsilon(\chi)$. It follows from the hypothesis that $\upsilon(\chi)$ consists of at most 3 G-classes. Then $G'x=x^G$, $G'y=y^G$ and $G'z=z^G$. Thus $|C_G(g)|=|C_{G/G'}(G'g)|=4$ for every $g\in G-G'$. Hence (G,G') is a Camina-pair. If G is not a 2-group, then by Lemmas 2 and 3, we see that $P\cong Q_8$ and G is a Frobenius group with kernel M and a complement isomorphic to Q_8. Thus G has an irreducible character χ with $\chi(1)=8$ such that $\upsilon(\chi)$ consists of at least 4 G-classes, and we obtain a contradiction. So if $|P|=8$, then $G=P$, and thus G satisfies (2) of the theorem.

Next suppose that $G/K\cong P$ is of order 4. Notice that $C_{G'}(P)\subseteq Z(G)\cap G'=1$ (See [1, Theorem 5.6]), we have $C_{G'}(P)=\{1\}$. Since G' is abelian and $[G:G']=4$, it follows by [1, Theorem 6.15] that $\chi(1)=2$ or 4 for every $\chi\in\mathrm{Irr}_1(G)$.

Assume that $\chi(1)=2$ for all $\chi\in\mathrm{Irr}_1(G)$. Then, since $C_{G'}(P)=\{1\}$, we obtain that $|Z(G)|=2$ (see [1, Theorem 12.5 and Lemma 12.12]) and thus $[G/Z(G):(G/Z(G))']=2$. Bearing in mind that $(G/Z(G))'\cong G'$ is abelian, we get from Lemma 1 that $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))'\cong G'$ and a complement of order 2. Thus G satisfies (3) of the theorem.

Assume that there exists $\chi\in\mathrm{Irr}_1(G)$ such that $\chi(1)=4$. Recall that $[G:G']=4$, we can easily conclude that $\chi=\lambda^G$, where λ is a linear character of G'. It follows that $G'x\cup G'y\cup G'z=G-G'=\upsilon(\chi)$, and thus $G'g$ is a G-class for all $g\in G-G'$. So, we conclude that $|C_G(g)|=[G:G']=4=|P|$ for all $g\in P-\{1\}$. It follows that $G=G'P$ is a Frobenius group with kernel G' and a cyclic complement of order 4 (see [1, Problems (7.1), p.121] and [4, V, theorem 8.7]), and thus G satisfies (4) of the theorem.

(III) Suppose that $[H:G']=3$ and $[G:H]=2$.

We have $G-H=G'x\cup G'y\cup G'z$, where $x,y,z\notin H$. Since $G-H\subseteq\upsilon(\varphi)$ and $\upsilon(\varphi)$ consists of at most 3 G-classes, we conclude that $G-H=\upsilon(\varphi)$ and $G'x=x^G$, $G'y=y^G$ and $G'z=z^G$. It follows that $|C_G(g)|=6$ for all $g\in G-H$.

Clearly, $G-H$ contains an involution t, and $|C_G(t)|=6$ implies that $|G|_2=2$. Thus $G=H\langle t\rangle$ and $|H|$ is odd.

On the other hand, by the second orthogonality relation we have

$$|C_G(g)| = |G/G'| + \sum\{|\chi(g)|^2 \mid \chi \in \mathrm{Irr}_1(G)\}$$

for all $g \in G - G'$. Then $\chi(g) = 0$ for all $g \in G - H$ and all $\chi \in \mathrm{Irr}_1(G)$, that is, $G - H \subseteq \upsilon(\chi)$ for all $\chi \in \mathrm{Irr}_1(G)$. So, we obtain that $G - H = \upsilon(\chi)$ for all $\chi \in \mathrm{Irr}_1(G)$, and hence θ_H is not irreducible for every $\theta \in \mathrm{Irr}_1(G)$.

Now we claim that H is abelian. If else, let $\alpha \in \mathrm{Irr}_1(H)$ and $\theta \in \mathrm{Irr}_1(G)$ with $[\theta_H, \alpha] \neq 0$. Then $\theta_H = \alpha + \alpha^t$ and $\alpha^G = \theta$. As G' is abelian, $\alpha_{G'} = \beta_1 + \beta_2 + \beta_3$ with $\beta_i \in \mathrm{Irr}(G')$. Then $I_H(\beta_1) = G'$ so that $(\beta_1)^H = \alpha$. Then $\theta = \alpha^G = ((\beta_1)^H)^G = (\beta_1)^G$. So $G - G' \subset \upsilon(\theta) = G - H$, a contradiction. Hence H is abelian.

Clearly, $C_G(t)$ contains an element u of order 3 so that $u \in H$ and as H is abelian, $\langle t, H\rangle \subset C_G(u)$ and so $u \in Z(G)$. As $G' \cap Z(G) = 1$ (see [1, Theorem 5.6]), $u \in H - G'$. Note that $C_G(t) = \langle t\rangle \times \langle u\rangle$ with $t, u \notin G'$. Thus $C_G(t) \cap G' = 1$ and so $G = G'C_G(t) = (G'\langle t\rangle)\langle u\rangle$. As $u \in Z(G)$, $G = (G'\langle t\rangle) \times \langle u\rangle$.

Finally, $C_G(t) \cap G' = 1$ implies that $G'\langle t\rangle$ is a Frobenius group with kernel G' and a complement of order 2. Hence G satisfies (5) of the theorem. The proof is completed. □

Remark As in the proof of Theorem, we see that for such a group G, there exists a normal subgroup N such that $G - N$ contains 2 G-classes or 3 G-classes. Qian treated such groups G in [6]. Of course, the set of such groups is big, and it is impossible to classify them completely.

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